EDGE-ISOPERIMETRIC PROBLEM FOR CAYLEY GRAPHS AND GENERALIZED TAKAGI FUNCTIONS

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Abstract. Let G be a finite abelian group of exponent $m \geq 2$. For subsets $A, S \subseteq G$, denote by $\partial_S(A)$ the number of edges from A to its complement $G \setminus A$ in the directed Cayley graph, induced by S on G. We show that if S generates G, and A is non-empty, then

$$\partial_S(A) \geq \tfrac{e}{m}\, |A| \ln \tfrac{|G|}{|A|}\,.$$

Here the coefficient $e = 2.718\dots$ is best possible and cannot be replaced with a number larger than e.

For homocyclic groups G of exponent m, we find an explicit closed-form expression for $\partial_S(A)$ in the case where S is the "standard" generating subset of G, and A is an initial segment of G with respect to the lexicographic order induced by S. Namely, we show that in this situation

$$\partial_S(A) = |G|\, \omega_m(|A|/|G|),$$

where ω_2 is the Takagi function, and ω_m for $m \geq 3$ is an appropriate generalization thereof. This particular case is of special interest, since for $m \in \{2,3,4\}$ it is known to yield the smallest possible value of $\partial_S(A)$, over all sets $A \subseteq G$ of given size. We give this classical result a new proof, somewhat different from the standard one.

We also give a new, short proof of the Boros–Páles inequality

$$\omega_2\left(\tfrac{x+y}{2}\right) \leq \tfrac{\omega_2(x)+\omega_2(y)}{2} + \tfrac{1}{2}\,|y-x|,$$

establish an extremal characterization of the Takagi function as the (pointwise) maximal function, satisfying this inequality and the boundary condition $\max\{\omega_2(0), \omega_2(1)\} \leq 0$, and obtain similar results for the 3-adic analogue ω_3 of the Takagi function.

1. Introduction: summary of results and background

The three tightly related objects of study in this paper are the edge-isoperimetric problem on Cayley graphs, a sequence of Takagi-style functions, and classes of functions satisfying a certain kind of convexity condition.

The edge-isoperimetric problem for a graph Γ on the vertex set V is to find, for every non-negative integer $n \leq |V|$, the smallest possible number of edges between

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an n-element set of vertices and its complement in V. This classical problem has received much attention in the literature; for the history, results, variations, and related problems, the reader can refer to the survey of Bezrukov [B96] or the monograph of Harper [H04].

In the present paper we are concerned with, arguably, the most studied case where Γ is a Cayley graph. We use the following notation. Given two subsets $S, A \subseteq G$ of a finite abelian group G, by $\Gamma_S(G)$ we denote the (directed) Cayley graph, induced by S on G, and we write $\partial_S(A)$ for the number of edges in $\Gamma_S(G)$ from an element of A to an element in its complement $G \setminus A$; that is,

$$\partial_S(A) := |\{(a, s) \in A \times S \colon a + s \notin A\}|.$$

It is easily seen that if S is symmetric (meaning that $S = -S$, where $-S := \{-s \colon s \in S\}$), then $\partial_S(A)$ can be equivalently defined as the number of edges of the corresponding *undirected* Cayley graph, with one of the incident vertices in A and another one in $G \setminus A$. As a less trivial fact, we have

$$\partial_{-S}(A) = \partial_S(G \setminus A) = \partial_S(A);$$

consequently, if S is antisymmetric (that is, $S \cap (-S) = \varnothing$), then $\partial_S(A)$ is half the number of edges, joining a vertex from A with a vertex from $G \setminus A$, in the undirected Cayley graph, induced on G by the set $S \cup (-S)$. We omit detailed explanations since none of these observations are used below.

Up until now, most of the research we are aware of has focused on particular families of Cayley graphs.[1] In contrast, our first principal result addresses the general situation.

Recall that the exponent of an abelian group is the maximum of the orders of its elements.

Theorem 1.1. *Let $m \geq 2$ be an integer, and suppose that G is a finite abelian group of exponent m. Then for any non-empty subset $A \subseteq G$ and any generating subset $S \subseteq G$ we have*

$$\partial_S(A) \geq \frac{e}{m}\, |A| \ln \frac{|G|}{|A|}$$

(where $e = 2.718...$ is Euler's number).

The estimate of Theorem 1.1 is sharp in the sense that the coefficient e cannot be replaced with a larger number.

[1]Classical results on the isoperimetric constant of a graph, presented, for instance, in the survey paper [HLW06], provide a noticeable exception. However, they apply to undirected graphs only and, specified to Cayley graphs, yield weaker bounds than Theorems 1.1 and 1.6 below.

Example 1.2. For integer $r \geq 1$ and $m \geq 2$, let G be the homocyclic group of exponent m and rank r (which is the direct product of r cyclic groups of order m). Fix arbitrarily a generating subset $S = \{s_1, \dots, s_r\} \subseteq G$ and integer $k \in [1, r]$ and $t \in [1, m-1]$, and consider the set

$$A := \{x_1 s_1 + \dots + x_r s_r \colon 0 \leq x_1, \dots, x_k \leq t-1,\ 0 \leq x_{k+1}, \dots, x_r \leq m-1\}.$$

Write $\alpha := t/m$. Then $|A| = t^k m^{r-k}$, $\ln(|G|/|A|) = k \ln(1/\alpha)$, and

$$\partial_S(A) = k t^{k-1} m^{r-k} = \frac{c(\alpha)}{m}\, |A| \ln \frac{|G|}{|A|},$$

where $c(\alpha) = \frac{1/\alpha}{\ln(1/\alpha)}$ can be made arbitrarily close to e by choosing t and m appropriately.

The proofs of Theorem 1.1 and most of the other results, presented in the introduction, are postponed to subsequent sections.

Below we use the standard notation C_m^r for the homocyclic group of exponent m and rank r. In the case where $m \in \{2, 3, 4\}$, and $S \subseteq C_m^r$ is a generating set of size $|S| = r$, the minimum of $\partial_S(A)$ over all sets A of prescribed size is known to be realized when A is the set of the lexicographically smallest group elements; this basic fact due to Harper [H64] (the case $m \in \{2, 4\}$) and Lindsey [Li64] (the case $m = 3$) follows also from our present results, as explained below. To put things formally, for a finite, totally ordered set T and a non-negative integer $n \leq |T|$, denote by $\mathcal{I}_n(T)$ the length-n initial segment of T; that is, the set of the n smallest elements of T. Consider the group C_m^r along with a fixed generating subset $S \subseteq C_m^r$ of size $|S| = r$. We assume that S is totally ordered, inducing a lexicographic order on C_m^r; thus, $\mathcal{I}_n(C_m^r)$ is the set of the n lexicographically smallest elements of C_m^r. As we have just mentioned, if $m \in \{2, 3, 4\}$, then

$$\min\{\partial_S(A) \colon A \subseteq C_m^r,\ |A| = n\} = \partial_S(\mathcal{I}_n(C_m^r)), \quad 0 \leq n \leq m^r. \tag{1.1}$$

Surprisingly, to our knowledge, no explicit closed-form expression for the quantity $\partial_S(\mathcal{I}_n(C_m^r))$ has ever been obtained (although in the case $m = 2$, an asymptotic formula has been established in [G00]; see the remark following Theorem 1.3). We give such an expression in terms of the Takagi function for $m = 2$, and an appropriate m-adic version thereof for $m \geq 3$.

For real x, let $\|x\|$ denote the distance from x to the nearest integer. The Takagi function, first introduced by Teiji Takagi in 1903 as an example of an everywhere continuous but nowhere differentiable function, is defined by

$$\omega(x) := \sum_{k=0}^{\infty} 2^{-k} \|2^k x\|.$$

Numerous remarkable properties of this function, applications, and relations in various fields of mathematics can be found in the recent survey papers by Allaart and Kawamura [AK11] and Lagarias [La12]. For the generalization we need, for real x and α let $\|x\|_\alpha := \min\{\|x\|, \alpha\}$ (the distance from x to the nearest integer, truncated at α), and set

$$\omega_m(x) := \sum_{k=0}^{\infty} m^{-k} \|m^k x\|_{1/m}, \quad m \ge 2. \tag{1.2}$$

Thus, ω_2 is just the regular Takagi function. Moreover, as Pieter Allaart has kindly brought to our attention, ω_3 coincides with the function $-h$ (for $q = 3$) from [D75].

Since the series in (1.2) is uniformly convergent, the functions ω_m are well-defined and continuous on the whole real line. Furthermore, they are even functions, periodic with period 1, vanishing at integers, strictly positive for non-integer values of the argument, and satisfying

$$\max \omega_m \le \sum_{k=0}^{\infty} m^{-(k+1)} = \frac{1}{m-1}. \tag{1.3}$$

The reader is invited to compare our second major result against (1.1).

Theorem 1.3. *For integer $r \ge 1$ and $m \ge 2$, let S be an r-element generating subset of the homocyclic group C_m^r. Suppose that an ordering of S is fixed, inducing a lexicographic ordering of C_m^r. Then for any non-negative integer $n \le m^r$, the set $\mathcal{I}_n(C_m^r)$ of the n lexicographically smallest elements of C_m^r satisfies*

$$\partial_S(\mathcal{I}_n(C_m^r)) = m^r \omega_m(n/m^r).$$

Notice that for $m \in \{2, 3, 4\}$, Theorem 1.3 together with (1.1) and continuity of ω_m readily shows that for any fixed $x \in (0, 1)$, if $n_r = (1 + o(1))m^r x$ as $r \to \infty$, then

$$\min\{\partial_S(A) \colon A \subseteq C_m^r,\ |A| = n_r\} = (1 + o(1))\, m^r \omega_m(x).$$

The particular case $m = 2$ and $n_r = \lfloor 2^r x \rfloor$ is the main result of [G00].

In the appendix we establish some estimates for the growth rate of the functions ω_m: specifically, we show that

$$x \log_2(1/x) \le \omega_2(x) \le x \log_2(4/3x), \tag{1.4}$$

$$x \log_3(1/x) \le \omega_3(x) \le x \log_3(3/2x), \tag{1.5}$$

$$x \log_4(1/x) \le \omega_4(x) \le x \log_4(4/3x), \tag{1.6}$$

and for $m \ge 5$,

$$x \log_m(e/mx) \le \omega_m(x) \le x \log_m(3/2x) \tag{1.7}$$

for any $x \in (0, 1]$.

Estimates (1.4)–(1.6) are sharp: the lower bound in (1.4) and (1.6) is attained for $x = 2^{-k}$ and the upper bound for $x = 2^{1-k}/3$, the lower bound in (1.5) is attained for $x = 3^{-1-k}$ and the upper bound for $x = 3^{-k}/2$, for any integer $k \ge 0$. In contrast, estimate (1.7) is not sharp; it is provided, essentially, as a "proof of concept" and can easily be improved. However, as $x \to 0$, the lower and upper bounds in (1.7) coincide up to lower-order terms, and it may well be impossible to obtain both sharp and explicit bounds of this sort for $m \ge 5$.

We remark that the lower bound in (1.4) is not original; see, for instance, [Kr07, Lemma 3.1].

The graphs of the functions ω_2, ω_3, and ω_5, along with the functions representing the corresponding lower and upper bounds, are shown in Figure 1.

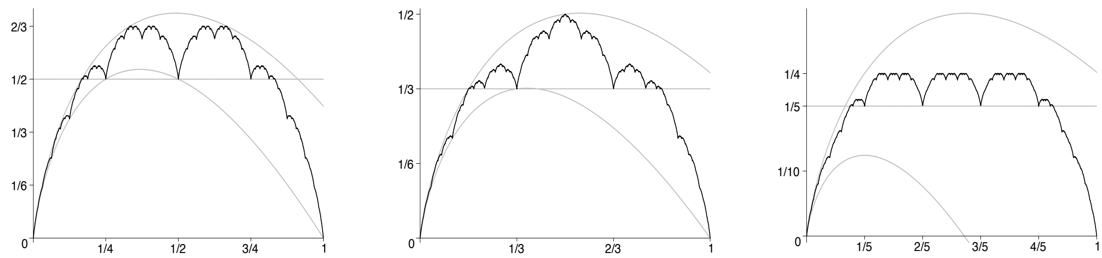

FIGURE 1. The graphs of the functions ω_2 (Takagi function), ω_3, and ω_5.

The reason for omitting the graph of ω_4 is that this function turns out to be identical, up to a constant factor, to the Takagi function; namely, we have

$$\omega_4 = \frac{1}{2}\,\omega_2. \tag{1.8}$$

To prove this somewhat surprising relation, it suffices to show that for any real x and integer $k \ge 0$ we have

$$2^{-2k}\|2^{2k}x\| + 2^{-2k-1}\|2^{2k+1}x\| = 2 \cdot 4^{-k}\|4^k x\|_{1/4}.$$

Indeed, letting $z := 2^{2k}x$ and multiplying by 2^{2k+1}, we can rewrite this equality as

$$2\|z\| + \|2z\| = 4\|z\|_{1/4},$$

and this is readily verified by restricting z to the range $0 \le z \le 1/2$ and considering separately the cases $z \le 1/4$ and $z \ge 1/4$.

It was conjectured by Páles [P04] and proved by Boros [B08] that the Takagi function satisfies

$$\omega_2\left(\frac{x_1 + x_2}{2}\right) \le \frac{\omega_2(x_1) + \omega_2(x_2)}{2} + \frac{1}{2}\,(x_2 - x_1) \tag{1.9}$$

for any real $x_1 \le x_2$. Combining this inequality with a result of Házy and Páles [HP05, Theorem 4], one immediately derives the following stronger version:

$$\omega_2(\lambda x_1 + (1-\lambda)x_2) \le \lambda\omega_2(x_1) + (1-\lambda)\omega_2(x_2) + \omega_2(\lambda)|x_2 - x_1| \tag{1.10}$$

for any real x_1, x_2, and $\lambda \in [0,1]$. We give short proofs to (1.9) and (1.10), which seem to be genuinely different from the original proofs, and establish the 3-adic analogues, as follows.

Theorem 1.4. *We have*

$$\omega_3\left(\frac{x_1+x_2+x_3}{3}\right) \le \frac{\omega_3(x_1)+\omega_3(x_2)+\omega_3(x_3)}{3} + \frac{1}{3}(x_3 - x_1)$$

for any real $x_1 \le x_2 \le x_3$.

Theorem 1.5. *We have*

$$\omega_3(\lambda x_1 + (1-\lambda)x_2) \le \lambda\omega_3(x_1) + (1-\lambda)\omega_3(x_2) + \omega_3(\lambda)\,|x_2 - x_1|$$

for any real x_1, x_2, *and* $\lambda \in [0,1]$.

The importance of Boros–Páles inequality (1.9) and Theorem 1.4 for our present purposes, and the way they are applied in this paper, will be explained shortly. Inequality (1.10) and Theorem 1.5 are derived as particular cases of a more general result, presented below (Theorem 1.12).

Back to Theorem 1.1, we actually prove a more versatile and precise result, with the improvement being particularly significant for small values of m. To state it we bring into consideration the classes of functions, defined as follows. For integer $m \ge 2$, let $\mathcal{F}_m$ consist of all real-valued functions f, defined on the interval $[0,1]$, satisfying the boundary condition

$$\max\{f(0), f(1)\} \le 0, \tag{1.11}$$

and such that for any $x_1, \dots, x_m \in [0,1]$ with $\min_i x_i = x_1$ and $\max_i x_i = x_m$, we have

$$f\left(\frac{x_1+\dots+x_m}{m}\right) \le \frac{f(x_1)+\dots+f(x_m)}{m} + (x_m - x_1). \tag{1.12}$$

Condition (1.12) can be understood as a "relaxed convexity" and in fact, any convex function satisfying the boundary condition (1.11) is contained in every class $\mathcal{F}_m$.

We notice that if $l, m \ge 2$ are integers with $l \mid m$, then $\mathcal{F}_m \subseteq \mathcal{F}_l$, for, given a function $f \in \mathcal{F}_m$ and a system of l numbers in $[0,1]$, we can "blow up" this system to get a system of m numbers (where every original number is repeated m/l times) and then apply (1.12) to this new system to obtain the analogue of (1.12) for the original l numbers. For $l = 2$ and $m = 4$ the inverse inclusion holds, too, so that we have

$\mathcal{F}_4 = \mathcal{F}_2$; to prove this, fix $f \in \mathcal{F}_2$ and $x_1, x_2, x_3, x_4 \in [0,1]$ with $x_1 \le x_2 \le x_3 \le x_4$, and observe that then

$$\begin{aligned} f\left(\frac{x_1+x_2+x_3+x_4}{4}\right) &\le \frac{1}{2}\left(f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_3+x_4}{2}\right)\right) \\ &\qquad\qquad\qquad + \frac{x_3+x_4}{2} - \frac{x_1+x_2}{2} \\ &\le \frac{1}{4}\big(f(x_1)+f(x_2)+f(x_3)+f(x_4)\big) + (x_4 - x_1), \end{aligned}$$

whence $f \in \mathcal{F}_4$.

It is not difficult to see, however, that, say, the class $\mathcal{F}_6$ is distinct from each of the classes $\mathcal{F}_2$ and $\mathcal{F}_3$, and the class $\mathcal{F}_8$ is distinct from the class $\mathcal{F}_2$. Indeed, a straightforward numerical verification confirms that the functions $F_2 \in \mathcal{F}_2$ and $F_3 \in \mathcal{F}_3$, introduced below in this section, satisfy $F_2 \notin \mathcal{F}_6$, $F_3 \notin \mathcal{F}_6$, and $F_2 \notin \mathcal{F}_8$.

We notice that Boros–Páles inequality (1.9) can be interpreted as $2\omega_2 \in \mathcal{F}_2$, and Theorem 1.4 gives $3\omega_3 \in \mathcal{F}_3$. (In fact, it is the *restrictions* of the functions $2\omega_2$ and $3\omega_3$ onto the interval $[0,1]$ that belong to the classes $\mathcal{F}_2$ and $\mathcal{F}_3$, respectively. However, this little abuse of notation does not lead to any confusion.)

Theorem 1.6. *Let $m \ge 2$ be an integer. Suppose that $f \in \mathcal{F}_m$, and G is a finite abelian group, the exponent of which divides m. Then for any subset $A \subseteq G$ and any generating subset $S \subseteq G$ we have*

$$\partial_S(A) \ge \frac{1}{m}\,|G|\,f(|A|/|G|).$$

Theorem 1.1 is an immediate consequence of Theorem 1.6 and the following proposition.

Proposition 1.7. *Let $f(x) = ex\ln(1/x)$ for $x \in (0,1]$, and $f(0) = 0$. Then $f \in \mathcal{F}_m$ for any integer $m \ge 2$.*

To use Theorem 1.6 more efficiently, we must choose the function f in an optimal way for every particular value of m. Our next result shows that for each $m \ge 2$, there is a "universal" choice which does not depend on the density $|A|/|G|$.

Theorem 1.8. *For any $m \ge 2$ there is a (unique) function $F_m \in \mathcal{F}_m$ such that for any other function $f \in \mathcal{F}_m$ and any $x \in [0,1]$ we have $F_m(x) \ge f(x)$. The function F_m is continuous on $[0,1]$, strictly positive on $(0,1)$, and satisfies $F_m(0) = F_m(1) = 0$ and $F_m(x) = F_m(1-x)$ for $x \in [0,1]$.*

From now on we adopt F_m as a standard notation for the functions of Theorem 1.8.

We were able to find the functions F_m explicitly for $m \in \{2,3,4\}$ and estimate them for $m \ge 5$. Determining F_m for $m \ge 5$ seems to be a non-trivial and challenging problem; we have done some work toward the case $m = 5$, and the results may appear elsewhere.

Theorem 1.9. *For any $m \geq 2$ we have $F_m \leq m\omega_m$, with equality if and only if $m \in \{2,3,4\}$.*

The case $m \in \{2,3,4\}$ of Theorem 1.9 will be derived from (1.9), Theorem 1.4, and (1.8).

As remarked above, (1.1) follows from the results of the present paper; indeed, the reader can now see that it is an immediate corollary of Theorems 1.3, 1.6, and 1.9.

Combining Theorems 1.6 and 1.9 and estimates (1.4)–(1.6) we obtain the following result.

Corollary 1.10. *If G is a finite abelian group of exponent $m \in \{2,3,4\}$, then for any non-empty subset $A \subseteq G$ and generating subset $S \subseteq G$ we have*

$$\partial_S(A) \geq |G|\,\omega_m(|A|/|G|) \geq |A| \log_m \frac{|G|}{|A|}.$$

We remark that in the case $m = 2$, the resulting estimate $\partial_S(A) \geq |A| \log_2(|G|/|A|)$ is well-known, the first appearance in the literature we are aware of being [CFGS88, Lemma 4.1].

Theorem 1.9 can be considerably improved for large values of m.

Proposition 1.11. *For any integer $m \geq 2$ and real $x \in (0,1]$ we have*

$$F_m(x) \leq \frac{m}{m-1}\, ex \ln(e/x).$$

For the lower bound, we notice that Proposition 1.7 yields

$$F_m(x) \geq ex \ln(1/x)$$

for each $m \geq 2$ and $x \in (0,1]$.

We deduce Proposition 1.11 from the following result, which, in view of Theorem 1.9, also implies (1.10) and Theorem 1.5, and indeed provides a common generalization to both of them.

Theorem 1.12. *Let $m \geq 2$ be an integer. Then for any function $f \in \mathcal{F}_m$ and any $\lambda, x, y \in [0,1]$ with $x \leq y$ we have*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) + (y-x) F_m(\lambda). \tag{1.13}$$

Moreover, if f is a function, defined on the whole real line and satisfying (1.12) *for any real $x_1, \dots, x_m$ with $\min_i x_i = x_1$ and $\max_i x_i = x_m$, then* (1.13) *holds for any real $x \leq y$ and $\lambda \in [0,1]$.*

The rest of the paper, devoted to the proof of the results discussed above, is partitioned into three sections and an appendix. In Section 2 we study the generalized Takagi functions, proving Boros–Páles inequality (1.9) and its 3-adic analog, Theorem 1.4, and establishing an important lemma used in both proofs and also in the proofs of Theorems 1.3 and 1.9. Section 3 deals with the isoperimetric problem: we prove here Theorems 1.3 and 1.6. In Section 4 we investigate the classes $\mathcal{F}_m$ and the functions F_m, and prove Propositions 1.7 and 1.11 and Theorems 1.8, 1.9, and 1.12. As remarked above, Theorem 1.1 is an immediate consequence of Theorem 1.6 and Proposition 1.7, while (1.10) and Theorem 1.5 (in view of Theorem 1.9) are particular cases of Theorem 1.12; hence no additional proofs are needed. In the appendix we prove estimates (1.4)–(1.7).

2. The generalized Takagi functions: proofs of the Boros–Páles inequality and Theorem 1.4

The following lemma, used in the proofs of the Boros–Páles inequality and Theorems 1.3, 1.4, and 1.9, is known in the case $m=2$; see [HY83] or [AK11, Theorem 5.1].

Lemma 2.1. *Let $m \ge 2$ be an integer. Then for any integer $r \ge 1$ and n, the latter of which is not divisible by m, writing $n = tm + \rho$ with integer t and $\rho \in [1, m-1]$, we have*

$$\omega_m\left(\frac{n}{m^r}\right) = \left(1 - \frac{\rho}{m}\right)\omega_m\left(\frac{t}{m^{r-1}}\right) + \frac{\rho}{m}\,\omega_m\left(\frac{t+1}{m^{r-1}}\right) + \frac{1}{m^r}.$$

Proof. We want to prove that

$$\sum_{k=0}^{\infty} m^{-k}\left(\|m^{k-r}n\|_{1/m} - \left(1-\frac{\rho}{m}\right)\|m^{k+1-r}t\|_{1/m} - \frac{\rho}{m}\,\|m^{k+1-r}(t+1)\|_{1/m}\right) = \frac{1}{m^r}.$$

We notice that all the summands in the left-hand side, corresponding to $k \ge r$, vanish, while the summand, corresponding to $k = r-1$, contributes $m^{-(r-1)}(1/m) = m^{-r}$ to the sum. Consequently, to complete the proof it suffices to show that

$$m\|m^{k-r}n\|_{1/m} = (m-\rho)\|m^{k+1-r}t\|_{1/m} + \rho\,\|m^{k+1-r}(t+1)\|_{1/m}, \quad k \in [0, r-2].$$

To this end we prove that the interval $(m^{k+1-r}t, m^{k+1-r}(t+1))$ (of which $m^{k-r}n$ is an internal point) does not contain any number of the form $N + \varepsilon/m$ with integer N and $\varepsilon \in \{0, \pm 1\}$, and therefore $\|x\|_{1/m}$ is a linear function of x on this interval. Indeed, if we had

$$m^{k+1-r}t < N + \varepsilon/m < m^{k+1-r}(t+1),$$

then, multiplying by m^{r-k-1}, we would get $t < Nm^{r-k-1} + \varepsilon m^{r-k-2} < t+1$, which cannot hold since the midterm is an integer. □

For the rest of this section, for integer n and $m \geq 2$ we let

$$\delta_m(n) := \begin{cases} 0 & \text{if } m \text{ divides } n, \\ 1 & \text{if } m \text{ does not divide } n. \end{cases}$$

We record the following immediate corollary of Lemma 2.1.

Corollary 2.2. *For any integer $r \geq 1$ and n, we have*

$$\omega_2\left(\frac{n}{2^r}\right) = \frac{1}{2}\,\omega_2\left(\frac{\lfloor n/2 \rfloor}{2^{r-1}}\right) + \frac{1}{2}\,\omega_2\left(\frac{\lceil n/2 \rceil}{2^{r-1}}\right) + \frac{\delta_2(n)}{2^r},$$

where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor and the ceiling functions, respectively.

Proof of Boros–Páles inequality (1.9). Since ω_2 is a continuous function, it suffices to show that

$$\omega_2\left(\frac{x+y}{2^r}\right) \leq \frac{1}{2}\,\omega_2\left(\frac{x}{2^{r-1}}\right) + \frac{1}{2}\,\omega_2\left(\frac{y}{2^{r-1}}\right) + \frac{1}{2^r}\,|y-x|,$$

for any integer x, y, and $r \geq 1$. We use induction on r. The case $r = 1$ is immediate since $\omega_2(x) = \omega_2(y) = 0$ and $\omega_2((x+y)/2)$ is equal to 0 or 1 depending on whether x and y are of the same or of distinct parity. Thus, we assume that $r \geq 2$. Moreover, we assume that x is odd; clearly, this does not restrict the generality.

Applying Corollary 2.2 with $n = x + y$ and using the induction hypothesis, we get

$$\begin{aligned}
\omega_2\left(\frac{x+y}{2^r}\right) &= \frac{1}{2}\,\omega_2\left(\frac{\lfloor (x+y)/2 \rfloor}{2^{r-1}}\right) + \frac{1}{2}\,\omega_2\left(\frac{\lceil (x+y)/2 \rceil}{2^{r-1}}\right) + \frac{1}{2^r}\,\delta_2(x+y) \\
&= \frac{1}{2}\,\omega_2\left(\frac{(x-1)/2 + \lfloor (y+1)/2 \rfloor}{2^{r-1}}\right) + \frac{1}{2}\,\omega_2\left(\frac{(x+1)/2 + \lceil (y-1)/2 \rceil}{2^{r-1}}\right) \\
&\qquad\qquad + \frac{1}{2^r}\,\delta_2(x+y) \\
&\leq \frac{1}{4}\left(\omega_2\left(\frac{x-1}{2^{r-1}}\right) + \omega_2\left(\frac{x+1}{2^{r-1}}\right)\right) \\
&\quad + \frac{1}{4}\left(\omega_2\left(\frac{2\lfloor (y+1)/2 \rfloor}{2^{r-1}}\right) + \omega_2\left(\frac{2\lceil (y-1)/2 \rceil}{2^{r-1}}\right)\right) \\
&\quad + \frac{1}{2^r}\left|\left\lfloor\frac{y+1}{2}\right\rfloor - \frac{x-1}{2}\right| + \frac{1}{2^r}\left|\left\lceil\frac{y-1}{2}\right\rceil - \frac{x+1}{2}\right| + \frac{1}{2^r}\,\delta_2(x+y).
\end{aligned}$$

We now notice that, by Corollary 2.2,

$$\frac{1}{2}\left(\omega_2\left(\frac{x-1}{2^{r-1}}\right) + \omega_2\left(\frac{x+1}{2^{r-1}}\right)\right) = \omega_2\left(\frac{x}{2^{r-1}}\right) - \frac{1}{2^{r-1}}$$

and

$$\frac{1}{2}\left(\omega_2\left(\frac{2\lfloor (y+1)/2 \rfloor}{2^{r-1}}\right) + \omega_2\left(\frac{2\lceil (y-1)/2 \rceil}{2^{r-1}}\right)\right) = \omega_2\left(\frac{y}{2^{r-1}}\right) - \frac{\delta_2(y)}{2^{r-1}},$$

as it follows easily by considering separately the cases of even and odd y. Consequently,

$$\omega_2\left(\frac{x+y}{2^r}\right) \le \frac{1}{2}\,\omega_2\left(\frac{x}{2^{r-1}}\right) + \frac{1}{2}\,\omega_2\left(\frac{y}{2^{r-1}}\right) + \frac{1}{2^r}\left(\delta_2(x+y) - \delta_2(y) - 1\right)$$
$$+ \frac{1}{2^r}\left|\left|\left\lfloor\frac{y-x+2}{2}\right\rfloor\right|\right| + \frac{1}{2^r}\left|\left|\left\lceil\frac{y-x-2}{2}\right\rceil\right|\right|.$$

To complete the proof we observe that $\delta_2(x+y) - \delta_2(y) - 1 \le 0$ and that if $x \ne y$ (in which case the assertion is trivial), then $\lfloor (y-x+2)/2 \rfloor$ and $\lceil (y-x-2)/2 \rceil$ are of the same sign, whence

$$\begin{aligned} |\,\lfloor (y-x+2)/2 \rfloor\,| + |\,\lceil (y-x-2)/2 \rceil\,| &= |\,\lfloor (y-x+2)/2 \rfloor + \lceil (y-x-2)/2 \rceil\,| \\ &= |\,\lfloor (y-x)/2 \rfloor + \lceil (y-x)/2 \rceil\,| \\ &= |y-x|. \end{aligned}$$

□

To prove Theorem 1.4 we need yet another corollary of Lemma 2.1.

Corollary 2.3. *Let $r \ge 1$ and n be integers. If $\xi_n \in \{-1,0,1\}$ and $\zeta_n \in \{-2,0,2\}$ are defined by $n \equiv \xi_n \equiv \zeta_n \pmod 3$, then*

$$\omega_3\left(\frac{n}{3^r}\right) = \frac{2}{3}\,\omega_3\left(\frac{n-\xi_n}{3^r}\right) + \frac{1}{3}\,\omega_3\left(\frac{n-\zeta_n}{3^r}\right) + \frac{\delta_3(n)}{3^r}.$$

Observe that, with ξ_n and ζ_n defined as in Corollary 2.3, we have

$$2\xi_n + \zeta_n = 0. \tag{2.1}$$

Proof of Theorem 1.4. By continuity of ω_3, it suffices to show that

$$\omega_3\left(\frac{x+y+z}{3^r}\right) \le \frac{1}{3}\,\omega_3\left(\frac{x}{3^{r-1}}\right) + \frac{1}{3}\,\omega_3\left(\frac{y}{3^{r-1}}\right) + \frac{1}{3}\,\omega_3\left(\frac{z}{3^{r-1}}\right) + \frac{1}{3^r}\,(z-x)$$

for any integer $r \ge 1$ and $x \le y \le z$.

For integer $r \ge 0$ and n, let

$$T_r(n) := \sum_{k=1}^{r} 3^k\,\|3^{-k}n\|_{1/3}.$$

Thus, $T_0(n) = 0$, $T_1(n) = \delta_3(n)$, $T_r(-n) = T_r(n)$, and $T_r(3n) = 3T_{r-1}(n)$; these simple observations may be used below without special references. Furthermore,

$$3^r\omega_3\left(\frac{n}{3^r}\right) = \sum_{k=0}^{r-1} 3^{r-k}\|3^{k-r}n\|_{1/3} = T_r(n);$$

therefore, keeping the notation of Corollary 2.3, we can rewrite its conclusion as

$$T_r(n) = \frac{2}{3}\,T_r(n-\xi_n) + \frac{1}{3}\,T_r(n-\zeta_n) + \delta_3(n), \tag{2.2}$$

and the estimate we have to prove as

$$T_r(x+y+z) \le T_{r-1}(x) + T_{r-1}(y) + T_{r-1}(z) + (z-x). \tag{2.3}$$

To establish (2.3) we use induction on r. For $r=1$ the assertion is easy to verify in view of $T_0 = 0$ and $T_1(x+y+z) = \delta_3(x+y+z)$, and we assume that $r \ge 2$. We also assume that x is strictly smaller than z, for if $x=y=z$, then (2.3) is immediate from $T_r(3x) = 3T_{r-1}(x)$.

If $x, y,$ and z are all divisible by 3, then the assertion follows easily from the induction hypothesis. Otherwise, changing (simultaneously) the signs of $x, y,$ and z, if necessary, we can assume that one of the following holds:

(i) $x \equiv y \equiv z \equiv 1 \pmod 3$;
(ii) two of the numbers $x, y,$ and z are congruent to 1 modulo 3, and the third is divisible by 3;
(iii) the numbers $x, y,$ and z are pairwise incongruent modulo 3;
(iv) two of the numbers $x, y,$ and z are divisible by 3, and the third is congruent to 1 modulo 3;
(v) two of the numbers $x, y,$ and z are congruent to 1 modulo 3, and the third is congruent to 2 modulo 3.

We consider these five cases separately.

Case (i): $x \equiv y \equiv z \equiv 1 \pmod 3$. In this case, using the induction hypothesis we get

$$\begin{aligned} T_r(x+y+z) &= 3T_{r-1}\left(\frac{x-1}{3} + \frac{y-1}{3} + \frac{z+2}{3}\right) \\ &\le 3T_{r-2}\left(\frac{x-1}{3}\right) + 3T_{r-2}\left(\frac{y-1}{3}\right) + 3T_{r-2}\left(\frac{z+2}{3}\right) \\ &\qquad\qquad\qquad\qquad\qquad\qquad + (z-x+3) \\ &= T_{r-1}(x-1) + T_{r-1}(y-1) + T_{r-1}(z+2) + (z-x+3). \end{aligned} \tag{2.4}$$

Similarly,

$$T_r(x+y+z) \le T_{r-1}(x-1) + T_{r-1}(y+2) + T_{r-1}(z-1) + (z-x) \tag{2.5}$$

and

$$T_r(x+y+z) \le T_{r-1}(x+2) + T_{r-1}(y-1) + T_{r-1}(z-1) + (z-x-3), \tag{2.6}$$

except that we must add 3 to the right-hand side of (2.5) if $y=z$ and to the right-hand side of (2.6) if $x=y$. Averaging (2.4)–(2.6) and taking into account the observation just made and the fact that if $n \equiv 1 \pmod 3$, then

$$\frac{2}{3}T_{r-1}(n-1) + \frac{1}{3}T_{r-1}(n+2) = T_{r-1}(n) - 1$$

(as it follows from (2.2)), we get (2.3).

Case (ii): two of x, y, and z are congruent to 1 modulo 3, and the third is divisible by 3. Denote by w the element of the set $\{x, y, z\}$ which is divisible by 3, and let u be the smallest and v the largest of the two other elements. By (2.2), we have

$$\begin{aligned} T_r(x+y+z) &= \frac{2}{3}\,T_r\left(x+y+z+1\right) + \frac{1}{3}\,T_r\left(x+y+z-2\right) + 1 \\ &= 2T_{r-1}\left(\frac{u+v+w+1}{3}\right) + T_{r-1}\left(\frac{u+v+w-2}{3}\right) + 1. \end{aligned} \tag{2.7}$$

By the induction hypothesis,

$$\begin{aligned} T_{r-1}\left(\frac{u+v+w+1}{3}\right) &= T_{r-1}\left(\frac{u-1}{3} + \frac{v+2}{3} + \frac{w}{3}\right) \\ &\le T_{r-2}\left(\frac{u-1}{3}\right) + T_{r-2}\left(\frac{v+2}{3}\right) + T_{r-2}\left(\frac{w}{3}\right) \\ &\qquad + \frac{z-x+3}{3} \\ &= \frac{1}{3}\,T_{r-1}(u-1) + \frac{1}{3}\,T_{r-1}(v+2) + \frac{1}{3}\,T_{r-1}(w) \\ &\qquad + \frac{z-x+3}{3} \end{aligned} \tag{2.8}$$

and similarly,

$$\begin{aligned} T_{r-1}\left(\frac{u+v+w+1}{3}\right) &\le \frac{1}{3}\,T_{r-1}(u+2) + \frac{1}{3}\,T_{r-1}(v-1) + \frac{1}{3}\,T_{r-1}(w) \\ &\qquad + \frac{z-x+3}{3}. \end{aligned} \tag{2.9}$$

Also,

$$\begin{aligned} T_{r-1}\left(\frac{u+v+w-2}{3}\right) &= T_{r-1}\left(\frac{u-1}{3} + \frac{v-1}{3} + \frac{w}{3}\right) \\ &\le T_{r-2}\left(\frac{u-1}{3}\right) + T_{r-2}\left(\frac{v-1}{3}\right) + T_{r-2}\left(\frac{w}{3}\right) \\ &\qquad + \frac{z-x+1}{3} \\ &= \frac{1}{3}\,T_{r-1}(u-1) + \frac{1}{3}\,T_{r-1}(v-1) + \frac{1}{3}\,T_{r-1}(w) \\ &\qquad + \frac{z-x+1}{3}. \end{aligned} \tag{2.10}$$

In fact, we need a slight refinement of (2.8)–(2.10) which can be obtained by distinguishing the subcases where $w = x$ (meaning that it is the smallest of the numbers

x, y, z that is divisible by 3), $w = y$ (the middle one is divisible by 3), and $w = z$ (the largest one is divisible by 3). The reader will check easily that in the first case ($w = x$), both last summands in the right-hand sides of (2.8) and (2.9) can be replaced with $(z - x + 2)/3$, and the last summand in the right-hand side of (2.10) can be replaced with $(z - x - 1)/3$. Similarly, in the second case ($w = y$), we can replace the last summands in the right-hand sides of both (2.9) and (2.10) with $(z-x)/3$, and in the third case ($w = z$), both last summands in the right-hand sides of (2.8) and (2.9) can be replaced with $(z - x + 1)/3$. In any case, the sum of the three summands does not exceed $z - x + 1$. Taking this into account, adding up (2.8)–(2.10), and substituting the result into (2.7), we get

$$\begin{aligned} T_r(x+y+z) \le \left(\frac{2}{3}\, T_{r-1}(u-1) + \frac{1}{3}\, T_{r-1}(u+2) \right) & \\ + \left(\frac{2}{3}\, T_{r-1}(v-1) + \frac{1}{3}\, T_{r-1}(v+2) \right) & + T_{r-1}(w) + (z-x) + 2. \end{aligned}$$

The result now follows from (2.2).

Case (iii): x, y, and z are pairwise incongruent modulo 3. Using the induction hypothesis and the fact that $\xi_x + \xi_y + \xi_z = \zeta_x + \zeta_y + \zeta_z = 0$ we obtain in this case

$$\begin{aligned} T_r(x+y+z) &= 3T_{r-1}\left(\frac{x-\xi_x}{3} + \frac{y-\xi_y}{3} + \frac{z-\xi_z}{3} \right) \\ &\le 3T_{r-2}\left(\frac{x-\xi_x}{3} \right) + 3T_{r-2}\left(\frac{y-\xi_y}{3} \right) + 3T_{r-2}\left(\frac{z-\xi_z}{3} \right) \\ &\qquad\qquad + (z - x - \xi_z + \xi_x) \\ &= T_{r-1}(x-\xi_x) + T_{r-1}(y-\xi_y) + T_{r-1}(z-\xi_z) \\ &\qquad\qquad + (z - x - \xi_z + \xi_x). \end{aligned} \tag{2.11}$$

Similarly,

$$\begin{aligned} T_r(x+y+z) \le T_{r-1}(x-\zeta_x) + T_{r-1}(y-\zeta_y) + T_{r-1}(z-\zeta_z) & \\ + (z - x - \zeta_z + \zeta_x + 6), & \end{aligned} \tag{2.12}$$

for $\max\{x-\zeta_x, y-\zeta_y, z-\zeta_z\} \le z - \zeta_z + 3$ and $\min\{x-\zeta_x, y-\zeta_y, z-\zeta_z\} \ge x - \zeta_x - 3$. The assertion follows by averaging (2.11) and (2.12) with the weights $2/3$ and $1/3$, respectively, using (2.2), and noticing that

$$\begin{aligned} - \delta_3(x) - \delta_3(y) - \delta_3(z) + \frac{2}{3}\,(-\xi_z + \xi_x) + \frac{1}{3}\,(-\zeta_z + \zeta_x + 6) & \\ = \frac{1}{3}\,(\zeta_x + 2\xi_x) - \frac{1}{3}\,(\zeta_z + 2\xi_z) = 0. & \end{aligned}$$

Case (iv): two of x, y, and z are divisible by 3, and the third is congruent to 1 modulo 3. By (2.2), we have

$$T_r(x+y+z) = 2T_{r-1}\left(\frac{x+y+z-1}{3}\right) + T_{r-1}\left(\frac{x+y+z+2}{3}\right) + 1. \tag{2.13}$$

By the induction hypothesis,

$$\begin{aligned}
T_{r-1}\left(\frac{x+y+z-1}{3}\right) &= T_{r-1}\left(\frac{x-\xi_x}{3} + \frac{y-\xi_y}{3} + \frac{z-\xi_z}{3}\right) \\
&\le T_{r-2}\left(\frac{x-\xi_x}{3}\right) + T_{r-2}\left(\frac{y-\xi_y}{3}\right) + T_{r-2}\left(\frac{z-\xi_z}{3}\right) \\
&\qquad + \frac{z-x-\xi_z+\xi_x}{3} \\
&= \frac{1}{3}T_{r-1}(x-\xi_x) + \frac{1}{3}T_{r-1}(y-\xi_y) + \frac{1}{3}T_{r-1}(z-\xi_z) \\
&\qquad + \frac{z-x-\xi_z+\xi_x}{3}
\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}
T_{r-1}\left(\frac{x+y+z+2}{3}\right) &= T_{r-1}\left(\frac{x-\zeta_x}{3} + \frac{y-\zeta_y}{3} + \frac{z-\zeta_z}{3}\right) \\
&\le T_{r-2}\left(\frac{x-\zeta_x}{3}\right) + T_{r-2}\left(\frac{y-\zeta_y}{3}\right) + T_{r-2}\left(\frac{z-\zeta_z}{3}\right) \\
&\qquad + \frac{z-x-\zeta_z+\zeta_x}{3} \\
&= \frac{1}{3}T_{r-1}(x-\zeta_x) + \frac{1}{3}T_{r-1}(y-\zeta_y) + \frac{1}{3}T_{r-1}(z-\zeta_z) \\
&\qquad + \frac{z-x-\zeta_z+\zeta_x}{3}.
\end{aligned} \tag{2.15}$$

The result follows from (2.13)–(2.15), (2.2), and (2.1).

Case (v): two of x, y, and z are congruent to 1 modulo 3, and the third is congruent to 2 modulo 3. It is not difficult to verify that this case (2.13) and (2.14) remain valid, while (2.15) is to be replaced with

$$\begin{aligned}
T_{r-1}\left(\frac{x+y+z+2}{3}\right) \le \frac{1}{3}T_{r-1}(x-\zeta_x) + \frac{1}{3}T_{r-1}(y-\zeta_y) + \frac{1}{3}T_{r-1}(z-\zeta_z) \\
+ \frac{z-x-\zeta_z+\zeta_x+6}{3}.
\end{aligned}$$

The proof can now be completed as in Case (iv). □

3. The isoperimetric problem: proofs of Theorems 1.3 and 1.6

Proof of Theorem 1.3. We assume that m is fixed and use induction on r, for each r proving the equality

$$\partial_S(\mathcal{I}_n(C_m^r)) = m^r\omega_m(n/m^r)$$

for all $n \in [0, m^r]$. The case $r = 1$ is easy in view of $\omega_m(0) = \omega_m(1) = 0$ and since $\omega_m(n/m) = 1/m$ for $n = 1, \dots, m-1$; suppose, therefore, that $r \ge 2$.

Let s_0 be the smallest element of S. Denote by H the subgroup of C_m^r, generated by the set $S_0 := S \setminus \{s_0\}$, and for brevity, write $A := \mathcal{I}_n(C_m^r)$. For $i = 0, \dots, m-1$, let $A_i := A \cap (is_0 + H)$ and $n_i = |A_i|$. Notice that if $n = tm + \rho$ with integer $t \ge 0$ and $\rho \in [1, m]$, then

$$n_0 = \dots = n_{\rho-1} = t+1 \quad \text{and} \quad n_\rho = \dots = n_{m-1} = t. \tag{3.1}$$

We have

$$\partial_S(A) = \partial_{S_0}(A_0) + \dots + \partial_{S_0}(A_{m-1}) + (n_0 - n_{m-1}),$$

the first m summands counting those pairs (a, s) with $a \in A$ and $s \in S_0$ such that $a+s \notin A$, and the last summand counting pairs (a, s_0) with $a \in A$ such that $a+s_0 \notin A$. By the induction hypothesis, as applied to the subsets $A_i - is_0$ of the group H with the generating subset S_0, we then have

$$\partial_S(A) = m^{r-1}\omega_m\left(\frac{n_0}{m^{r-1}}\right) + \dots + m^{r-1}\omega_m\left(\frac{n_{m-1}}{m^{r-1}}\right) + (n_0 - n_{m-1}).$$

Now if m divides n, then $n_0 = \dots = n_{m-1} = n/m$ and the assertion follows immediately. If, on the other hand, m does not divide n, then in view of (3.1) and by Lemma 2.1, the right-hand side is equal to

$$m^r\left(\frac{\rho\,\omega_m((t+1)/m^{r-1}) + (m-\rho)\,\omega_m(t/m^{r-1})}{m} + \frac{1}{m^r}\right) = m^r\omega_m(n/m^r),$$

completing the proof. □

To prove Theorem 1.6 we need the following simple lemma.

Lemma 3.1. *For any integer $m \ge 2$ and real $x_1, \dots, x_m$, we have*

$$|x_2 - x_1| + |x_3 - x_2| + \dots + |x_m - x_{m-1}| + |x_1 - x_m| \ge 2\left(\max_i x_i - \min_i x_i\right).$$

Proof. Assume, without loss of generality, that x_1 is the smallest of the numbers $x_1, \dots, x_m$, and let $j \in [1, m]$ be chosen so that x_j is the largest of these numbers.

Then, by the triangle inequality,

$$\begin{aligned}|x_2-x_1|+|x_3-x_2|+\dots+|x_m-x_{m-1}|+|x_1-x_m|\\ =|x_2-x_1|+|x_3-x_2|+\dots+|x_j-x_{j-1}|\\ +|x_{j+1}-x_j|+\dots+|x_m-x_{m-1}|+|x_1-x_m|\\ \ge |x_j-x_1|+|x_1-x_j|=2(x_j-x_1).\end{aligned}$$

□

For further references, we record the following observation: if $m\ge 2$ and $f\in\mathcal{F}_m$, then, choosing in (1.12) some of the numbers x_i equal to 0, and the rest equal to 1, in view of the boundary condition (1.11) we get

$$f(n/m)\le 1;\ n=1,\dots,m-1. \tag{3.2}$$

Proof of Theorem 1.6. We fix m and use induction on $|G|$: assuming that the assertion is true for any abelian group, the order of which is smaller than $|G|$ (and the exponent of which divides m), we show that it is true for the group G.

Without loss of generality, we assume that S is a minimal (under inclusion) generating subset of G. Fix an element $s_0\in S$ and write $S_0:=S\setminus\{s_0\}$. If $S_0=\varnothing$, then G is cyclic of exponent $|G|$, whence $|G|$ divides m and therefore $f\in\mathcal{F}_{|G|}$; consequently, $f(|A|/|G|)\le 1$ by (3.2) and the assertion follows. Assuming now that $S_0\ne\varnothing$, let H be the subgroup of G, generated by S_0; thus, H is proper and non-trivial. Let $l:=[G:H]$. Observe that the quotient group G/H is cyclic, generated by s_0+H; hence the exponent of G/H is equal to its order l and therefore divides m. For $i=1,\dots,l$ set $x_i:=|A\cap(is_0+H)|/|H|$.

Fix $i\in[1,l]$. By the induction hypothesis (as applied to the subset $(A-is_0)\cap H$ of the group H with the generating subset S_0), the number of edges of $\Gamma_S(G)$ from an element of $(is_0+H)\cap A$ to an element of $(is_0+H)\setminus A$ is at least $\frac1m|H|f(x_i)$. Furthermore, the number of edges from $(is_0+H)\cap A$ to $((i+1)s_0+H)\setminus A$ is at least

$$\begin{aligned}\max\{|(is_0+H)\cap A|-|((i+1)s_0+H)\cap A|,0\}\\ =|H|\max\{x_i-x_{i+1},0\}=\frac12|H|\big(|x_i-x_{i+1}|+x_i-x_{i+1}\big)\end{aligned}$$

(where x_{i+1} is to be replaced with x_1 for $i=l$). It follows that

$$\begin{aligned}\partial_S(A)\ge\frac1m|H|\big(f(x_1)+\dots+f(x_l)\big)\\ +\frac12|H|\big(|x_1-x_2|+\dots+|x_{l-1}-x_l|+|x_l-x_1|\big).\end{aligned}\tag{3.3}$$

Choose $i, j \in [1, l]$ so that x_i is the smallest and x_j is the largest of the numbers $x_1, \dots, x_l$. By Lemma 3.1 and (3.3) we then have

$$\begin{aligned}\partial_S(A) &\ge \frac{1}{m}\,|G|\,\frac{f(x_1)+\dots+f(x_l)}{l} + |H|(x_j - x_i)\\ &\ge \frac{1}{m}\,|G|\left(\frac{f(x_1)+\dots+f(x_l)}{l} + (x_j - x_i)\right).\end{aligned}$$

Recalling that $f \in \mathcal{F}_m$ implies $f \in \mathcal{F}_l$ in view of $l \mid m$, we get

$$\begin{aligned}\partial_S(A) &\ge \frac{1}{m}\,|G|\,f\left(\frac{x_1+\dots+x_l}{l}\right)\\ &= \frac{1}{m}\,|G|\,f(|A|/|G|),\end{aligned}$$

as wanted. □

It may be worth noting that the proof of Theorem 1.6 relies on the normality of the subgroup H introduced in the course of the proof. For this reason, the proof fails to go through for non-abelian group; indeed, there are examples showing that one cannot drop the requirement that G is abelian in the statements of Theorem 1.6 and Theorem 1.1 depending on it.

4. The classes $\mathcal{F}_m$: proofs of Propositions 1.7 and 1.11 and Theorems 1.8, 1.9, and 1.12

Our proof of Proposition 1.7 is based on the following lemma (which we recommend the reader to compare with Theorem 1.12).

Lemma 4.1. *Suppose that f is a real-valued function, defined and concave on the interval $[0, 1]$ and satisfying the boundary condition (1.11). If the estimate*

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2) + (x_2 - x_1) \tag{4.1}$$

holds for all $\lambda, x_1, x_2 \in [0, 1]$ with $x_1 \le x_2$, then for any integer $m \ge 2$ we have $f \in \mathcal{F}_m$.

Proof. We fix integer $m \ge 2$ and real $x_1, \dots, x_m \in [0, 1]$ with $\min_i x_i = x_1$ and $\max_i x_i = x_m$, and, assuming (4.1), show that (1.12) holds true. For $i = 1, \dots, m$ define $\lambda_i \in [0, 1]$ by $x_i = \lambda_i x_1 + (1-\lambda_i)x_m$ and let $\lambda := (\lambda_1 + \dots + \lambda_m)/m$, so that

$$f(x_i) \ge \lambda_i f(x_1) + (1-\lambda_i) f(x_m)$$

by concavity and, consequently,

$$\frac{f(x_1)+\dots+f(x_m)}{m} \ge \lambda f(x_1) + (1-\lambda) f(x_m). \tag{4.2}$$

On the other hand, we have

$$\begin{aligned} f\left(\frac{x_1+\dots+x_m}{m}\right) &= f(\lambda x_1+(1-\lambda)x_m) \\ &\le \lambda f(x_1)+(1-\lambda)f(x_m)+(x_m-x_1) \end{aligned} \tag{4.3}$$

by (4.1), and the result follows by comparing (4.2) and (4.3). □

Proof of Proposition 1.7. Since f is concave on $[0,1]$, by Lemma 4.1 it suffices to prove (4.1) assuming $0 \le x_1 \le x_2 \le 1$ and $\lambda \in [0,1]$. The case $\lambda \in \{0,1\}$ is trivial, and we assume below that $0 < \lambda < 1$. Denote by $\Delta_\lambda(x_1,x_2)$ the difference of the left-hand side and the right-hand side of (4.1). Since the second partial derivative of $\Delta_\lambda(x_1,x_2)$ with respect to x_2 is

$$\frac{\lambda(1-\lambda)ex_1}{(\lambda x_1+(1-\lambda)x_2)x_2} \ge 0, \quad x_2 \in (0,1),$$

the largest value of $\Delta_\lambda(x_1,x_2)$ for any fixed λ and x_1 is attained either for $x_2 = x_1$ or for $x_2 = 1$; consequently, we can confine to these two cases. Indeed, (4.1) holds true in a trivial way for $x_2 = x_1$, and we therefore assume that $x_2 = 1$; thus, it remains to prove that

$$\Delta_\lambda(x_1,1) = f(\lambda x_1+1-\lambda) - \lambda f(x_1) - 1 + x_1 \le 0, \; x_1 \in [0,1].$$

To this end we just observe that the second derivative of $\Delta_\lambda(x_1,1)$ with respect to x_1 is

$$\frac{\lambda(1-\lambda)e}{(\lambda x_1+1-\lambda)x_1} > 0, \quad x_1 \in (0,1),$$

and that $\Delta_\lambda(0,1) = f(1-\lambda) - 1 \le 0$ and $\Delta_\lambda(1,1) = 0$. □

We now turn to the proof of Theorem 1.8.

Lemma 4.2. *For every integer $m \ge 2$, all functions from the class $\mathcal{F}_m$ are continuous on $(0,1)$.*

Proof. We fix an integer $m \ge 2$, a function $f \in \mathcal{F}_m$, and a number $x_0 \in (0,1)$, and we show that f is continuous at x_0. Let $l := \min\{\liminf_{x\to x_0} f(x), f(x_0)\}$ and $L := \max\{\limsup_{x\to x_0} f(x), f(x_0)\}$. It suffices to prove that $l \ge L$. For this, choose two sequences $\{\xi_k\}_{k=1}^\infty$ and $\{\zeta_k\}_{k=1}^\infty$ with all terms in $(0,1)$, converging to x_0, and satisfying $f(\xi_k) \to l$ and $f(\zeta_k) \to L$. In addition, we request $m\zeta_k - (m-1)\xi_k \in (0,1)$ to hold for any integer $k \ge 1$; in view of $m\zeta_k - (m-1)\xi_k \to x_0$, this can be arranged simply by dropping a finite number of terms from each sequence. By (1.12) we then have

$$f(\zeta_k) \le \frac{(m-1)f(\xi_k) + f(m\zeta_k - (m-1)\xi_k)}{m} + o(1)$$

as $k \to \infty$, and it remains to observe that the left-hand side is $L + o(1)$, while the right-hand side is at most $((m-1)l + L)/m + o(1)$. $\square$

We remark that the functions from the classes $\mathcal{F}_m$ are not necessarily continuous at the endpoints of the interval $[0,1]$. Indeed, for any $f \in \mathcal{F}_m$ and $a > 0$, letting

$$f_a(x) = \begin{cases} f(x) & \text{if } x \in \{0,1\}, \\ f(x) - a & \text{if } x \in (0,1), \end{cases}$$

we have $f_a \in \mathcal{F}_m$, and either f or f_a is discontinuous at 0 and 1. However, a slight modification of the proof of Lemma 4.2 shows that the potential discontinuities of a function $f \in \mathcal{F}_m$ at the endpoints of $[0,1]$ are removable; that is, the limits $\lim_{x\to 0+} f(x)$ and $\lim_{x \to 1-} f(x)$ exist and are finite.

The following corollary follows readily from Theorem 1.9 and the estimate (1.3). However, since we have not proved Theorem 1.9 yet, we use here an independent argument.

Corollary 4.3. *For any integer $m \ge 2$ and any function $f \in \mathcal{F}_m$, we have* $\sup f \le m/(m-1)$.

Proof. By Lemma 4.2, it suffices to show that $f(n/m^r) \le m/(m-1)$ holds for all integer $r \ge 0$ and $n \in [0, m^r]$. Indeed, using induction on r, we prove the slightly stronger estimate

$$f(n/m^r) \le 1 + 1/m + \dots + 1/m^{r-1}.$$

For $r = 0$ this reduces to the boundary condition (1.11). Assuming that $r \ge 1$ and n is not divisible by m, write $n = tm + \rho$ with integer t and $\rho \in [1, m-1]$. Then

$$\frac{n}{m^r} = \frac{(m-\rho)(t/m^{r-1}) + \rho\,((t+1)/m^{r-1})}{m}$$

so that by (1.12) and the induction hypothesis,

$$\begin{aligned} f\left(\frac{n}{m^r}\right) &\le \left(1 - \frac{\rho}{m}\right) f\left(\frac{t}{m^{r-1}}\right) + \frac{\rho}{m} f\left(\frac{t+1}{m^{r-1}}\right) + \frac{1}{m^{r-1}} \\ &\le \left(1 - \frac{\rho}{m}\right)\left(1 + \frac{1}{m} + \dots + \frac{1}{m^{r-2}}\right) \\ &\qquad + \frac{\rho}{m}\left(1 + \frac{1}{m} + \dots + \frac{1}{m^{r-2}}\right) + \frac{1}{m^{r-1}} \\ &= 1 + \frac{1}{m} + \dots + \frac{1}{m^{r-1}}. \end{aligned}$$

$\square$

Proof of Theorem 1.8. With Corollary 4.3 in mind, we set

$$F_m(x) := \sup\{f(x) \colon f \in \mathcal{F}_m\}, \quad x \in [0,1].$$

In view of Proposition 1.7, we have $F_m(0) \geq 0$, $F_m(1) \geq 0$, and $F_m(x) > 0$ for $x \in (0,1)$; indeed, $F_m(0) = F_m(1) = 0$ by (1.11). We now show that

$$F_m \in \mathcal{F}_m; \tag{4.4}$$

this will immediately imply continuity of F_m on $(0,1)$ (by Lemma 4.2) and show that $F_m(x) = F_m(1-x)$ (since if f belongs to $\mathcal{F}_m$, then so does the function $x \mapsto f(1-x)$).

To prove (4.4) we notice that, given $\varepsilon > 0$ and $x_1, \dots, x_m \in [0,1]$ with $\min_i x_i = x_1$ and $\max_i x_i = x_m$, we can find $f \in \mathcal{F}_m$ such that

$$F_m\left(\frac{x_1 + \dots + x_m}{m}\right) \leq f\left(\frac{x_1 + \dots + x_m}{m}\right) + \varepsilon,$$

and then, by (1.12),

$$\begin{aligned} F_m\left(\frac{x_1 + \dots + x_m}{m}\right) &\leq \frac{f(x_1) + \dots + f(x_m)}{m} + (x_m - x_1) + \varepsilon \\ &\leq \frac{F_m(x_1) + \dots + F_m(x_m)}{m} + (x_m - x_1) + \varepsilon. \end{aligned}$$

Taking the limits as $\varepsilon \to 0$ gives

$$F_m\left(\frac{x_1 + \dots + x_m}{m}\right) \leq \frac{F_m(x_1) + \dots + F_m(x_m)}{m} + (x_m - x_1),$$

whence $F_m \in \mathcal{F}_m$.

To complete the proof it remains to show that F_m is continuous at the endpoints of the interval $[0,1]$. As remarked above, a slight modification of the proof of Lemma 4.2 shows, in view of (4.4), that the limits $\lim_{x\to 0+} F_m(x)$ and $\lim_{x \to 1-} F_m(x)$ exist and are finite. Moreover, from $F_m(x) = F_m(1-x)$ it follows that these limits are equal to the same number L, and we want to show that $L = 0$. Since F_m is positive on $(0,1)$, we have $L \geq 0$. To show, on the other hand, that $L \leq 0$, we observe that if $\{\xi_k\}_{k=1}^{\infty}$ is a sequence satisfying $\xi_k \to 0$ and $\xi_k \in (0, 1/m]$ for any $k \geq 1$, then, by (4.4) and (1.12),

$$L + o(1) = F_m(\xi_k) \leq \frac{(m-1)F_m(0) + F_m(m\xi_k)}{m} + o(1) = \frac{1}{m} L + o(1)$$

as $k \to \infty$. □

Proof of Theorem 1.12. Considering $x < y$ fixed, let

$$f_{x,y}(\lambda) := \frac{1}{y-x}\big(f(\lambda x + (1-\lambda)y) - \lambda f(x) - (1-\lambda) f(y)\big).$$

Fix arbitrarily $\lambda_1, \dots, \lambda_m \in [0,1]$ with $\min_i \lambda_i = \lambda_1$ and $\max_i \lambda_i = \lambda_m$, and write $x_i := \lambda_i x + (1-\lambda_i)y;\ i \in [1,m]$. Notice that $\min_i x_i = x_m$ and $\max_i x_i = x_1$, and if

$x,y\in[0,1]$, then also $x_1,\dots,x_m\in[0,1]$. Hence, by (1.12),

$$\begin{aligned}
(y-x)&f_{x,y}\left(\frac{\lambda_1+\dots+\lambda_m}{m}\right)\\
&= f\left(\frac{x_1+\dots+x_m}{m}\right)-\frac1m\sum_{i=1}^m(\lambda_i f(x)+(1-\lambda_i)f(y))\\
&\le \frac1m\sum_{i=1}^m\big(f(x_i)-\lambda_i f(x)-(1-\lambda_i)f(y)\big)+(x_1-x_m)\\
&= \frac{y-x}{m}\sum_{i=1}^m f_{x,y}(\lambda_i)+(y-x)(\lambda_m-\lambda_1).
\end{aligned}$$

This shows that $f_{x,y}\in\mathcal{F}_m$. Consequently, $f_{x,y}(\lambda)\le F_m(\lambda)$ by the extremal property of the function F_m (cf. Theorem 1.8) and the assertion follows. □

Proof of Theorem 1.9. By continuity of the functions ω_m and F_m (see Theorem 1.8), to show that $F_m\le m\omega_m$ it suffices to prove that for any integer $r\ge0$ and $n\in[0,m^r]$, we have $F_m(n/m^r)\le m\omega_m(n/m^r)$. We use induction on r, and for each r we prove the assertion for all $n\in[0,m^r]$.

The case $r=0$ is immediate from $F_m(0)=0=m\omega_m(0)$ and $F_m(1)=0=m\omega_m(1)$. For $r\ge1$ we assume, without loss of generality, that n is not divisible by m, and we write $n=mt+\rho$ with integer t and $\rho\in[1,m-1]$. From $F_m\in\mathcal{F}_m$, the induction hypothesis, and Lemma 2.1 we then have

$$\begin{aligned}
F_m(n/m^r)&\le\frac{(m-\rho)\,F_m(t/m^{r-1})+\rho\,F_m((t+1)/m^{r-1})}{m}+\frac{1}{m^{r-1}}\\
&\le(m-\rho)\,\omega_m\left(\frac{t}{m^{r-1}}\right)+\rho\,\omega_m\left(\frac{t+1}{m^{r-1}}\right)+\frac{1}{m^{r-1}}\\
&=m\omega_m\left(\frac{n}{m^r}\right),
\end{aligned}$$

as wanted.

Next, we prove that $F_m=m\omega_m$ for $m\in\{2,3,4\}$. The case $m=2$ follows from the estimate $F_2\le2\omega_2$ which we have just obtained and Boros–Páles inequality (1.9), showing that $2\omega_2\in\mathcal{F}_2$ and, therefore, $F_2\ge2\omega_2$. Similarly, the case $m=3$ follows from $F_3\le3\omega_3$ and Theorem 1.4 showing that $3\omega_3\in\mathcal{F}$. For the case $m=4$ we notice that, in view of $\mathcal{F}_4=\mathcal{F}_2$ and (1.8),

$$F_4=F_2=2\omega_2=4\omega_4.$$

It remains to show that $F_m\ne m\omega_m$ for $m\ge5$. To this end we observe that in this case $4/m^2\le1/m\le4/m\le1-1/m$, whence

$$m\omega_m(4/m^2)=m\|4/m^2\|_{1/m}+\|4/m\|_{1/m}=5/m,$$

whereas, by (3.2), (1.12), and $F_m(0) = 0$,

$$F_m\left(\frac{4}{m^2}\right) \le \frac{(m-2)F_m(0) + 2F_m(2/m)}{m} + \frac{2}{m} \le \frac{4}{m}.$$

□

In connection with Theorem 1.9 we remark that the estimate $F_m \le m\omega_m$ and the inequality $F_m \ne m\omega_m$ for $m \ge 5$ also follow from Theorems 1.3 and 1.6, the latter of them applied with $f = F_m$, and the well-known and easy-to-verify fact that the sets $A = \mathcal{I}_n(C_m^r)$ do *not* minimize the quantity $\partial_S(A)$ for $m \ge 5$. This is yet another indication of the intrinsic relation between the discrete isoperimetric problem and the functions ω_m and F_m.

Finally, we prove Proposition 1.11.

Proof of Proposition 1.11. Suppose that f is a real-valued function, defined on the interval $[0,1]$ and satisfying the boundary condition (1.11) and the inequality (4.1) for all $\lambda, x_1, x_2 \in [0,1]$ with $x_1 \le x_2$. For real $\xi \in [0,1]$ and integer $k \ge 1$, applying (4.1) with $x_1 = 0,\ x_2 = \xi^{k-1}$, and $\lambda = 1 - \xi$, we obtain

$$f(\xi^k) = f(\lambda x_1 + (1-\lambda)x_2) \le (1-\lambda)f(x_2) + x_2 = \xi^{k-1} + \xi f(\xi^{k-1});$$

iterating,

$$f(\xi^k) \le 2\xi^{k-1} + \xi^2 f(\xi^{k-2}) \le \dots \le k\xi^{k-1}. \tag{4.5}$$

For $x \in (0,1)$, we use the resulting estimate with $k := \lceil \ln(1/x) \rceil$ and $\xi := x^{1/k}$ to get

$$f(x) < (1 + \ln(1/x))\, x \cdot x^{-1/k} \le ex\ln(e/x).$$

To complete the proof it remains to observe that, by Corollary 4.3, we have $F_m \le m/(m-1)$, and therefore Theorem 1.12 shows that the function $f = (1 - m^{-1})F_m$ satisfies (4.1). □

Appendix: proof of inequalities (1.4)–(1.7).

We prove here inequalities (1.4), (1.5), and (1.7); inequality (1.6) is immediate from (1.4) and (1.8). The proofs use the identities

$$\omega_m(x) = \|x\|_{1/m} + \frac{1}{m}\|mx\|_{1/m} + \dots + \frac{1}{m^k}\|m^k x\|_{1/m} + \frac{1}{m^{k+1}}\,\omega_m(m^{k+1}x) \tag{4.6}$$

and

$$\omega_m(n \pm x) = \omega_m(x), \tag{4.7}$$

valid for any integer $m \ge 2$, $k \ge 0$, and n, and any choice of the sign.

Proof of the inequality (1.4). As an immediate corollary of (4.6), for each $x \in [0, 1/2]$ we have $\omega_2(x) = x + \frac{1}{2}\,\omega_2(2x)$. On the other hand, for any fixed $C > 0$, the function $f_C(x) := x\log_2(C/x)$ satisfies the very same functional equation: $f_C(x) = x + \frac{1}{2}\,f_C(2x)$. Hence,

$$f_C(x) - \omega_2(x) = \frac{1}{2}\,\big(f_C(2x) - \omega_2(2x)\big),\ x \in (0, 1/2], \tag{4.8}$$

showing that it suffices to prove the estimates in question in the range $x \in [1/2, 1]$. To establish the lower bound we now observe that

$$\omega_2(x) \geq \|x\| + \frac{1}{2}\,\|2x\| = \begin{cases} 1/2 & \text{if} \quad 1/2 \leq x \leq 3/4, \\ 2 - 2x & \text{if} \quad 3/4 \leq x \leq 1, \end{cases}$$

and using some basic calculus, one verifies easily that the function in the right-hand side is at least as large as $x\log_2(1/x)$ for all $x \in [1/2, 1]$.

Turning to the upper bound, we notice that the function $f_{4/3}$ is concave and satisfies $f_{4/3}(1/3) = f_{4/3}(2/3) = 2/3$, and that the largest value attained by the Takagi function is known to be $\max\omega_2 = 2/3$ (see [AK11] or [La12]). As a result,

$$\omega_2(x) \leq 2/3 \leq f_{4/3}(x),\ x \in [1/3, 2/3].$$

In view of the functional equation (4.8), the resulting estimate $\omega_2(x) \leq f_{4/3}(x)$ extends onto the intervals $[2/3, 1]$ and $[1/6, 1/3]$, and then consequently onto the intervals $[1/12, 1/6]$, $[1/24, 1/12]$, etc. To complete the proof we just notice that the union of all these intervals (including the original interval $[1/3, 2/3]$) is the whole interval $(0, 1]$. □

Proof of the inequality (1.5). Similarly to the proof of (1.4), writing $f_C(x) := x\log_3(C/x)$, for every $x \in (0, 1/3]$ we have $\omega_3(x) = x + \frac{1}{3}\,\omega_3(3x)$ and also $f_C(x) = x + \frac{1}{3}\,f_C(3x)$. Hence,

$$f_C(x) - \omega_3(x) = \frac{1}{3}\,\big(f_C(3x) - \omega_3(3x)\big),\ x \in (0, 1/3],$$

showing that we can assume $x \in [1/3, 1]$.

Observing that if $x \in [1/3, 4/9]$, then

$$\omega_3(x) \geq \|x\|_{1/3} + \frac{1}{3}\,\|3x\|_{1/3} = \frac{1}{3} + \frac{1}{3}\,(3x - 1) = x \geq x\log_3(1/x),$$

and if $x \in [4/9, 1]$, then

$$\omega_3(x) \geq \|x\|_{1/3} \geq x\log_3(1/x)$$

(straightforward verification is left to the reader), we get the lower bound.

For the upper bound, we can further restrict the range to consider from $[1/3, 1]$ to $[1/3, 1/2]$, for once the estimate is established in this narrower range, it readily extends onto the interval $[1/2, 2/3]$ in view of

$$\omega_3(x) = \omega_3(1-x),\ f_{3/2}(x) \le f_{3/2}(1-x),\ 0 < x \le 1/2,$$

and onto the interval $[2/3, 1]$ since for any x in this interval, by (4.6) and (4.7) we have

$$\omega_3\left(\frac{2-x}{3}\right) = \frac{1}{3} + \frac{1}{3}\,\omega_3(x),$$

whence (assuming the upper bound is proved in $[1/3, 1/2]$)

$$\omega_3(x) = 3\omega_3\left(\frac{2-x}{3}\right) - 1 \le (2-x)\log_3\frac{9}{2(2-x)} - 1 \le x\log_3\frac{3}{2x},\ \ x \in [2/3, 1].$$

(For the last inequality observe that both sides are equal for $x = 1$, and compare the derivatives.)

Thus, it remains to prove the upper bound for $x \in [1/3, 1/2]$. To this end, for integer $r \ge 1$ we let

$$b_r := \frac{1}{3} + \dots + \frac{1}{3^r}$$

and use induction on r to show that $\omega_3(x) \le f_{3/2}(x)$ for all $x \in [b_r, b_{r+1}]$. If $r = 1$, then $x \in [1/3, 4/9]$; in view of (1.3), in this range we have

$$\begin{aligned}
\omega_3(x) &= \|x\|_{1/3} + \frac{1}{3}\,\|3x\|_{1/3} + \frac{1}{9}\,\|9x\|_{1/3} + \frac{1}{27}\,\omega_3(27x) \\
&\le \frac{1}{3} + \frac{1}{3}\,(3x-1) + \frac{1}{9}\,\min\left\{\frac{1}{3}, 4-9x\right\} + \frac{1}{54} \\
&= \min\left\{x + \frac{1}{18}, \frac{25}{54}\right\},
\end{aligned}$$

and a simple verification confirms that the expression in the right-hand side is smaller than $f_{3/2}(x)$ for $x \in [1/3, 4/9]$. Assuming now that $r \ge 2$, we observe that $x \in [b_r, b_{r+1}]$ implies $3x - 1 \in [b_{r-1}, b_r]$; hence, by the induction hypothesis, for all x in this range we have

$$\begin{aligned}
\omega_3(x) &= \|x\|_{1/3} + \frac{1}{3}\,\omega_3(3x-1) \\
&\le \frac{1}{3} + \frac{1}{3}\,(3x-1)\log_3\frac{3}{2(3x-1)} \\
&\le x\log_3\frac{3}{2x}.
\end{aligned}$$

(For the last inequality compare the values of both sides at $1/2$ and their derivatives for $1/3 < x < 1/2$.) This completes the proof. $\square$

Proof of the inequality (1.7). As in the proofs of (1.4) and (1.5), we can confine to the range $x \in [1/m, 1]$ where the upper bound readily follows from (1.3):

$$\omega_m(x) \le \frac{1}{m-1} \le x \log_m(3/2x), \ \ x \in [1/m, 1].$$

(Notice that the right-hand side is a concave function and hence attains its minimum at an endpoint.) For the lower bound we observe that the function $x \log_m(e/mx)$ is decreasing for $x \ge 1/m$, whence

$$\omega_m(x) \ge \frac{1}{m} \ge x \log_m(e/mx), \quad x \in [1/m, 1 - 1/m]$$

and

$$\omega_m(x) \ge 0 > x \log_m(e/mx), \quad x \in [1 - 1/m, 1].$$

□

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References

[A11] P.C. Allaart, An inequality for sums of binary digits, with application to Takagi functions, *J. Math. Anal. Appl.* **381** (2) (2011), 689–694.

[AK11] P.C. Allaart and K. Kawamura, The Takagi function: a survey, *Real Anal. Exchange* **37** (1) (2011/12), 1–54.

[B96] S.L. Bezrukov, Edge isoperimetric problems on graphs, Graph theory and combinatorial biology (Balatonlelle, 1996), 157–197. *Bolyai Soc. Math. Stud.* **7**, János Bolyai Math. Soc., Budapest, 1999.

[B08] Z. Boros, An inequality for the Takagi function, *Math. Inequal. Appl.* **11** (4) (2008), 757–765.

[CFGS88] F.R.K. Chung, Z. Füredi, R.L. Graham, and P. Seymour, On induced subgraphs of the cube, *J. Combin. Theory, Ser. A* **49** (1) (1988), 180–187.

[D75] H. Delange, Sur la fonction sommatoire de la fonction "somme des chiffres" (French), *Enseignement Math. (2)* **21** (1) (1975), 31–47.

[G00] C.J. Guu, The McFunction, *Discrete mathematics* **213** (2000), 163–167.

[H64] L.H. Harper, Optimal assignments of numbers to vertices, *J. Soc. Indust. Appl. Math.* **12** (1964), 131–135.

[H04] L.H. Harper, *Global methods for combinatorial isoperimetric problems*, Cambridge Studies in Advanced Mathematics **90**, Cambridge University Press, Cambridge, 2004.

[HLW06] S. Hoory, N. Linial, and A. Wigderson, Expander graphs and their applications, *Bull. Amer. Math. Soc.* **43** (4) (2006), 439–561.

[HY83] M. Hata and M. Yamaguti, Takagi function and its generalization, *Japan J. Appl. Math.* **1** (1984), 183–199.

[HP05] A. Házy and Z. Páles, On approximately midconvex functions, *Bull. London Math. Soc.* **36** (2004), 339–350.

[Kr07] M. Krüppel, On the extrema and the improper derivatives of Takagi's continuous nowhere differentiable function, *Rostock. Math. Kolloq.* **62** (2007), 41-59.

[La12] J. Lagarias, The Takagi function and its properties, *RIMS Kokyuroku Bessatsu, B34, Res. Inst. Math. Sci. (RIMS), Kyoto, 2012*, 153–189.

[Li64] Lindsey, J. H., II, Assignment of numbers to vertices, *Amer. Math. Monthly* **71** (1964), 508–516.

[P04] Z. Páles, 7. Problem *in* Report of meeting. The Forty-first International Symposium on Functional Equations, *Aequationes Math.* **67** (2004), 307.
5 *Math. Z.* **32** (1930), 474–475.

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