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The Connectivity of addition Cayley graphs

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Abstract

For any finite abelian group G and any subset $S \subseteq G$, we determine the connectivity of the addition Cayley graph induced by S on G. Moreover, we show that if this graph is not complete, then it possesses a minimum vertex cut of a special, explicitly described form.

Keywords: Addition Cayley Graphs, Connectivity, Critical pairs, Kemperman Structure Theorem.

1 Addition Cayley Graphs

For a subset S of the abelian group G, we denote by $\mathrm{Cay}_G^+(S)$ the addition Cayley graph induced by S on G; recall, this is the graph with the vertex set G and the edge set $\{(g_1, g_2) \in G \times G\colon g_1 + g_2 \in S\}$. Twins of the usual Cayley graphs, addition Cayley graphs (also called *sum graphs*) received much less attention in the literature; indeed, [A] (independence number), [CGW03]

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and [L] (hamiltonicity), [C92] (expander properties), and [Gr05] (clique number) is a nearly complete list of papers, known to us, where addition Cayley graphs are addressed. To some extent, this situation may be explained by the fact that addition Cayley graphs are rather difficult to study. For instance, it is well-known and easy to prove that any connected Cayley graph on a finite abelian group with at least three elements is hamiltonian, see [Mr83]; however, apart from the results of [CGW03], nothing seems to be known on hamiltonicity of *addition* Cayley graphs on finite abelian groups. Similarly, the connectivity of a Cayley graph on a finite abelian group is easy to determine, while determining the connectivity of an *addition* Cayley graph is a non-trivial problem, to which we will devote our focus.

Let Γ be a graph on the finite set V. The (vertex) connectivity of Γ, denoted by $\kappa(\Gamma)$, is the smallest number of vertices which are to be removed from V so that the resulting graph is either disconnected or trivial.

Our goal is to determine the connectivity of the addition Cayley graphs, induced on finite abelian groups by their subsets, and accordingly we use additive notation for the group operation. In particular, for subsets A and B of an abelian group we write

$$A \pm B := \{a \pm b \colon a \in A,\, b \in B\},$$

which is abbreviated by $A \pm b$ in the case where $B = \{b\}$ is a singleton subset.

It is immediate from the definition that for a subset $A \subseteq G$, the neighborhood of A in $\mathrm{Cay}^+_G(S)$ is the set $S - A$, and it is easy to derive that $\mathrm{Cay}^+_G(S)$ is complete if and only if either $S = G$, or $S = G \setminus \{0\}$ and G is an elementary abelian 2-group (possibly of zero rank). Also, since maximum degree in $\mathrm{Cay}^+_G(S)$ is at most $|S|$, we have the trivial bound $\kappa(\mathrm{Cay}^+_G(S)) \le |S|$.

If H is a subgroup of G satisfying $S + H \ne G$, and g is an element of G with $2g \in S + H$, then $g + H \subseteq S - (g + H)$. Consequently, the boundary of $g + H$ in $\mathrm{Cay}^+_G(S)$ has size $|(S - (g + H)) \setminus (g + H)| = |S + H| - |H|$, and $(S-(g+H)) \cup (g+H) = S+H-g \ne G$, implying $\kappa(\mathrm{Cay}^+_G(S)) \le |S+H|-|H|$. Set

$$2 * G := \{2g \colon g \in G\},$$

so that the existence of $g \in G$ with $2g \in S + H$ is equivalent to the condition $(S + 2 * G) \cap H \ne \varnothing$. Motivated by the above observation, we define

$$\mathcal{H}_G(S) := \{H \le G \colon (S + 2 * G) \cap H \ne \varnothing,\, S + H \ne G\},$$

and let

$$\eta_G(S) := \min\{|S + H| - |H| \colon H \in \mathcal{H}_G(S)\}.$$

Another important family of sets with small boundary is obtained as follows. Suppose that the subgroups $L \le G_0 \le G$ and the element $g_0 \in G_0$ satisfy

(i) $|G_0/L|$ is even and larger than 2;

(ii) $S + L = (G \setminus G_0) \cup (g_0 + L)$.

Fix $g \in G_0 \setminus L$ with $2g \in L$ and consider the set $A := (g + L) \cup (g + g_0 + L)$. The neighborhood of this set in $\mathrm{Cay}_G^+(S)$ is

$$S - A = (G \setminus G_0) \cup (g + L) \cup (g + g_0 + L) = (G \setminus G_0) \cup A,$$

whence

$$(S - A) \cup A \neq G, \ |(S - A) \setminus A| = |G \setminus G_0| = |S + L| - |L|.$$

Consequently, $\kappa(\mathrm{Cay}_G^+(S)) \le |S + L| - |L|$. With this construction in mind, we define $\mathcal{L}_G(S)$ to be the family of all those subgroups $L \le G$ for which a subgroup $G_0 \le G$, lying above L, and an element $g_0 \in G_0$, can be found so that the properties (i) and (ii) hold, and we let

$$\lambda_G(S) := \min\{|S + L| - |L| \colon L \in \mathcal{L}_G(S)\}.$$

Thus, $\kappa(\mathrm{Cay}_G^+(S)) \le \lambda_G(S)$.

Our first principal result is

Theorem 1.1 *If S is a proper subset of the finite abelian group G, then*

$$\kappa(Cay_G^+(S)) = \min\{\eta_G(S), \lambda_G(S), |S|\}.$$

Theorem 1.1 is an immediate corollary of Theorem 1.2 below, which actually shows that the minimum in the statement of Theorem 1.1 is attained, with just one exception, on either $\eta_G(S)$ or $|S|$. Being much subtler, Theorem 1.2 is also more technical, and to state it we have to bring into consideration a special sub-family of $\mathcal{L}_G(S)$. Specifically, let $\mathcal{L}_G^*(S)$ be the family of those subgroups $L \le G$ such that for some $G_0, G_1 \le G$, lying above L, and some $g_0 \in G_0$, the following conditions hold:

(L1) $G/L = (G_0/L) \oplus (G_1/L)$;

(L2) G_0/L is a cyclic 2-group of order $|G_0/L| \ge 4$, and $\langle g_0 \rangle + L = G_0$;

(L3) G_1/L is an elementary abelian 2-group (possibly of zero rank);

(L4) $S + L = (G \setminus G_0) \cup (g_0 + L)$ and $S \cap (g_0 + L)$ is not contained in a proper coset of L.

Theorem 1.2 *Let S be a proper subset of the finite abelian group G. There exists at most one subgroup $L \in \mathcal{L}^*_G(S)$ with $|S+L|-|L| \leq |S|-1$. Moreover,*

(i) if L is such a subgroup, then $\kappa(Cay^+_G(S)) = \lambda_G(S) = |S+L|-|L|$ and $\eta_G(S) \geq |S|$;

(ii) if such a subgroup does not exist, then $\kappa(Cay^+_G(S)) = \min\{\eta_G(S), |S|\}$.

Our last result shows that under the extra assumption $\kappa(\mathrm{Cay}^+_G(S)) < |S|$, the conclusion of Theorem 1.1 can be greatly simplified.

Theorem 1.3 *Let S be a proper subset of the finite abelian group G. If $\kappa(Cay^+_G(S)) < |S|$, then*

$$\kappa(Cay^+_G(S)) = \min\{|S+H|-|H| \colon H \leq G,\ S+H \neq G\}.$$

References

[A] N. Alon, Large sets in finite fields are sumsets, *J. Number Theory*, to appear.

[CGW03] B. Cheyne, V. Gupta, and C. Wheeler, Hamilton Cycles in Addition Graphs, *Rose-Hulman Undergraduate Math. Journal* **1** (4) (2003) (electronic).

[C92] F.R.K. Chung, Diameters and eigenvalues, *J. Amer. Math. Soc.* **2** (2) (1989), 187–196.

[Gr05] B.J. Green, Counting sets with small sumset, and the clique number of random Cayley graphs, *Combinatorica* **25** (2005), 307–326.

[Gk01] D.J. Grynkiewicz, Quasi-periodic decompositions and the Kemperman structure theorem, *European Journal of Combinatorics* **26** (5) (2005), 559–575.

[Gk02] D.J. Grynkiewicz, Sumsets, Zero-Sums and Extremal Combinatorics, *Ph.D. Dissertation*, Caltech (2005).

[Km60] J.H.B. Kemperman, On small subsets in an abelian group, *Acta Mathematica* **103** (1960), 63–88.

[Kn53] M. Kneser, Abschätzung der asymptotischen Dichte von Summenmengen, *Math. Z.* **58** (1953), 459–484.

[Kn55] M. Kneser, Ein Satz über abelsche Gruppen mit Anwendungen auf die Geometrie der Zahlen, *Math. Z.* **61** (1955), 429–434.

[L05] V.F. Lev, Critical pairs in abelian groups and Kemperman's structure theorem, *Internation Journal of Number Theory* **2** (3) (2006), 379–396.

[L] V.F. Lev, Sums and differences along Hamiltonian cycles, *Submitted*.

[Mn76] H.B. Mann, Addition theorems: the addition theorems of group theory and number theory. Reprint, with corrections, of the 1965 original. Robert E. Krieger Publishing Co., Huntington, N.Y., 1976.

[Mr83] D. Marušič, Hamiltonian circuits in Cayley graphs, *Discrete Math.* **46** (1) (1983), 49–54.