Graphs that admit polyline drawings with few crossing angles*

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Abstract

We consider graphs that admit polyline drawings where all crossings occur at the same angle $\alpha \in (0, \frac{\pi}{2}]$. We prove that every graph on n vertices that admits such a polyline drawing with at most two bends per edge has $O(n)$ edges. This result remains true when each crossing occurs at an angle from a small set of angles. We also provide several extensions that might be of independent interest.

1 Introduction

Graphs that admit polyline drawings with few bends per edge and such that every crossing occurs at a large angle have received some attention lately, since cognitive experiments [8, 9] indicate that such drawings are almost as readable as planar drawings. That is, one can easily track the edges in such drawings, even though some edges may cross.

A *topological graph* is a graph drawn in the plane where the vertices are represented by distinct points, and edges by Jordan arcs connecting the incident vertices but not passing through any other vertex. A *polyline drawing* of a graph G is a topological graph where each edge is drawn as a simple polygonal arc between the incident vertices but not passing through any bend point of other arcs. In a polyline drawing, every crossing occurs in the relative interior of two segments of the two polygonal arcs, and so they have a well-defined crossing angle in $(0, \frac{\pi}{2}]$.

Didimo *et al.* [6] introduced *right angle crossing* (*RAC*) drawings, which are polyline drawings where all crossings occur at right angle. They proved that a graph with $n \geq 3$ vertices that admits a straight line RAC drawing has at most $4n - 10$ edges, and this bound is the best possible. A different proof of the same upper bound was later found by Dujmović *et al.* [7]. It is not hard to show that every graph admits a RAC drawing with three bends per edge (see Figure 1 for an example). Arikushi *et al.* [4] have recently proved, improving previous results by Didimo *et al.* [6], that if a graph with n vertices admits a RAC drawing with at most two bends per edge, then it has $O(n)$ edges.

Dujmović *et al.* [7] generalized RAC drawings, allowing crossings at a range of angles rather than at right angle. They considered αAC drawings, which are polyline drawings where every crossing occurs at some angle *at least* α. They showed that any straight line αAC drawing of a graph with n vertices has at most $\frac{\pi}{\alpha}(3n-6)$ edges, by partitioning the graph into $\frac{\pi}{\alpha}$ planar graphs.

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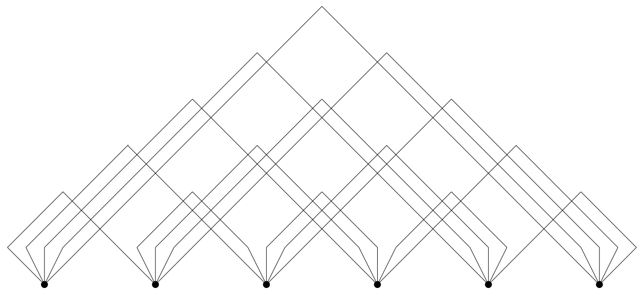


Figure 1: A RAC drawing of K_6 with 3 bends per edge.

They also proved that their bounds are essentially optimal for $\alpha = \frac{\pi}{k} - \varepsilon$, with $k = 2, 3, 4, 6$ and sufficiently small $\varepsilon > 0$.

Results for polyline drawings. We first consider polyline drawings where every crossing occurs at the same angle $\alpha \in (0, \frac{\pi}{2}]$. An $\alpha AC_b^=$ drawing of a graph is a polyline drawing where every edge is a polygonal arc with at most b bends and every crossing occurs at angle *exactly* α. It is easy to see that every graph with $n > 2$ vertices that admits an $\alpha AC_0^=$ drawing has at most $3(3n-6)$ edges (see Lemma 2.1 below). Every graph admits an $\alpha AC_3^=$ drawing for every $\alpha \in (0, \frac{\pi}{2}]$: Didimo *et al.* [6] constructed a RAC drawing of the complete graph with three bends per edge (see also Figure 1), where every crossing occurs between a pair of orthogonal segments of the same orientation, so an affine transformation deforms all crossing angles uniformly. It remains to consider graphs that admit $\alpha AC_1^=$ or $\alpha AC_2^=$ drawings. We prove the following.

Theorem 1.1. *For every $\alpha \in (0, \frac{\pi}{2}]$, a graph on n vertices that admits an $\alpha AC_2^=$ drawing has $O(n)$ edges. Specifically, a graph on n vertices has*

(a) *at most $27n$ edges if it admits an $\alpha AC_1^=$ drawing; and*
(b) *at most $385n$ edges if it admits an $\alpha AC_2^=$ drawing.*

For $\alpha = \frac{\pi}{2}$, slightly better bounds have been derived by Arikushi *et al.* [4]: they proved that if a graph on n vertices admits a RAC drawing with at most one (resp., two) bends per edge, then it has at most $6.5n$ (resp., $74.2n$) edges. Their proof techniques, however, do not generalize to all $\alpha \in (0, \frac{\pi}{2}]$.

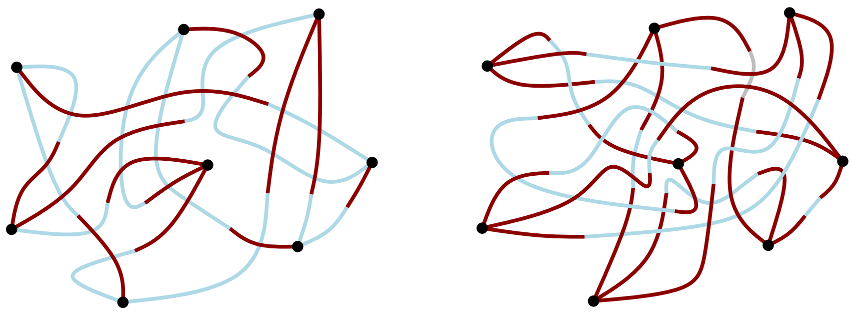
A straightforward generalization of $\alpha AC_1^=$ and $\alpha AC_2^=$ drawings are polyline drawings where each crossing occurs at an angle from a list of k distinct angles.

Theorem 1.2. *Let $A \subset (0, \frac{\pi}{2}]$ be a set of k angles, $k \in \mathbb{N}$, and let G be a graph on n vertices that admits a polyline drawing with at most b bends per edge such that every crossing occurs at some angle from A. Then,*

(a) *G has $O(kn)$ edges if $b = 1$;*
(b) *G has $O(k^2n)$ edges if $b = 2$.*

Generalizations to topological graphs. Suppose that every edge in a topological graph is partitioned into edge segments, such that all crossings occur in the relative interior of the segments. The bends in polyline drawings, for example, naturally define such edge partitions. An *end segment* is an edge segment incident to a vertex of the edge, while a *middle* segment is an edge segment not incident to any vertex. The key idea in proving Theorems 1.1 and 1.2

is to consider the crossings that involve either two end segments, or an end segment and a middle segment. This idea extends to topological graphs whose edge segments satisfy a few properties, which automatically hold for polyline drawings with same angle crossings (perhaps after removing a constant fraction of the edges). We obtain the following results, which might be of independent interest.



(a) (b)

Figure 2: (a) A 3-regular topological graph satisfying the conditions of Theorem 1.3 for $k = 2$. (b) A 3-regular topological graph satisfying the conditions of Theorem 1.4 for $k = 2$.

Theorem 1.3. *Let $G = (V, E)$ be a topological graph on n vertices, in which every edge can be partitioned into two end segments, one colored red and the other colored blue, such that (see Figure 2(a))*

(1) *no two end segments of the same color cross;*

(2) *every pair of end segments intersects at most once; and*

(3) *no blue end segment is crossed by more than k red end segments that share a vertex.*

Then G has $O(kn)$ edges.

We show that the above theorem implies the following stronger result.

Theorem 1.4. *Let $G = (V, E)$ be a topological graph on n vertices. Suppose that every edge of G can be partitioned into two end segments and one middle segment such that (Figure 2(b))*

(1) *each crossing involves one end segment and one middle segment;*

(2) *each middle segment and end segment intersect at most once; and*

(3) *each middle segment crosses at most k end segments that share a vertex.*

Then G has $O(kn)$ edges.

Note that Theorem 1.4 implies Theorem 1.3. Indeed, given a graph that satisfies the constraints in Theorem 1.3, one can partition every edge e into three parts as follows: its two end segments are the red segment and a crossing-free portion of the blue segment incident to a vertex, while the rest of the blue segment is the middle segment of e. Such a partition clearly satisfies the constraints in Theorem 1.4 with the same parameter k.

Organization. We begin with a few preliminary observations in Section 2. In Section 3, we consider polyline drawings with one possible crossing angle and prove Theorem 1.1. Then we extend the proof of Theorem 1.1(a) allowing up to k possible crossing angles and prove Theorem 1.2(a). We also show that Theorem 1.1(b) can be generalized to a weaker version of Theorem 1.2(b) with an upper bound of $O(k^4 n)$ (rather than $O(k^2 n)$). In Section 4, we generalize the crossing conditions from angle constraints to colored segments in topological graphs, and prove Theorems 1.3 and 1.4. Theorem 1.2(b) is derived from these general results at the end of Section 4. We conclude with some lower bound constructions and open problems in Section 5.

2 Preliminaries

In a polyline drawing of a graph, the edges are simple polygonal paths, consisting of line segments. We start with a few initial observations about line segments and polygonal paths. We say that two line segments *cross* if their relative interiors intersect in a single point. (In our terminology, intersecting segments that share an endpoint or are collinear do not cross.)

The following lemma is about the crossing pattern of line segments: if any two crossing segments cross at the same angle $\alpha \in (0, \frac{\pi}{2}]$, then a constant fraction of the segments are pairwise noncrossing. This lemma will be instrumental when applied to specific edge segments of an $\alpha AC_\infty^=$ drawing D.

Lemma 2.1. *Let $\alpha \in (0, \frac{\pi}{2}]$ and let S be a finite set of line segments in the plane such that any two segments may cross only at angle α. Then S can be partitioned into at most three subsets of pairwise noncrossing segments. Moreover, if $\frac{\pi}{\alpha}$ is irrational or if $\frac{\pi}{\alpha} = \frac{p}{q}$, where $\frac{p}{q}$ is irreducible and q is even, then S can be partitioned into at most two subsets of pairwise noncrossing segments.*

Proof. Partition S into maximal subsets of pairwise parallel line segments. Let $\mathcal{S}$ denote the subsets of S. We define a graph $G_\mathcal{S} = (\mathcal{S}, E_\mathcal{S})$, in which two subsets $S_1, S_2 \in \mathcal{S}$ are joined by an edge if and only if their respective directions differ by angle α. Clearly, the maximum degree of a vertex in $G_\mathcal{S}$ is at most two, and so $G_\mathcal{S}$ is 3-colorable. In any proper 3-coloring of $G_\mathcal{S}$, the union of each color class is a set of pairwise noncrossing segments in S, since they do not meet at angle α.

If $\frac{\pi}{\alpha}$ is irrational, then $G_\mathcal{S}$ is cycle-free. If $\frac{\pi}{\alpha} = \frac{p}{q}$, where $\frac{p}{q}$ is irreducible and q is even, then $G_\mathcal{S}$ can only have even cycles. In both cases, $G_\mathcal{S}$ is 2-colorable, and S has a partition into two subsets of pairwise noncrossing segments. □

The first claim in Lemma 2.1 can easily be generalized to finite sets of crossing angles [5].

Lemma 2.2 ([5])**.** *Let $A \subset (0, \frac{\pi}{2}]$ be a set of k angles, $k \in \mathbb{N}$, and let S be a finite set of line segments in the plane such that any two segments may cross only at an angle in A. Then S can be partitioned into at most $2k+1$ subsets of pairwise noncrossing segments.*

Proof. Partition S into maximal subsets of pairwise parallel line segments. Let $\mathcal{S}$ denote the subsets of S. We define a graph $G_\mathcal{S} = (\mathcal{S}, E_\mathcal{S})$, in which two subsets $S_1, S_2 \in \mathcal{S}$ are joined by an edge if and only if their respective directions differ by an angle in A. Clearly, the maximum degree of a vertex in $G_\mathcal{S}$ is at most $2k$, and so $G_\mathcal{S}$ is $(2k+1)$-colorable. In any proper $(2k+1)$-coloring of $G_\mathcal{S}$, the union of each color class is a set of pairwise noncrossing segments in S, since they do not meet at an angle in A. □

Polylines with restricted turning angles. In the proof of Theorem 1.1, rather than counting the edges in an graph with an $\alpha AC_1^=$ (resp., $\alpha AC_2^=$) drawing, we estimate the number of edges in an auxiliary multigraph, called a *red* graph. The edges of the red graph closely follow the edges of the $\alpha AC_1^=$ (resp., $\alpha AC_2^=$) drawing, and each bend lies at a crossing point. This ensures that the red graph has a polyline drawing where the angle between any two consecutive segments of an edge is exactly α. Since we direct the red edges, it will be necessary to distinguish between counterclockwise angles α and clockwise angles $-\alpha$.

Consider a simple open polygonal path $\gamma = (v_0, v_1, \ldots, v_n)$ in the plane. Refer to Figure 3(a). At every interior vertex v_i, $1 \le i \le n-1$, the *turning angle* $\angle(\gamma, v_i)$ is the directed angle in $(-\pi, \pi)$ (the counterclockwise direction is positive) from ray $\overrightarrow{v_{i-1}v_i}$ to $\overrightarrow{v_iv_{i+1}}$. The *turning angle* of the polygonal path γ is the sum of turning angles over all interior vertices $\sum_{i=1}^{n-1} \angle(\gamma, v_i)$. We say that two line segments have a *common tail* if they share an endpoint and one of them is contained in the other (*e.g.*, segments pu_1 and pv_1 have a common tail in Figure 3(c)).

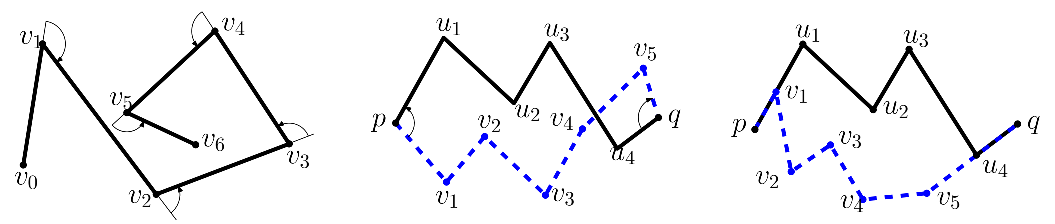


Figure 3: (a) The turning angles of a polygonal path. (b) Two crossing polygonal paths with the same turning angle between p and q. (c) Two noncrossing polygonal paths with the same turning angle between p and q.

We will use the next lemma to bound the maximal multiplicity of an edge in a red multigraph.

Lemma 2.3. *Let p and q be two points in the plane. Let γ_1 and γ_2 be two directed simple polygonal paths from p to q. If γ_1 and γ_2 have the same turning angle and they do not cross, then the first segment of γ_1 shares a common tail with the first segment of γ_2 and the last segment of γ_1 shares a common tail with the last segment of γ_2.*

Proof. Let $\gamma_1 = (u_0, u_1, \ldots, u_m)$ and $\gamma_2 = (v_0, v_1, \ldots, v_n)$, with $p = u_0 = v_0$ and $q = u_m = v_n$. Let β be their common turning angle. Since γ_1 and γ_2 do not cross, they enclose a weakly simple polygon P with $m+n$ vertices (Figure 3(c)). Suppose w.l.o.g. that the vertices of P in clockwise order are $v_0 = u_0, u_1, \ldots, u_m = v_n, v_{n-1} \ldots, v_1$. Every interior angle of P is in $[0, 2\pi]$, and the sum of interior angles is $(m+n-2)\pi$. The sum of interior angles at the vertices $u_1, \ldots, u_{m-1}$ is $(m-1)\cdot\pi + \beta$; and the sum of interior angles at $v_1, \ldots, v_{n-1}$ is $(n-1)\cdot\pi - \beta$. Hence the interior angles at p and q are both 0. $\square$

Overlapping edges. Let G be a topological multigraph. We say that two edges *overlap* if their intersection contains a connected set of more than one point. A maximal connected component of the intersection of two edges is called an *overlap* of the two edges. A *common tail* is an overlap of two edges that contains a common endpoint of the two edges. In Sections 3 and 4, we construct topological multigraphs whose edges may overlap, but only in common tails.

Lemma 2.4. *Let G be a topological multigraph in which some edges may overlap, but only in common tails. Then the edges of G can be slightly perturbed such that all overlaps are removed and no new crossings are introduced.*

Proof. We successively perturb G and decrease the number of edge pairs that have a common tail. Let $e = (u, v)$ be an edge in G, and let $e_1, e_2, \ldots, e_k$ be edges in G that have a common tail with e such that their overlaps with e contain the vertex u. Direct all these edges away from u. Then every edge e_i, $i = 1, 2, \ldots, k$, follows an initial portion of e, and then turns either right or left at some *turning point* p_i. Assume without loss of generality that there is at least one right turning point, and let p_j be the last such point. (Observe that the common tails of e incident to u and v, respectively, are disjoint, otherwise two edges that share common tails with e would overlap in an arc that is not a common tail. It follows that the part of e between u and p_j is disjoint from any common tail incident to v.)

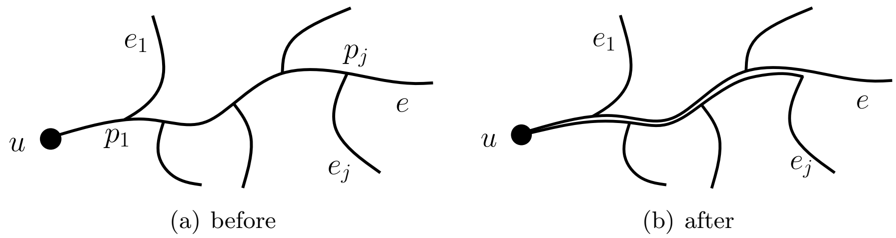


Figure 4: Removing overlaps.

Redraw all the edges e_i with a right turning point such that they closely follow e on the right. See Figure 4. We have removed the overlap between e and e_j, and decreased the number of edge pairs that have a common tail. □

In the sequel we will use the following upper bound (Theorem 2.5) for the maximum number of edges in a simple quasi-planar graph by Ackerman and Tardos [1]. A topological graph is *simple* if any two of its edges meet at most once, either at a common endpoint or at a crossing. A topological graph is *quasi-planar* if it has no three pairwise crossing edges.

Theorem 2.5 ([1]). *A simple quasi-planar graph on* $n \geq 4$ *vertices has at most* $6.5n - 20$ *edges.*

3 Polyline drawings with one crossing angle

In this section, we prove Theorem 1.1. Our proof technique can be summarized as follows. Consider an $\alpha AC_\infty^=$ drawing D of a graph $G = (V, E)$, where each edge has an arbitrary number of edge segments, and any two edges cross at angle α. For a constant fraction of the edges $(u, v) \in E$, we draw a new directed "red" edge that connects u to another vertex in V (which is not necessarily v). The red edges follow some edges in D, and they only turn at edge crossings of D. Some of the red edges may be parallel (even though G is a simple graph), but none of them is a loop, and some of them may have a common tail. The vertex set V and the red edges form a topological multigraph, which we call the "red graph." Every edge in the red graph is a polyline where the turning angles at each bend is $\pm\alpha$ or $\pm(\pi - \alpha)$. The multiplicity of the red edges can be bounded using Lemma 2.3. By Lemma 2.1, a constant fraction of the red edges form a crossing-free multigraph, and overlaps can be removed using Lemma 2.4. We continue with the details.

An $\alpha AC_\infty^=$ drawing of a graph G is a polyline drawing with an arbitrary number of bends where every crossing occurs at angle α. Every edge is a polygonal arc that consists of line segments. The first and last segments of each edge are called *end segments*, all other segments are called *middle segments*. Note that each end segment is incident to a vertex of G. Let $G=(V,E)$ be a graph with an $\alpha AC^{=}_{\infty}$ drawing. It is clear that G has at most $3n-6$ crossing-free edges, since they form a plane graph. All other edges have some crossings. We distinguish several cases below depending on whether the edges have crossings along their end segments.

3.1 Crossings between end segments

Lemma 3.1. *Let $\alpha \in (0, \frac{\pi}{2}]$ and $G=(V,E)$ be a graph on $n \geq 4$ vertices that admits an $\alpha AC^{=}_{\infty}$ drawing such that an end segment of every edge $e \in E$ crosses an end segment of some other edge in E. Then $|E| \leq 36n$. Moreover, the number of edges in E whose both end segments cross some end segments is at most $18n$.*

Proof. Let D be an $\alpha AC^{=}_{\infty}$ drawing of G as above. Let S be the set of end segments that cross some other end segments in D. We have $|E| \leq |S| \leq 2|E|$. Direct each segment $s \in S$ from an incident vertex in V to the other endpoint (which is either a bend point or another vertex in V). For a straight line edge, choose the direction arbitrarily.

We construct a directed multigraph $G'=(V,\Gamma)$. We call the edges in Γ *red*, to distinguish them from the edges of E. For every end segment $s \in S$, we construct a red edge $\gamma(s)$, which is a polygonal path with one bend between two vertices in V. For a segment $s \in S$, the path $\gamma(s)$ is constructed as follows (refer to Figure 5).

Let $u_s \in V$ denote the starting point of s (along its direction). Let c_s be the first crossing of s with an end segment, which we denote by t_s. Let $v_s \in V$ be a vertex incident to the end segment t_s. Now let $\gamma(s)=(u_s,c_s,v_s)$.

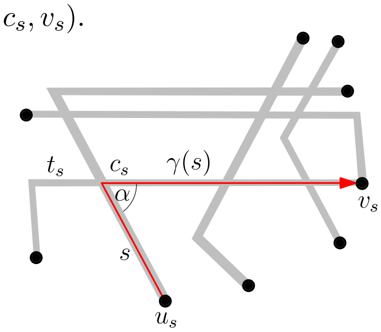


Figure 5: Construction of a red edge $\gamma(s)=(u_s,c_s,v_s)$.

Note that for every $s \in S$, the first segment of $\gamma(s)$ is part of the segment s and does not cross any segment in S. Hence the first segments of the red edges $\gamma(s)$ are distinct and do not cross other red edges. However, the second segment of $\gamma(s)$ may cross other red edges. Since the edges of G cross at angle α and c_s is a crossing, the turning angle of $\gamma(s)$ is $\pm\alpha$ or $\pm(\pi-\alpha)$. Note also that red edges may have common tails (which can be removed using Lemma 2.4).

We show that for any two vertices $u, v \in V$, there are at most 4 directed red edges from u to v. The red edges from u to v cannot cross, since their first segments are crossing-free, and their second segments are all incident to the same point v. By Lemma 2.3, any two noncrossing paths of the same turning angle between u and v must overlap in the first and last segments, however, the first segments of the red edges are pairwise non-overlapping. Since the red edges may have up to 4 distinct turning angles, there are at most 4 red edges from u to v.

We distinguish two types of red edges. Let $\Gamma_1 \subseteq \Gamma$ be the set of red edges whose second segment crosses some other red edge, and let $\Gamma_2 = \Gamma \setminus \Gamma_1$ be the set of red edges where both segments are crossing-free.

Note that two edges in Γ_1 cannot follow the same path γ in opposite directions because the first segments of every red edge is crossing-free. Hence, there are at most 4 red edges in Γ_1 between any two vertices in V. Let S_1 be the set of second segments of the red edges in Γ_1. By Lemma 2.1, there is a subset $S_1' \subseteq S_1$ of pairwise noncrossing segments of size at least $\frac{1}{3}|\Gamma_1|$. Let Γ_1' be the set of red edges containing the segments S_1', with $|\Gamma_1'| \geq \frac{1}{3}|\Gamma_1|$.

If Γ_2 contains two edges that follow the same path γ in opposite directions, then pick one arbitrarily, and let $\Gamma_2' \subseteq \Gamma_2$ be the selected red edges, with $|\Gamma_2'| \geq \frac{1}{2}|\Gamma_2|$. Now $(V, \Gamma_1' \cup \Gamma_2')$ is a crossing-free multigraph with maximum multiplicity 4, with at most $4(3n-6)$ edges. Note that any overlap between red end segments can be removed using Lemma 2.4, and so $(V, \Gamma_1' \cup \Gamma_2')$ becomes a planar multigraph with maximum multiplicity 4. It follows that $|\Gamma_1' \cup \Gamma_2'| \leq 4(3n-6)$, hence $|\Gamma| \leq 3 \cdot 4(3n-6) = 36n - 72$ for $n \geq 3$.

For the last part of the statement observe that in the above argument, an edge in E is counted twice if both of its end segments are in S. $\square$

We are now ready to prove part (a) of Theorem 1.1.

Lemma 3.2. *For any angle $\alpha \in (0, \frac{\pi}{2}]$, a graph on n vertices that admits an $\alpha AC_1^=$ drawing has at most $27n$ edges.*

Proof. Let $G = (V, E)$ be a graph with $n \geq 4$ vertices drawn in the plane with an $\alpha AC_1^=$ drawing. Let $E_1 \subseteq E$ denote the set of edges in E that have at least one crossing-free end segment. Let $G_1 = (V, E_1)$ and $G_2 = (V, E \setminus E_1)$.

It is easy to see that if $\alpha \neq \frac{\pi}{3}$, then G_1 is a simple quasi-planar graph and so it has at most $6.5n - 20$ edges by Theorem 2.5. If $\alpha = \frac{\pi}{3}$, let S_1 be the set of crossed end segments of edges in E_1. By Lemma 2.1, there is a subset $S_1' \subseteq S_1$ of pairwise noncrossing segments of size $\frac{1}{3}|E_1|$. The graph G_1' corresponding to these edges is planar, with at most $3n - 6$ edges. Hence E_1 contains at most $3 \cdot (3n - 6) = 9n - 18$ edges.

By Lemma 3.1, G_2 has at most $18n$ edges. Hence, G has at most $24.5n$ edges if $\alpha \neq \frac{\pi}{3}$ and at most $27n$ edges otherwise. $\square$

Remark. It is easy to generalize the proof of Lemma 3.1 to the case that every two polyline edges cross at one of k possible angles. The only difference is that the red edges may have up to $2k$ different turning angles.

Lemma 3.3. *Let $G = (V, E)$ be a graph on $n \geq 4$ vertices that admits a polyline drawing such that an end segment of every edge $e \in E$ crosses an end segment of some other edge in E at one of k possible angles. Then $|E| \leq 36kn$. Moreover, the number of edges in E whose both end segments cross some end segments is at most $18kn$.* $\square$

Corollary 3.4 (Theorem 1.2(a))**.** *Let $A \subset (0, \frac{\pi}{2}]$ be a set of k angles. If a graph G on n vertices admits a drawing with at most one bend per edge such that every crossing occurs at some angle from A, then G has at most $(18k + 3(2k + 1))n = (24k + 3)n$ edges.* $\square$

3.2 Polyline drawings with at most two bends per edge

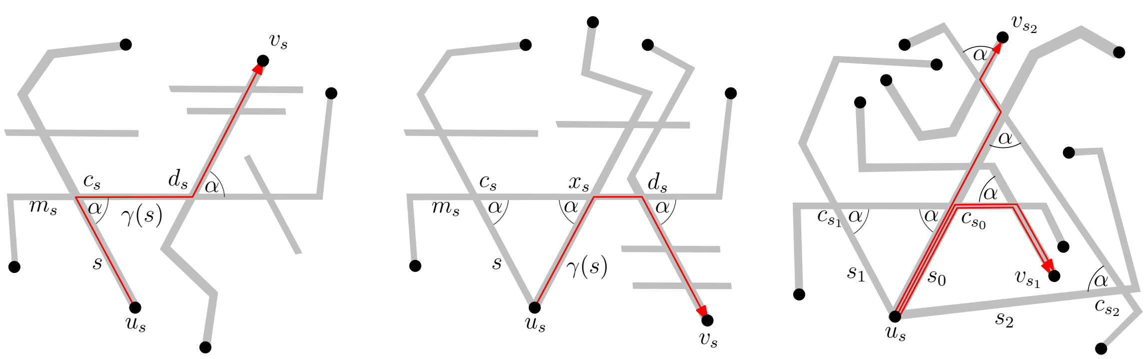
In the proof of Lemma 3.1, we constructed red edges in an $\alpha AC_1^=$ drawing of a graph G such that each red edge had a crossing-free first segment and one bend at a crossing. A similar strategy works for $\alpha AC_2^=$ drawings, but the red edges may now have up to two bends.

Lemma 3.5. *Let $\alpha \in (0, \frac{\pi}{2}]$ and $G = (V, E)$ be a graph on $n \geq 4$ vertices that admits an $\alpha AC_2^=$ drawing such that every crossing occurs between an end segment and a middle segment. Then $|E| \leq 116.14n$.*

Proof. Let D be an $\alpha AC_2^=$ drawing of G where every crossing occurs between an end segment and a middle segment. Every middle segment crosses at most two end segments incident to the same vertex in V since every crossing occurs at the same angle α. Let M be the set of middle segments that cross at least 3 end segments, and let S be the set of end segments that cross some middle segment in M. We distinguish two types of edges in G: let $E_1 \subseteq E$ be the set of edges with at least one end segment in S, and let $E_2 = E \setminus E_1$ be the set of edges with no end segment in S. Then $G_2 = (V, E_2)$ has at most $2|E_2|$ crossings in this drawing. The crossing number of a graph with n vertices and m edges is at least $0.032m^3/n^2 - 1.06n$ by a recent variant of the Crossing Lemma [10]. Applying this to (V, E_2), we have $2|E_2| \geq 0.032|E_2|^3/n^2 - 1.06n$, which gives $|E_2| \leq 8.14n$. In the remainder of the proof, we derive an upper bound for $|E_1|$.

We have $|E_1| \leq |S| \leq 2|E_1|$. Direct each segment $s \in S$ from an incident vertex in V to the other endpoint (which is either a bend or another vertex in V). We construct a directed multigraph (V, Γ), which we call the *red* graph. For every end segment $s \in S$, we construct a red edge $\gamma(s) \in \Gamma$, which is a polygonal path with two bends between two vertices in V. It is constructed as follows. Refer to Figure 6.

Let $u_s \in V$ denote the starting point of s (along its direction). Let c_s be the first crossing of s with a middle segment in M, which we denote by m_s. Recall that m_s crosses at least three end segments, at most two of which are incident to u_s. Let $d_s \in m_s$ be the closest crossing to c_s with an end segment that is not incident to u_s. Let $v_s \in V$ be a vertex incident to the end segment containing d_s. If c_s and d_s are consecutive crossings along m_s, then let $\gamma(s) = (u_s, c_s, d_s, v_s)$, see Figure 6(a). Otherwise, there is exactly one crossing x_s between c_s and d_s such that $u_s x_s$ is part of some end segment, and $\angle(u_s c_s, u_s x_s) = \pm(\pi - 2\alpha)$. In this case, let $\gamma(s) = (u_s, x_s, d_s, v_s)$, see Figure 6(b).



(a) (b) (c)

Figure 6: Construction of a red edge $\gamma(s)$. (a) c_s and d_s are consecutive crossings along m_s. (b) there is a crossing x_s between c_s and d_s. (c) The first segments of three red edges may overlap.

Every edge $\gamma(s) \in \Gamma$ has three segments: the first and third segments of $\gamma(s)$ lie along some end segments of edges in E, and the second segment of $\gamma(s)$ lies along a middle segment in M. By construction, the middle segment of $\gamma(s)$ is between two consecutive crossings along a middle segment in M, and so it does not cross any red edges. The two end segments of $\gamma(s)$

can cross only middle segments of red edges, however, the red middle segments are crossing-free. We conclude that no two red edges cross.

Since the bends c_s, d_s, and x_s are at crossings in an $\alpha AC_2^=$ drawing of G, the turning angle of $\gamma(s)$ must be among the 9 angles in $\{0, \pm\pi, \pm 2\alpha, \pm 2(\pi-\alpha), \pm(\pi-2\alpha)\}$. Note also that the red edges may have common tails (which can be removed using Lemma 2.4). Furthermore, the first segments of at most three red edges may overlap because the angle between s and the first segment of $\gamma(s)$ is 0 or $\pm(\pi-2\alpha)$. However, if the first segments of three red edges overlap, then at most two of these edges are parallel (that is, join the same two vertices in V), see Figure 6(c).

We show that there are at most 36 directed red edges between any two vertices $u, v \in V$. By Lemma 2.3, any two noncrossing paths of the same turning angle between u and v must overlap in the first and last segments. As noted above, the first segments of at most two parallel red edges overlap. Since the red edges may have up to 9 distinct turning angles, there are at most 18 red edges from u to v by Lemma 2.3. Hence there are at most 36 red edges between u and v (in either direction).

Since (V, Γ) is a planar multigraph with edge multiplicity at most 36, it has at most $|\Gamma| \leq 36(3n-6) < 108n$ edges. Altogether, we have $|E| = |E_1| + |E_2| \leq |E_1| + |\Gamma| \leq 8.14n + 108n = 116.14n$. □

We can now prove part (b) of Theorem 1.1.

Lemma 3.6. *For any angle $\alpha \in (0, \frac{\pi}{2}]$, a graph $G = (V, E)$ on n vertices that admits an $\alpha AC_2^=$ drawing has less than $385n$ edges.*

Proof. Consider an $\alpha AC_2^=$ drawing of G. Let E_0 be the set of edges which have an end segment crossing the end segment of another edge. By Lemma 3.1, we have $|E_1| \leq 36n$.

Consider the edges $E_1 = E \backslash E_0$. By Lemma 2.1, there is a partition $E_1 = E_{11} \cup E_{12} \cup E_{13}$ such that the middle segments of the edges in each subset are pairwise noncrossing. Suppose without loss of generality that $|E_{11}| = \max(|E_{11}|, |E_{12}|, |E_{13}|)$. By Lemma 3.5, we have $|E_{11}| \leq 116.14$. It follows that $|E| = |E_0| + |E_1| \leq |E_0| + 3|E_{11}| \leq (36 + 3 \cdot 116.14)n = 384.42n$. □

Remark. It is not difficult to generalize the proof of Lemma 3.5 to the case that every two polyline edges cross at one of k possible angles. The only difference is that the red edges may have up to $(4k)^2$ different turning angles, and that the first segment of a red edge may overlap at most $(2k-1)$ first segments of other red edges.

Lemma 3.7. *Let $\alpha \in (0, \frac{\pi}{2}]$ and $G = (V, E)$ be a graph on $n \geq 4$ vertices that admits a polyline drawing such that every crossing occurs between an end segment and a middle segment at one of k possible angles. Then $|E| = O(k^3 n)$.* □

Corollary 3.8. *Let $A \subset (0, \frac{\pi}{2}]$ be a set of k angles. If a graph G on n vertices admits a drawing with at most two bends per edge such that every crossing occurs at some angle from A, then G has $O(k^4 n)$ edges.* □

The dependence on k can be improved. In the next section, we reduce the upper bounds in Lemma 3.7 and Corollary 3.8 to $O(nk)$ and $O(nk^2)$, respectively.

4 Crossing between end segments in topological graphs

In this section, we prove Theorems 1.3 and 1.4, and then deduce part (b) of Theorem 1.2 from these general results. Our proof techniques are similar to the method in the previous section:

we construct a topological multigraph (V,Γ) whose edges are drawn along some edges in a given topological graph (V,E). The key difference is that we do not assume anything about the crossing angles, and so we cannot use Lemma 2.3 for bounding the edge multiplicity in (V,Γ). The greatest challenge in this section is to bound the edge multiplicity in the auxiliary graph (V,Γ) using solely combinatorial and topological conditions.

4.1 Proof of Theorem 1.3

We start with the proof of Theorem 1.3, which is the topological analogue of our result for $\alpha AC_1^=$ drawings.

Proof of Theorem 1.3: Let $G=(V,E)$ be a topological graph on n vertices, and assume that every edge in E is partitioned into a red end segment and a blue end segment, such that: (1) no two end segments of the same color cross; (2) every pair of end segments intersects at most once; and (3) no blue end segment is crossed by more than k red end segments that share a vertex. Assume further that G is drawn so that the number of edge crossings is minimized subject to the conditions (1)–(3). We show that G has $O(kn)$ edges.

By Theorem 2.5 the graph G has at most $6.5n-20$ edges with a crossing-free end segment, since these edges form a simple quasi-planar graph. Denote by $E_1 \subseteq E$ the set of the remaining edges of G.

For every edge $e \in E_1$ we draw a new edge $\gamma(e)$ as follows. Let s denote the red end segment of e, and let $u_s \in V$ be the vertex incident to s. Direct s from u_s to its other endpoint, and let c_s be the first crossing point along s. By condition (2), c_s is a crossing of s with a blue end segment s' of some edge e', where s' is incident to a unique vertex v_s. It is clear that $u_s \neq v_s$, since otherwise we can redraw the portion of e' between v_s and c_s so that it closely follows e and thereby reduce the total number of crossings in G without violating conditions (1)–(3). Let $\gamma(e)$ be the Jordan arc between u_s and v_s that follows the red segment s from u_s to c_s, and the blue segment s' from c_s to v_s. The new edges form a topological multigraph graph $G'=(V,\Gamma)$, where $\Gamma=\{\gamma(e) : e \in E_1\}$. We call the edges in Γ *red-blue* to distinguish them from the edges in E.

Note that G' is a plane multigraph. Indeed, crossings may occur only between a red end segment and a blue end segment, however, the red end segment of every edge in G' is crossing-free. G' might contain edges with a common tail, however, these overlaps may be removed using Lemma 2.4.

We define a *bundle of edges* in G' as a maximal set of parallel edges such that the interior of the region enclosed by the edges does not contain any vertex of V. Recall that a plane multigraph on n vertices has at most $3n$ edges if it has no face of size 2. Therefore, G' has at most $3n$ bundles.

Proposition 4.1. *Every bundle of edges of G' contains at most $4k+6$ edges.*

Proof. Let B be a bundle of edges between vertices $u,v \in V$. Let $B_1 \subseteq B$ be the set of red-blue edges in B whose red segment is incident to u, and assume without loss of generality that $|B_1| \geq |B|/2$.

Label the red-blue edges in B_1 by $\gamma_1,\gamma_2,\ldots,\gamma_\ell$ in the order they appear in the rotation system at u such that the closed region R enclosed by γ_1 and γ_ℓ contains all other edges of B_1. For $i=1,2,\ldots,\ell$, let $e_i \in E$ be the edge of the original graph G incident to u whose red segment contains the red segment of γ_i.

Let $Q = \{e_2, e_3, \ldots, e_{\ell-1}\} \subseteq E$ be a set of $\ell - 2$ edges in E_1 containing the red segments of $\gamma_2, \gamma_3, \ldots, \gamma_{\ell-1}$. By property (3), the red segments of at most k edges in Q cross the blue segment of γ_1. Similarly, the red segments of at most k edges in Q cross the blue segment of γ_ℓ. So the red segments of at least $\ell - (2k+2)$ edges in Q cross neither the red nor the blue segment of γ_1 and γ_ℓ. The relative interiors of these red segments lie in the interior of region R. The blue segment of such an edge in Q cannot cross the boundary of R (since the red segments of γ_1 and γ_ℓ are crossing-free, and blue segments do not cross each other), so this edge must connect u and v. The graph G has at most one edge between u and v, and so $\ell - (2k+2) \leq 1$. It follows that $|B_1| = \ell \leq 2k+3$, as required. $\square$

Therefore, G' has at most $3(4k+6)n$ edges. We conclude that $|E| \leq |E_1| + 6.5n = |E'| + 6.5n \leq (4k+6)3n + 6.5n = (12k+24.5)n$. $\square$

4.2 Proof of Theorem 1.4

We now prove Theorem 1.4.

Proof of Theorem 1.4: Let $G = (V, E)$ be a topological graph on n vertices. Assume that every edge of G is partitioned into two end segments and one middle segment such that (1) each crossing involves one end segment and one middle segment; (2) each middle segment and end segment intersect at most once; and (3) each middle segment crosses at most k end segments that share a vertex. Assume further that G is drawn in the plane so that the number of edge crossings is minimized subject to the constraints (1)–(3). We show that G has $O(kn)$ edges.

Observe that G has at most $3n-6$ edges whose both end segments are crossing-free, since two such edges cannot cross each other. Let $E_1 \subset E$ denote the set of edges with at least one crossed end segment. Let S be the set of end segments with at least one crossing each. It is clear that $|E_1| \leq |S| \leq 2|E_1|$. We construct a red edge for every end segment in S.

Constructing a red graph. For every end segment $s \in S$, let $u_s \in V$ be the vertex incident to s. Direct s from u_s to its other endpoint, and let c_s be the first crossing along s. Direct every middle segment arbitrarily. For every end segment $s \in S$, we construct a directed red edge $\gamma(s)$, which is a Jordan arc from u_s to another vertex in V. These edges form a directed topological multigraph (V, Γ) with $\Gamma = \{\gamma(s) : s \in S\}$. The edges in Γ are called *red* to distinguish them from the edges of E.

For $s \in S$, the red edge $\gamma(s)$ is constructed as follows. See Figure 7(a) for an example. Point c_s is the crossing of s with some middle segment m_s. Let d_s be the first intersection point along m_s after c_s (following the direction of m_s) with an end segment s' which is not adjacent to u_s. That is, d_s is either a crossing of m_s with an end segment or it is the endpoint of m_s (if c_s is the last crossing along m_s or all segments that m_s crosses after c_s are incident to u_s). At any rate, d_s lies on a unique end segment, which is incident to a unique vertex $v_s \in V$. Now let the directed edge $\gamma(s)$ follow segment s from u_s to c_s, the middle segment m_s from c_s to d_s, and the end segment s' from d_s to v_s.

Since G is drawn with the minimal number of crossings, we have $u_s \neq v_s$. Indeed, suppose that $u_s = v_s$. If d_s is the endpoint of the middle segment m_s, then we could redraw the edge $e_m \in E$ containing m_s so that the middle segment of the edge e_m ends right before reaching point c_s and then e_m continues to u_s closely following along s without crossings. By redrawing e_m this way, we reduce the total number of crossings without violating conditions (1)–(3).

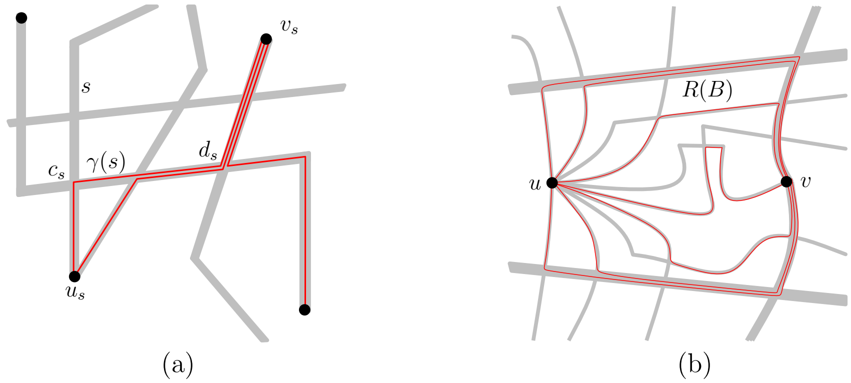


Figure 7: (a) Construction of a red edge $\gamma(s)$. (b) A bundle of 7 red edges from u to v.

Every red edge $\gamma(s)$ is naturally partitioned into three segments: a first, a middle, and a third segment. We briefly summarize the properties we have established for the three segments of the red edges.

(i) The first segments of the red edges are distinct, they lie along the end segments of G, and they are crossing-free.

(ii) The middle segment of each red edge lies along a middle segment in G, following its prescribed direction. A middle segment of a red edge $\gamma(s)$ may be crossed by the last segment of some other red edge $\gamma(s')$ if $\gamma(s')$ is incident to vertex $u_s \in V$.

(iii) The last segment of each $\gamma(s)$ lies along an end segment of G, and it possibly has a common tail with other red edges. The last segment of $\gamma(s)$ may cross middle segments of other red edges.

Observe that a crossing in the red graph can occur only between two red edges sharing a vertex. Note also that two red edges in Γ cannot follow the same Jordan arc in opposite directions (*e.g.*, (u, v) and (v, u)), since every red edge follows a prescribed direction along its middle segment. We show that (V, Γ) contains a plane subgraph having at least $|\Gamma|/4$ edges.

Label each vertex in V by either 0 or 1 as described below, and let $\Gamma_1 \subseteq \Gamma$ denote the set of red edges directed from a vertex labeled 0 to one labeled 1. If the labels are distributed uniformly at random, then every edge in Γ is in Γ_1 with probability 1/4. Hence there is a labeling such that $|\Gamma_1| \geq |\Gamma|/4$. Fix such a labeling for the remainder of the proof. By the properties of red edges noted above, no two edges in Γ_1 cross. If two red edges in Γ_1 overlap, then they have a common tail. By Lemma 2.4, overlaps along common tails can be removed, and so (V, Γ_1) is a directed plane multigraph.

Bundles of red edges. In (V, Γ_1), we define a *bundle* as a maximal set of directed parallel edges such that the interior of the region enclosed by the edges does not contain any vertex of V. See Figure 7(b). Let $\mathcal{B}$ denote the set of bundles of (V, Γ_1). For a bundle $B \in \mathcal{B}$, let $R(B)$ denote the region enclosed by the edges in B. Since Γ_1 is planar and each edge goes from a vertex labeled 0 to one labeled 1, the interior of the regions $R(B)$, $B \in \mathcal{B}$, are pairwise disjoint. Recall that a plane multigraph on n vertices has at most $3n$ edges if it has no face of size 2. Therefore, there are at most $3n$ bundles in $\mathcal{B}$.

Proposition 4.2. *Let $B = \{\gamma_1, \gamma_2, \ldots, \gamma_\ell\} \subseteq \Gamma_1$ be a bundle in $\mathcal{B}$ from u to v appearing in the rotation system at u in this order. For $i = 1, 2, \ldots, \ell$, let $e_i \in E$ be the edge of the original graph G incident to u whose end segment contains the first segment of γ_i. Then there are at least $\ell - (2k + 3)$ edges e_i, $1 \le i \le \ell$, whose first segment lies entirely in the region $R(B)$.*

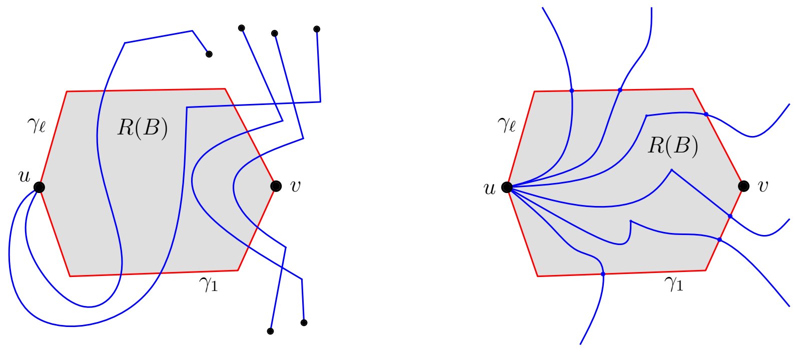
Proof. Let $E_{uv} = \{e_2, e_3, \ldots, e_{\ell-1}\} \subseteq E$ be a set of $\ell - 2$ edges in E containing the first segments of $\gamma_2, \gamma_3, \ldots, \gamma_{\ell-1} \in \Gamma_1$. By property (3), the first segments of at most k edges in E_{uv} cross the middle segment of γ_1. Similarly, the first segments of at most k edges in E_{uv} cross the middle segment of γ_ℓ. So at least $\ell - (2k+2)$ edges in E_{uv} cross neither the first nor the middle segment of γ_1 and γ_ℓ. At most one of these edges joins u and v, since G is a simple graph. So at least $\ell - (2k + 3)$ edges in E_{uv} has to cross the last segment of γ_1 or the last segment of γ_ℓ. However by property (1), only the middle segments of the edges in E_{uv} can cross the last segment of γ_1 or γ_ℓ. Hence, for at least $\ell - (2k+3)$ edges in E_{uv}, the first segment lies entirely in region $R(B)$. □

Partition Γ_1 into two two subsets $\Gamma_1 = \Gamma_2 \cup \Gamma_3$ as follows. Let Γ_2 contain all edges of all bundles of size at most $2k + 3$, as well as those edge $\gamma_i \in B$ of any larger bundle $B \in \mathcal{B}$ such that the first segment of the corresponding edge $e_i \in E$ is *not* contained in the region $R(B)$. Let $\Gamma_3 = \Gamma_1 \setminus \Gamma_2$, that is, Γ_3 contains all edges $\gamma_i \in \Gamma_1$ such that γ_i is part of some bundle B of size at least $2k + 4$, and the first segment of the corresponding edge $e_i \in E$ lies entirely in the region $R(B)$. By Proposition 4.2, each bundle in $\mathcal{B}$ contains at most $2k + 3$ edges of Γ_2. Since there are at most $3n$ bundles, we have $|\Gamma_2| \le (2k + 3)3n$. It remains to bound the number of edges in Γ_3.

Label each region $R(B)$ enclosed by a bundle $B \in \mathcal{B}$ by either 0 or 1 as described below. Let $\Gamma_4 \subseteq \Gamma_3$ be the set of edges $\gamma \in \Gamma_3$ such that the end segment of the edge $e \in E$ containing the first segment of γ lies in a region labeled 0, and the other end segment of e either lies in a region labeled 1 (including its boundary) or it does not lie in any such region. If the labels are distributed uniformly at random, then every edge in Γ_3 will be in Γ_4 with probability at least 1/4. Hence there is a labeling such that $|\Gamma_4| \ge |\Gamma_3|/4$. Fix such a labeling for the remainder of the proof. Let $E_4 \subseteq E$ denote the edges containing the first segments of the red edges in Γ_4.

Edges traversing a bundle. Consider a bundle $B \in \mathcal{B}$ of edges from u to v, and suppose that the region $R(B)$ is enclosed by the edges $\gamma_1, \gamma_\ell \in B$. We say that an edge $e \in E$ *traverses* $R(B)$ if a connected component of $e \cap R(B)$ intersects the interior of $R(B)$, but this component is incident to neither u nor v. See Figure 8(a). Suppose that $e \in E$ traverses $R(B)$. Then e crosses $\gamma_1 \cup \gamma_\ell$ twice. The first segments of γ_1 and γ_ℓ are crossing-free by property (i) of the red edges. The middle segments of γ_1 and γ_ℓ can cross only end segments incident to u by property (ii) of the red edges. By condition (3), at most $2k$ edges traverse $R(B)$ such that they intersect the middle segment of γ_1 or γ_ℓ. Denote by $T_1 \subset E$ the set of edges in E that traverse some bundle $B \in \mathcal{B}$ such that they cross a middle segment on the boundary of $R(B)$. Summing over all bundles, we have $|T_1| \le (3n)(2k) = 6kn$.

Assume now that $e \in E$ crosses the last segment of γ_1 and γ_ℓ. By conditions (1) and (2), the middle segment of e crosses the last segment of γ_1 and γ_ℓ. That is, a connected component of $e \cap R(B)$ is part of the middle segment of e. By condition (1) and (2), the middle segments that traverse $R(B)$ cannot cross each other inside $R(B)$. Among all middle segments that traverse $R(B)$, let $m(B)$ be the one whose intersection with γ_1 (and γ_ℓ) is closest to u. Recall that for a red edge $\gamma_i \in B \cap \Gamma_4$, we denote by $e_i \in E_4$ the edge containing the first segment of γ_i. By the



(a) (b)

Figure 8: (a) Four edges traverse $R(B)$, two of them cross some middle segments on the boundary of $R(B)$, and two of them cross end segments only. (b) Construction of two edges in $\widehat{E}$ by redrawing their portions within a region $R(B)$.

choice of $\Gamma_4 \subset \Gamma_3$, an end segment of e_i lies in the region $R(B)$. If the first segment of e_i crosses any middle segment that traverses $R(B)$, then e_i must cross $m(B)$. By condition (3), however, at most k edges $e_i \in E_4$ cross $m(B)$. Denote by $T_2 \subset E$ the set of edges in E such that one end segment lies entirely in the region $R(B)$ of some bundle $B \in \mathcal{B}$, and this end segment crosses a middle segment that traverses $R(B)$. Summing over all bundles, we have $|T_2| \leq (3n)k = 3kn$.

Let $\Gamma_5 \subseteq \Gamma_4$ be the set of edges $\gamma_i \in \Gamma_4$ such that the corresponding edge e_i is neither in T_1 nor in T_2. Let $E_5 \subseteq E$ denote the edges containing the first segments of the red edges in Γ_5. By the above argument, we have $|\Gamma_4| \leq |\Gamma_5| + |T_1| + |T_2| \leq |\Gamma_5| + 9kn$. It remains to derive an upper bound for $|\Gamma_5|$.

Bounding $|\Gamma_5|$. In order to bound $|\Gamma_5|$, we construct the new topological graph $(V, \widehat{E})$. For each $\gamma_i \in \Gamma_5$, we construct an edge $\hat{e}_i \in \widehat{E}$ as follows. Suppose that γ_i is in a bundle $B \in \mathcal{B}$ from u to v. Let $e_i \in E_5$ denote the edge that contains the first segment of γ_i. Suppose that $e_i = (u, w)$. By construction, the first segment of e_i lies in the region $R(B)$, and it does not cross any edge in E_5 that traverses $R(B)$. Indeed, the first segment of e_i cannot cross any edge in E_5 that traverses $R(B)$ since $E_5 \cap T_2 = \emptyset$, and the middle segment of e_i cannot cross any edge in E_5 that traverses $R(B)$ since $E_5 \cap T_1 = \emptyset$. Draw the edge $\hat{e}_i = (u, w)$ as follows (refer to Figure 8(b)): $\hat{e}_i$ starts from the vertex u, it goes to the first intersection point $e_i \cap \partial R(B)$ inside the region $R(B)$ as described bellow, then it follows e_i to the endpoint w outside of $R(B)$. Due to the 0-1 labeling of the regions $R(B)$, all edges of $\widehat{E}$ that intersect the interior of a region $R(B)$ are incident to only one vertex of the bundle B. Therefore, the portion of the edges in $\widehat{E}$ in each region $R(B)$, with $B \in \mathcal{B}$, can be drawn without crossings. We can now partition each edge $\hat{e}_i \in \widehat{E}_5$ into *two* segments: its *blue* segment consists of its part inside a region $R(B)$ and its part along the middle segment of e_i; its *red* segment is the last segment of e_i. As noted above, $\hat{e}_i$ does not cross any other edge of $\widehat{E}$ inside the region $R(B)$. By property (3), every blue segment crosses at most k red segments incident to the same vertex. We can apply Theorem 1.3 for the graph $(V, \widehat{E})$. It follows that $|\widehat{E}| = O(kn)$, hence $|\Gamma_5| = O(kn)$.

In summary, we have $|E| < 3n + |E_1| \leq 3n + |\Gamma| \leq 3n + 4|\Gamma_1| = 3n + 4(|\Gamma_2| + |\Gamma_3|) \leq 3n + 4(2k+3)3n + 4|\Gamma_3| \leq (8k+39)n + 16|\Gamma_4| \leq (8k+39)n + 16(9kn + |\Gamma_5|) = (152k+39)n + 16|\Gamma_5| \leq (152k+39)n + 16|\widehat{E}| \leq (152k+39)n + 16(12k+24.5)n = (344k+431)n$, as required. $\square$

4.3 Completing the proof of Theorem 1.2

We are now ready to prove Theorem 1.2(b).

Proof of Theorem 1.2(b): Let $A \subset (0, \frac{\pi}{2}]$ be a set of k angles. Let G be a graph on n vertices that admits a drawing with at most two bends per edge such that every crossing occurs at some angle from A. Partition the edges into two subsets $E = E_1 \cup E_2$, where E_1 is the set of edges which have an end segment crossing some other end segment, and E_2 contains all other edges in E. By Lemma 3.3, we have $|E_1| \leq 36kn$.

Let S_2 be the set of *middle* segments of all edges in E_2. By Lemma 2.2, there is a subset $S_2' \subseteq S_2$ of at least $\frac{1}{2k+1}|S_2| = \frac{1}{2k+1}|E_2|$ pairwise noncrossing segments. Let $E_2' \subseteq E_2$ be the set of edges whose middle segments are in S_2'. Note that in the graph (V, E_2'), every crossing is between an end segment and a middle segment. Moreover, no middle segment crosses more than $2k$ end segments that share a vertex, since there are k possible crossing angles, and for each angle $\alpha \in A$, two end segments incident to a vertex that meet a middle segment at the angle α form an isosceles triangle. Therefore, it follows from Theorem 1.4 that $|E_2'| = O(kn)$. Altogether, we have $|E| = |E_1| + |E_2| \leq 36kn + (2k+1)|E_2'| = O(k^2n)$. □

5 Discussion

We have shown that for every list A of k angles, a graph on n vertices that admits a polyline drawing with at most one (resp., two) bends per edge in which all crossings occur at an angle from A has at most $O(kn)$ (resp., $O(k^2n)$) edges. It is easy to construct a *straight line* graph with n vertices and $\Omega(kn)$ edges such that the edges cross in at most k different angles: Let the vertices $v_1, \ldots, v_n$ be equally spaced points along a circle in this order, add a straight line edge v_iv_j if and only if $|i-j| \leq k+1$.

With one bend per edge, one can construct slightly larger graphs on n vertices, but the number of edges remains $O(kn)$ by Theorem 1.2(a). However, we do not know whether the upper bound of $O(k^2n)$ in Theorem 1.2 is the best possible for polyline drawings with two bends per edge and k possible crossing angles.

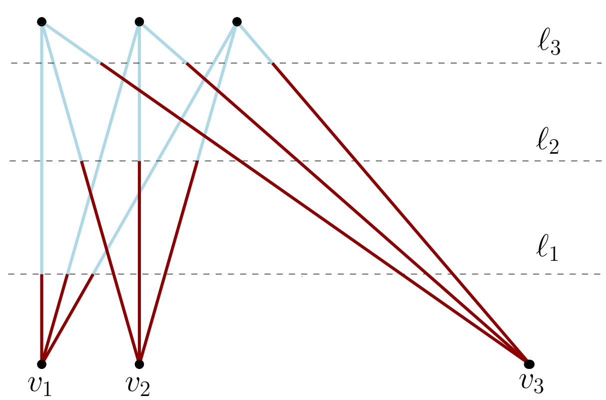


Figure 9: A straight line drawing of $K_{3,3}$ satisfying the conditions of Theorem 1.3 for $k = 2$.

In Theorems 1.3 and 1.4, the upper bound $O(kn)$ cannot be improved. For every $k \geq 0$ there is a straight line drawing of $K_{k+1,k+1}$ satisfying the conditions of Theorem 1.3, by the following construction which is due to Rom Pinchasi [11]. Place $k+1$ vertices on a horizontal line. Then add $k+1$ vertices $v_1, v_2, \ldots, v_{k+1}$ one by one on another horizontal line below the first line, and connect each of them to all the first $k+1$ vertices, as follows. Place v_1 arbitrarily on the bottom line, and partition each of its adjacent edges into red and blue segments such that the red segments are adjacent to v_1. For $i = 2, \ldots, k+1$, add v_i far enough to the right of v_{i-1} such that the edges between v_i and the first set of vertices cross only blue segments of previous edges. Let (x_i, y_i) be the highest crossing point on edges adjacent to v_i. Fix a horizontal line ℓ_i slightly above the line $y = y_i$ and partition every edge e adjacent to v_i into red and blue segments, such that endpoints of the red segment are v_i and $e \cap \ell_i$. See Figure 9 for an example. It is not hard to verify that this drawing satisfies the condition of Theorem 1.3. By taking $\left\lfloor \frac{n}{2(k+1)} \right\rfloor$ disjoint copies of $K_{k+1,k+1}$ drawn as above, we obtain a graph on n vertices and $\Omega(kn)$ edges satisfying the conditions of Theorem 1.3.

Finally, we mention a few related questions. How hard is it to determine whether a graph admits a polyline drawing with few bends per edge and few crossing angles? Recently, it was shown that recognizing straight line RAC graphs is NP-hard [3], so it is likely that recognizing graphs that admit αAC_2 drawings for a given α or just some α are hard as well. It might also be interesting to find or approximate the minimum value t for a given graph G such that G admits a polyline drawing with at most two bends per edge and t possible crossing angles.

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