A note on 1-planar graphs

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November 5, 2013

Abstract

A graph is 1-*planar* if it can be drawn in the plane such that each of its edges is crossed at most once. We prove a conjecture of Czap and Hudák [6] stating that the edge set of every 1-planar graph can be decomposed into a planar graph and a forest. We also provide simple proofs for the following recent results: (i) an n-vertex graph that admits a 1-planar drawing with straight-line edges has at most $4n - 9$ edges [7]; and (ii) every drawing of a *maximally dense right angle crossing* graph is 1-planar [12].

1   Introduction

In a *drawing* of a graph in the plane its vertices are represented as distinct points and its edges as Jordan arcs that connect corresponding points and do not contain any other vertex as an interior point. Any two edges in a drawing of a graph have a finite number of intersection points. Every intersection point of two edges is either a vertex that is common to both edges, or a crossing point at which one edge passes from one side of the other edge to its other side. A drawing of a graph is 1-*planar* if each of the edges is crossed at most once. If a graph has a 1-planar drawing then it is 1-*planar*.

The notion of 1-planarity was introduced in 1965 by Ringel [18], and since then many properties of 1-planar graphs have been studied (see, e.g., [1, 2, 3, 4, 5, 6, 7, 11, 13, 15, 16]). It is known that the maximum number of edges in an n-vertex 1-planar graph is $4n - 8$ [13, 17, 19], and that this bound is tight, that is, for any $n \geq 12$ there exists an n-vertex 1-planar graph with $4n - 8$ edges [17].

Czap and Hudák [6] showed that if an n-vertex 1-planar graph has the maximum number of edges, namely $4n - 8$, then its edge set can be decomposed into two subsets, such that one of them induces a planar graph and the other induces a forest. We prove their conjecture that this holds for every 1-planar graph.

Theorem 1. *Let $G = (V, E)$ be a 1-planar graph. Then there is a partition of E into two subsets A and B such that A induces a planar graph and B induces a forest.*

Note that it is not always possible to add an edge to a 1-planar graph with less than $4n - 8$ edges. Indeed, Brandenburg et al. [2] showed that there are 1-planar graphs with n vertices and only $\frac{45}{17}n + O(1)$ edges, such that adding an edge to any such graph results in a graph that is no longer 1-planar. Therefore, one cannot conclude Theorem 1 simply because it holds for 1-planar graphs with the maximum possible number of edges.

Apart from Theorem 1, this note contains simple proofs of two recent results related to 1-planar graphs. The first result is due to Didimo [7]:

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Theorem 2 ([7]). *A graph on $n \geq 3$ vertices that can be drawn in the plane with straight-line edges such that every edge is crossed at most once has at most $4n - 9$ edges.*

The second result, due to Eades and Liotta [12], concerns drawing of graphs with right angle crossings. A *right angle crossing* (RAC) drawing of a graph is a drawing with straight-line edges that may cross each other only at a right angle. A RAC graph is a graph that admits a RAC drawing. The class of RAC graphs was introduced by Didimo et al. [8], following experiments showing that large angle crossings are visually appealing [14]. They proved that an n-vertex RAC graph has at most $4n - 10$ edges and that this bound is tight, namely, there are RAC graphs with that many edges. We say that such RAC graphs are *maximally dense*.

Eades and Liotta [12] recently showed that every RAC drawing of a maximally dense RAC graph must be 1-planar. They also showed that there are RAC graphs that do not admit 1-planar RAC drawings, and that there are graphs with $4n - 10$ edges that admit 1-planar drawings but no RAC drawing. For further results on RAC graphs and related problems see a recent survey of Didimo and Liotta [9]. Here we provide a different and shorter proof of the main theorem in [12]. Moreover, our result is a bit more general.

Theorem 3. *Let D be a RAC drawing of an n-vertex graph $G = (V, E)$, $n \geq 3$, such that there is an edge $e \in E$ which is crossed $k \geq 1$ times in D. Then $|E| \leq 4n - 9 - k$.*

Corollary 4 ([12]). *If G is a maximally dense RAC graph then every RAC drawing of G is 1-planar.*

Organization. Theorem 1 is proved in Section 2, while Theorems 2 and 3 are proved in Section 3.

2 Proof of Theorem 1

Let $G = (V, E)$ be a 1-planar graph drawn in the plane such that no edge is crossed more than once. Henceforth we do not distinguish G from its drawing. We may assume without loss of generality that if two edges cross in G, then they do not share a common vertex. Indeed, otherwise these edges can be redrawn such that this crossing is eliminated and no new crossing is introduced (and the abstract graph remains the same).

Therefore, every crossing involves two edges with four distinct endpoints. In such a case we show that E can be partitioned into two subsets A and B, such that A induces a plane graph and B induces a plane forest. Note that this proves a slightly stronger statement than the one stated in Theorem 1.

Let p be a crossing point and let u and v be two vertices of the edges that cross at p, such that (u, v) is not one of these edges. Then we can draw a new edge (u, v) such that it is crossing-free by following the two edges that cross at p from u and v until they meet in a close neighborhood of p. For every crossing point p and every such u and v, we draw a new edge (u, v) as described. Note that the new drawing might contain parallel edges. Denote the new (multi)graph by $G' = (V, E')$ and let $E'_i \subseteq E'$ be the edges in E' that are crossed exactly i times, for $i = 0, 1$.

Call a face whose boundary is a simple cycle of length four a *quadrangle*. A *chord* is a new edge that is drawn within a face and connects two of its vertices that are not consecutive on the boundary of the face. The proof of Theorem 1 follows from the next claim.

Proposition 2.1. *For every plane multigraph $H = (V, E)$ and every pair of adjacent vertices $x, y \in V$ it is possible to add a chord to every quadrangle in H, such that the graph induced by the chords is a forest in which there is no path between x and y.*

(a) there is no face but f that is incident to both v_0 and v_2.

(b) Merge v_0 and v_2 into a new vertex v and apply induction.

(c) Split v back into v_0 and v_2 and add the chord (v_0, v_2).
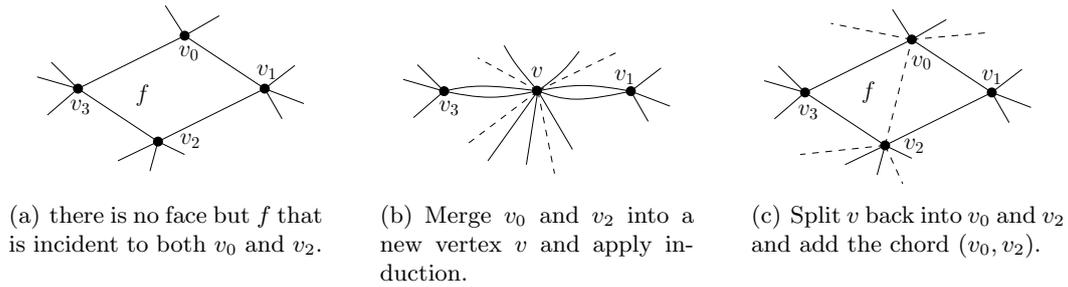
Figure 1: Case 1 of Proposition 2.1

Indeed, denote by G_0' the plane multigraph induced by E_0'. Then every pair of crossing edges in G are the possible chords of a distinct quadrangle in G_0'. Applying Proposition 2.1 to G_0' we obtain a set $B \subset E_1'$ of edges that induce a plane forest. Setting $A = (E_0' \cap E) \cup (E_1' \setminus B)$ we obtain the subsets A and B as required. It remains to prove Proposition 2.1.

Proof of Proposition 2.1: We may assume without loss of generality that H does not contain faces of size greater than four, for such faces can be triangulated without changing the set of quadrangles in H. We prove the claim by induction on the number of quadrangles in H. If there are no quadrangles then the claim trivially holds. Otherwise, let f be a quadrangle and let v_0, v_1, v_2, v_3 be the vertices on the boundary of f, listed in their clockwise order around f. We consider several cases.

Case 1: There is no other face but f that is incident to both v_0 and v_2. Note that this implies that v_0 and v_2 are not adjacent. We add the edge (v_0, v_2) within f and immediately contract it. Denote by H' the resulting graph, and let v be the vertex into which v_0 and v_3 are merged. Comparing H and H' we observe that f was replaced by two faces of size two, and the size of every other face and the number of vertices on its boundary have not changed (see Figure 1). We now apply the induction hypothesis on H' with the same pair of adjacent vertices x, y (if one of v_0 or v_2 is in $\{x, y\}$, then v 'plays' their role in H'). Then, we 'uncontract' the edge (v_0, v_2) and add it to the set of chords (see Figure 1(c)). Since every quadrangle in H except f is also a quadrangle in H', there is indeed a chord now in every quadrangle. Note also that if the graph induced by the chords of H contains a cycle or an x–y path, then so does the graph induced by the chords of H'.

Note that if there was another face but f that is incident to both v_0 and v_2, then merging them would change the number of (distinct) vertices on the boundary of that face. Specifically, we could destroy two quadrangles instead of just one. Therefore, such a case should be handled with care.

Case 2: There is a face different from f that is incident to both v_0 and v_2 and there is no other face but f that is incident to both v_1 and v_3. This case is similar to the previous case. The new chord will be (v_1, v_3).

Case 3: There is a face $f' \neq f$ that is incident to both v_0 and v_2, and there is a face $f'' \neq f$ that is incident to both v_1 and v_3. Observe that it must be that $f' = f''$. Indeed, otherwise we could have drawn the edges (v_0, v_2) and (v_1, v_3) as chords in f' and f'', respectively, such that they do not cross. Then add a vertex inside f and connect it to $v_0, \ldots, v_3$, and thus get a plane drawing of K_5, which is impossible.
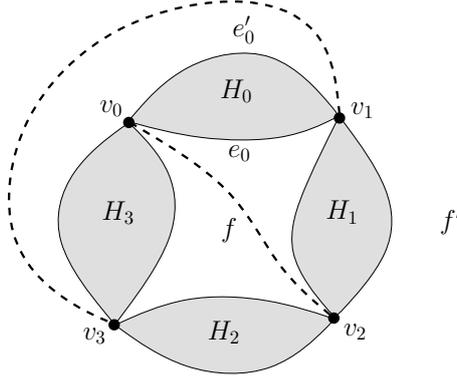
Figure 2: Case 3 of Proposition 2.1: both pairs v_0, v_2 and v_1, v_3 are incident to another face but f.

Note that f' is a quadrangle and that v_0, v_1, v_2, v_3 appear in this counter-clockwise order around f'. In order to simplify the presentation, we assume that f' is the outer face of H (recall that a planar graph can always be redrawn such that any given face becomes the outer face). For $i = 0, \ldots, 3$, denote by e_i and e_i' the edges (v_i, v_{i+1}) (addition is modulo 4) of f and f', respectively, and let H_i be the subgraph of H that is induced by all the vertices that lie in the bounded region whose boundary consists of e_i and e_i' (including v_i and v_{i+1}). Refer to Figure 2.

Since x and y are adjacent, there must some subgraph H_i that contains both of them. Assume without loss of generality it is H_0. We proceed by applying induction on each of the graphs H_i: for H_0 we apply induction with the vertices x and y, while for $i = 1, 2, 3$, we apply induction on H_i with the vertices v_i and v_{i+1}. Finally, we add the new chords (v_0, v_2) and (v_1, v_3) in f and f', respectively.

It is not hard to see that the set of chords does not induce a cycle or an x–y path. Indeed, a cycle or an x–y path of chords cannot go through H_i, for $i = 1, 2, 3$, since by the induction hypothesis there is no path of chords between v_i and v_{i+1}. Therefore, a cycle or an x–y path of chords if exists must lie inside H_0, but that also contradicts the induction hypothesis. □

Algorithmic aspects. Note that the inductive proof of Theorem 1 implies a polynomial-time (in fact, linear) algorithm for partitioning the edge set of an embedded 1-planar graph into a plane graph and a forest.

3 Proof of Theorems 2 and 3

Both proofs of Theorems 2 and 3 use the *discharging method* and are very similar. Therefore, we rephrase and prove Theorems 2 and 3 as a single theorem.

Theorem 5. *Let $G = (V, E)$ be an n-vertex graph, $n \geq 3$, and let D be a drawing of G in the plane with straight-line edges such that every edge is crossed at most k times. Then:*

(1) if $k = 1$ then $|E| \leq 4n - 9$;

(2) if D is a RAC drawing then $|E| \leq 4n - 9 - k$.

Proof. We follow and refine the proof of Theorem 4.1 in [10]. For completeness we repeat some of the arguments therein.

We may assume without loss of generality that G is connected and maximal in the sense that no crossing-free edges can be added to D. Let $G' = (V', E')$ be the plane graph we

obtain by adding the crossing points in D as vertices, and subdividing every crossed edge accordingly. We denote by F' the set of faces of G'. Suppose that we assign *charges* to the vertices and faces of G', such that the charge of a vertex v, $\text{ch}(v)$, is $\deg(v) - 4$, and the charge of a face f is $\text{ch}(f) = |f| - 4$, where $|f|$ denotes the size of f. Note that the charge of a new vertex (a crossing point in D) is zero. It now follows from Euler's formula that the total charge is -8:

$$\sum_{v \in V'} \text{ch}(v) + \sum_{f \in F'} \text{ch}(f) = \sum_{v \in V'} (\deg(v) - 4) + \sum_{f \in F'} (|f| - 4) = 2|E'| - 4|V'| + 2|E'| - 4|F'| = -8.$$

In the *discharging* phase every original vertex $v \in V$ sends 0.5 units of charge to every face it is incident to. Denote by $\text{ch}'(\cdot)$ the new charge function, and let $\text{ch}'(F') = \sum_{f \in F'} \text{ch}'(f)$. Since the total charge has not changed we have:

$$-8 = \sum_{v \in V'} \text{ch}'(v) + \text{ch}'(F') = \sum_{v \in V} \left(\frac{\deg(v)}{2} - 4 \right) + \text{ch}'(F') = |E| - 4n + \text{ch}'(F').$$

Therefore, $|E| = 4n - 8 - \text{ch}'(F')$. Observe that the charge of every face of size at least four is non-negative and there are no faces of size smaller than three, since $n \geq 3$. Denote by f_{out} the outer face of G'. Since no crossing-free edge can be added to D, it follows that the boundary of f_{out} is a convex polygon whose vertices are the vertices of the convex hull of D. We now consider the two cases in the statement of the theorem.

(1) Suppose that $k = 1$. Then a face f of size three must be incident to at least two original vertices and therefore $\text{ch}'(f) \geq 0$. Moreover, $\text{ch}'(f_{\text{out}}) \geq |f_{\text{out}}| - 4 + |f_{\text{out}}|/2 \geq 0.5$. Thus, $\text{ch}'(F') \geq 0.5$ and $|E| \leq 4n - 8.5$. Since $|E|$ is an integer, the first part of the theorem follows.

(2) Suppose that D is a RAC drawing. Then a face of size three must be incident to at least two original vertices. Therefore, after the discharging phase every face has a non-negative charge, and hence $\text{ch}'(F') \geq 0$. To complete the proof it is enough to show that $\text{ch}'(F') > k$.

Call a face $f \neq f_{\text{out}}$ a *fence* face if it shares an edge (of G') with f_{out}. Perform a second discharging step in which every fence face with a positive charge sends 0.5 units of charge to f_{out}. Denote by $\text{ch}''(\cdot)$ the new charge function, and let $\text{ch}''(F') = \sum_{f \in F'} \text{ch}''(f)$. Clearly, $\text{ch}'(F') = \text{ch}''(F')$ and $\text{ch}''(f) \geq 0$ for every $f \in F'$. We first show that $\text{ch}''(f_{\text{out}}) \geq 1.5$. This clearly holds if $|f_{\text{out}}| \geq 4$, since then $\text{ch}'(f_{\text{out}}) \geq 2$. If $|f_{\text{out}}| = 3$ then $\text{ch}'(f_{\text{out}}) = 0.5$ and we will use the following simple fact.

Proposition 3.1. *Let $\triangle XYZ$ be a triangle and let Q and P be two distinct points inside it. If the interiors of the triangles $\triangle XQY$ and $\triangle XPZ$ are disjoint, then it is impossible that both angles $\angle XQY$ and $\angle XPZ$ are right angles.*

Proof. Suppose that $\angle XQY = \angle XPZ = \pi/2$. Then $\angle XYQ + \angle YXQ + \angle XZP + \angle ZXP = \pi$. However, since $\triangle XQY$ and $\triangle XPZ$ are disjoint it follows that $\angle YXZ > \angle YXQ + \angle ZXP$ and we get that the sum of angles in $\triangle XYZ$ is greater than π. $\qquad\square$

Suppose that $|f_{\text{out}}| = 3$. Since no crossing-free edges can be added to D there must be three distinct fence faces. By Proposition 3.1 at most one of them is a triangle with exactly two original vertices (and zero charge after the first discharging step). Therefore, at least two fence faces contribute 0.5 units of charge each to f_{out} and $\text{ch}''(f_{\text{out}}) \geq 1.5$.

So far we have shown that $\text{ch}''(F') \geq 1.5$, therefore $|E| \leq 4n - 10$, since $|E|$ is an integer. If $k = 1$, then we are done. Otherwise, consider an edge $e \in E$ that is crossed $k \geq 2$ times, and suppose without loss of generality that e is horizontal. Let $p_1, \ldots, p_k$ be the crossing points on e, ordered from left to right. Set $e_0 = (p_1, p_2)$ and let f_0 be the face that lies above

e_0. Observe that $|f_0| \geq 4$ and that $\mathrm{ch}'(f_0) = 0$ if and only if all the vertices of f_0 are crossing points in D.

For $i > 0$ define $e_i = e_{i-1}$ and $f_i = f_{i-1}$ if $\mathrm{ch}''(f_{i-1}) > 0$. Otherwise, f_{i-1} is a rectangle whose vertices are crossing points in D. In this case we define e_i to be the opposite edge to e_{i-1} in f_{i-1}, and f_i to be the face that lies above e_i. Since the graph is finite for some $j \geq 0$ we have $f_j = f_{j+1}$ (put differently, f_j is the first face with a non-negative charge that we encounter when climbing the ladder whose first step is e_0). We collect 0.5 units of charge from f_j, and in a similar way obtain 0.5 units of charge from a face that lies below e_0. Similarly, we collect one unit of charge for each of the edges $(p_2, p_3), \ldots, (p_{k-1}, p_k)$. An easy case analysis shows that the charge of every face remains non-negative. Therefore, $\mathrm{ch}''(F') \geq 1.5 + k - 1 > k$, and the second part of the theorem follows. $\qquad \square$

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