PROGRESSION-FREE SETS IN $\mathbb{Z}_4^n$ ARE EXPONENTIALLY SMALL

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Abstract. We show that for integer $n \ge 1$, any subset $A \subseteq \mathbb{Z}_4^n$ free of three-term arithmetic progressions has size $|A| \le 4^{\gamma n}$, with an absolute constant $\gamma \approx 0.926$.

1. Background and Motivation

In his influential papers [R52, R53], Roth has shown that if a set $A \subseteq \{1, 2, \dots, N\}$ does not contain three elements in an arithmetic progression, then $|A| = o(N)$ and indeed, $|A| = O(N/\log\log N)$ as N grows. Since then, estimating the largest possible size of such a set has become one of the central problems in additive combinatorics. Roth's original results were improved by Heath-Brown [H87], Szemerédi [S90], Bourgain [B99], Sanders [S12, S11], and Bloom [B16], the current record being $|A| = O(N(\log\log N)^4/\log N)$, due to Bloom.

It is easily seen that Roth's problem is essentially equivalent to estimating the largest possible size of a subset of the cyclic group $\mathbb{Z}_N$, free of three-term arithmetic progressions. This makes it natural to investigate other finite abelian groups.

We say that a subset A of an (additively written) abelian group G is *progression-free* if there do not exist pairwise distinct $a, b, c \in A$ with $a + b = 2c$, and we denote by $r_3(G)$ the largest size of a progression-free subset $A \subseteq G$. For abelian groups G of odd order, Brown and Buhler [BB82] and independently Frankl, Graham, and Rödl [FGR87] proved that $r_3(G) = o(|G|)$ as $|G|$ grows. Meshulam [M95], following the general lines of Roth's argument, has shown that if G is an abelian group of odd order, then $r_3(G) \le 2|G|/\operatorname{rk}(G)$ (where we use the standard notation $\operatorname{rk}(G)$ for the rank of G); in particular, $r_3(\mathbb{Z}_m^n) \le 2m^n/n$. Despite many efforts, no further progress was made for over 15 years, till Bateman and Katz in their ground-breaking paper [BK12] proved that $r_3(\mathbb{Z}_3^n) = O(3^n/n^{1+\varepsilon})$ with an absolute constant $\varepsilon > 0$.

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Abelian groups of even order were first considered in [L04] where, as a further elaboration on the Roth-Meshulam proof, it is shown that $r_3(G) < 2|G|/\operatorname{rk}(2G)$ for any finite abelian group G; here $2G = \{2g \colon g \in G\}$. For the homocyclic groups of exponent 4 this result was improved by Sanders [S09] who proved that $r_3(\mathbb{Z}_4^n) = O(4^n/n(\log n)^\varepsilon)$ with an absolute constant $\varepsilon > 0$. The goal of this paper is to further improve Sanders's result, as follows.

Let H denote the binary entropy function; that is,

$$H(x) = -x\log_2 x - (1-x)\log_2(1-x), \quad x \in (0,1),$$

where $\log_2 x$ is the base-2 logarithm of x. For the rest of the paper, we set

$$\gamma := \max\left\{\frac{1}{2}\big(H(0.5-\varepsilon) + H(2\varepsilon)\big) \colon 0 < \varepsilon < 0.25\right\} \approx 0.926.$$

Theorem 1. *If $n \geq 1$ and $A \subseteq \mathbb{Z}_4^n$ is progression-free, then $|A| \leq 4^{\gamma n}$.*

The proof of Theorem 1 is presented in the next section.

We note that the exponential reduction in Theorem 1 is the first of its kind for problems of this sort.

Starting from Roth, the standard way to obtain quantitative estimates for $r_3(G)$ involves a combination of the Fourier analysis and the density increment technique; the only exception is [L12] where for the groups $G \cong \mathbb{Z}_q^n$ with a prime power q, the above-mentioned Meshulam's result is recovered using a completely elementary argument. In contrast, in the present paper we use the polynomial method, without resorting to the familiar Fourier analysis – density increment strategy.

For a finite abelian group $G \cong \mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_k}$ with positive integer $m_1 \mid \dots \mid m_k$, denote by $\operatorname{rk}_4(G)$ the number of indices $i \in [1,k]$ with $4 \mid m_i$. Since, writing $n := \operatorname{rk}_4(G)$, the group G is a union of $4^{-n}|G|$ cosets of a subgroup isomorphic to $\mathbb{Z}_4^n$, as a direct consequence of Theorem 1 we get the following corollary.

Corollary 1. *If A is a progression-free subset of a finite abelian group G then, writing $n := \operatorname{rk}_4(G)$, we have $|A| \leq 4^{-(1-\gamma)n}|G|$.*

2. Proof of Theorem 1

We recall that the degree of a multivariate polynomial is the largest sum of the exponents of all of its monomials. The polynomial is *multilinear* if it is linear in every individual variable.

The proof of Theorem 1 is based on the following lemma.

Lemma 1. *Suppose that $n \geq 1$ and $d \geq 0$ are integers, P is a multilinear polynomial in n variables of total degree at most d over a field $\mathbb{F}$, and $A \subseteq \mathbb{F}^n$ is a set with $|A| > 2\sum_{0\leq i\leq d/2}\binom{n}{i}$. If $P(a-b) = 0$ for all $a, b \in A$ with $a \neq b$, then also $P(0) = 0$.*

Proof. Let $m := \sum_{0\le i\le d/2}\binom{n}{i}$, and let $\mathcal{K}=\{K_1,\dots,K_m\}$ be the collection of all sets $K\subseteq[n]$ with $|K|\le d/2$. Writing for brevity

$$x^I := \prod_{i\in I} x_i, \quad x=(x_1,\dots,x_n)\in\mathbb{F}^n,\ I\subseteq[n],$$

there exist coefficients $C_{I,J}\in\mathbb{F}$ $(I,J\subseteq[n])$ depending only on the polynomial P, such that for all $x,y\in\mathbb{F}^n$ we have

$$\begin{aligned}P(x-y) &= \sum_{\substack{I,J\subseteq[n]\\ I\cap J=\varnothing\\ |I|+|J|\le d}} C_{I,J}\,x^I y^J\\ &= \sum_{I\in\mathcal{K}} x^I \sum_{\substack{J\subseteq[n]\setminus I\\ |J|\le d-|I|}} C_{I,J}\,y^J + \sum_{J\in\mathcal{K}}\Bigg(\sum_{\substack{I\subseteq[n]\setminus J\\ d/2<|I|\le d-|J|}} C_{I,J}\,x^I\Bigg) y^J.\end{aligned}$$

The right-hand side can be interpreted as the scalar product of the vectors $u(x),v(y)\in\mathbb{F}^{2m}$ defined by

$$u_i(x)=x^{K_i},\quad u_{m+i}(x)=\sum_{\substack{I\subseteq[n]\setminus K_i\\ d/2<|I|\le d-|K_i|}} C_{I,K_i}\,x^I$$

and

$$v_i(y)=\sum_{\substack{J\subseteq[n]\setminus K_i\\ |J|\le d-|K_i|}} C_{K_i,J}\,y^J,\quad v_{m+i}(y)=y^{K_i}$$

for all $1\le i\le m$. Consequently, if we had $P(a-b)=0$ for all $a,b\in A$ with $a\ne b$, while $P(0)\ne 0$, this would imply that the vectors $u(a)$ and $v(b)$ are orthogonal if and only if $a\ne b$. As a result, the vectors $u(a)$ would be linearly independent (an equality of the sort $\sum_{a\in A}\lambda_a u(a)=0$ with the coefficients $\lambda_a\in\mathbb{F}$ after a scalar multiplication by $v(b)$ yields $\lambda_b=0$, for any $b\in A$). Finally, the linear independence of $\{u(a)\colon a\in A\}\subseteq\mathbb{F}^{2m}$ implies $|A|\le 2m$, contrary to the assumptions of the lemma. □

Remark. It is easy to extend the lemma relaxing the multilinearity assumption to the assumption that P has bounded degree in each individual variable. Specifically, denoting by $f_\delta(n,d)$ the number of monomials $x_1^{i_1}\dots x_n^{i_n}$ with $0\le i_1,\dots,i_n\le\delta$ and $i_1+\dots+i_n\le d$, if P has all individual degrees not exceeding δ, and the total degree not exceeding d, then $|A|>2f_\delta(n,\lfloor d/2\rfloor)$ along with $P(a-b)=0$ $(a,b\in A,\ a\ne b)$ imply $P(0)=0$. Moreover, taking $\delta=d$, or $\delta=|\mathbb{F}|-1$ for $\mathbb{F}$ finite, one can drop the individual degree assumption altogether.

We will use the estimate

$$\sum_{0\le i\le z}\binom{n}{i} < 2^{nH(z/n)} \tag{1}$$

valid for all integer $n \ge 1$ and real $0 < z \le n/2$; see, for instance, [McWS77, Ch. 10, §11, Lemma 8].

Recall, that for integers $n \ge d \ge 0$, the sum $\sum_{i=0}^{d}\binom{n}{i}$ is the dimension of the vector space of all multilinear polynomials in n variables of total degree at most d over the two-element field $\mathbb{F}_2$. In particular, the dimension of the vector space of *all* multilinear polynomials in n variables over $\mathbb{F}_2$ is equal to the dimension of the vector space of all $\mathbb{F}_2$-valued functions on $\mathbb{F}_2^n$, and it follows that any non-zero multilinear polynomial represents a non-zero function. These basic facts are used in the proof of Proposition 1 below.

For an integer $n \ge 1$, denote by F_n the subgroup of the group $\mathbb{Z}_4^n$ generated by its involutions; thus, F_n is both the image and the kernel of the doubling endomorphism of $\mathbb{Z}_4^n$ defined by $g \mapsto 2g$ ($g \in \mathbb{Z}_4^n$), and we have $F_n \cong \mathbb{Z}_2^n$.

Proposition 1. *Suppose that $n \ge 1$ and $A \subseteq \mathbb{Z}_4^n$ is progression-free. Then for every $0 < \varepsilon < 0.25$, the number of F_n-cosets containing at least $2^{nH(0.5-\varepsilon)+1}$ elements of A is less than $2^{nH(2\varepsilon)}$.*

Proof. Let $\mathcal{R}$ be the set of all those F_n-cosets containing at least $2^{nH(0.5-\varepsilon)+1}$ elements of A, and for each coset $R \in \mathcal{R}$ let $A_R := A \cap R$; thus, $\cup_{R\in\mathcal{R}} A_R \subseteq A$ (where the union is disjoint), and

$$|A_R| \ge 2^{nH(0.5-\varepsilon)+1}, \quad R \in \mathcal{R}. \tag{2}$$

For a subset $S \subseteq \mathbb{Z}_4^n$, write

$$2\cdot S := \{s'+s'' \colon (s',s'') \in S\times S,\ s' \ne s''\} \quad \text{and} \quad 2*S := \{2s \colon s \in S\}.$$

The assumption that A is progression-free implies that the sets

$$B := \cup_{R\in\mathcal{R}}(2\cdot A_R) \subseteq F_n \quad \text{and} \quad C := \cup_{R\in\mathcal{R}}(2*R) \subseteq F_n$$

are disjoint: this follows by observing that if $2r \in 2\cdot A$ with some $r \in R$, then for each $a \in r+F_n$ we have $2a = 2r \in 2\cdot A$. Furthermore, the sets $2*R$ are in fact pairwise distinct singletons (for $2r_1 = 2r_2$ is equivalent to $r_1 - r_2 \in F_n$ and thus to $r_1 + F_n = r_2 + F_n$), whence $|C| = |\mathcal{R}|$.

Let $d = n - \lceil 2\varepsilon n\rceil$ so that, in view of (2) and (1),

$$2\sum_{0\le i\le d/2}\binom{n}{i} < 2^{nH(0.5-\varepsilon)+1} \le |A_R|, \quad R \in \mathcal{R}. \tag{3}$$

Denoting by $\overline{C}$ the complement of C in F_n, and assuming, contrary to what we want to prove, that $|\mathcal{R}| \ge 2^{nH(2\varepsilon)}$, from (1) we get

$$\sum_{i=0}^{d} \binom{n}{i} = 2^n - \sum_{i=0}^{\lceil 2\varepsilon n\rceil -1} \binom{n}{i} > 2^n - 2^{nH(2\varepsilon)} \ge 2^n - |\mathcal{R}| = 2^n - |C| = |\overline{C}|.$$

(This is the computation where the assumption $\varepsilon < 0.25$ is used.) Consequently, identifying F_n with the additive group of the vector space $\mathbb{F}_2^n$, and accordingly considering B and C as subsets of $\mathbb{F}_2^n$, we conclude that the dimension of the vector space of all multilinear n-variate polynomials over the field $\mathbb{F}_2$ exceeds the dimension of the vector space of all $\mathbb{F}_2$-valued functions on $\overline{C}$. Thus, the evaluation map, associating with every polynomial the corresponding function, is degenerate. As a result, there exists a non-zero multilinear polynomial $P \in \mathbb{F}_2[x_1, \dots, x_n]$ of total degree $\deg P \le d$ such that P vanishes on $\overline{C}$. In particular, P vanishes on $B \subseteq \overline{C}$, and therefore on each set $2 \cdot A_R$, for all $R \in \mathcal{R}$. Fixing arbitrarily an element $r \in R$, the polynomial $P(2r + x)$ thus vanishes whenever $x \in 2 \cdot (A_R - r)$. Hence, also $P(2r) = 0$ by Lemma 1 (which is applicable in view of (3)); that is, P also vanishes on each singleton set $2 * A_R$, for all $R \in \mathcal{R}$. It follows that P vanishes on C. However, P was chosen to vanish on $\overline{C}$. Therefore, P vanishes on all of $\mathbb{F}_2^n$, and it follows that P is the zero polynomial. This is a contradiction showing that $|\mathcal{R}| < 2^{nH(2\varepsilon)}$, and thus completing the proof. □

Proof of Theorem 1. For $x \ge 0$, let $N(x)$ denote the number of F_n-cosets containing at least x elements of A; thus $N(x) = 0$ for $x > 2^n$, and we can write

$$|A| = \int_0^{2^{n+1}} N(x)\, dx. \tag{4}$$

Trivially, we have $N(x) \le 2^n$ for all $x \ge 0$, so that

$$\int_0^{2^{nH(1/4)+1}} N(x)\, dx \le 2^{(H(1/4)+1)n+1} < 2 \cdot 4^{\gamma n}. \tag{5}$$

On the other hand, the substitution $x = 2^{nH(0.5-\varepsilon)+1}$ gives

$$\int_{2^{nH(1/4)+1}}^{2^{n+1}} N(x)\, dx = n \int_0^{1/4} 2^{nH(0.5-\varepsilon)+1} N(2^{nH(0.5-\varepsilon)+1}) \log \frac{0.5+\varepsilon}{0.5-\varepsilon}\, d\varepsilon, \tag{6}$$

and applying Proposition 1, the integral in the right-hand side can be estimated as

$$2n \int_0^{1/4} 2^{n(H(0.5-\varepsilon)+H(2\varepsilon))} \log \frac{0.5+\varepsilon}{0.5-\varepsilon}\, d\varepsilon < 3n \int_0^{1/4} 2^{n(H(0.5-\varepsilon)+H(2\varepsilon))}\, d\varepsilon < n \cdot 4^{\gamma n}. \tag{7}$$

From (4)–(7) we get $|A| < (n+2) \cdot 4^{\gamma n}$, and to conclude the proof we use the tensor power trick: for an integer $k \ge 1$, the set $A \times \cdots \times A \subseteq \mathbb{Z}_4^{kn}$ is progression-free and therefore

$$|A|^k < (kn+2) \cdot 4^{\gamma kn}$$

by what we have just shown. This readily implies the result. □

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