

Minimising the Sum of Projections of a Finite Set

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Abstract Consider the projections of a finite set $A \subset \mathbb{R}^n$ onto the coordinate hyperplanes; how small can the sum of the sizes of these projections be, given the size of A ? In a different form, this problem has been studied earlier in the context of edge-isoperimetric inequalities on graphs, and it can be derived from the known results that there is a linear order on the set of n -tuples with non-negative integer coordinates, such that the sum in question is minimised for the initial segments with respect to this order. We present a new, self-contained and constructive proof, enabling us to obtain a stability result and establish algebraic properties of the smallest possible projection sum. We also solve the problem of minimising the sum of the sizes of the one-dimensional projections.

Keywords Isoperimetric problem · Loomis–Whitney inequality · Projections

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1 Preliminaries

Given an integer $n \geq 1$, for each $i \in [1, n]$ denote by π_i the orthogonal projection of the vector space \mathbb{R}^n onto the coordinate hyperplane $\{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i = 0\}$.

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For a finite set $A \subset \mathbb{R}^n$, as a simple consequence of the Loomis–Whitney inequality (see, for instance, [2,3,5]), we have

$$\prod_{i=1}^n |\pi_i(A)| \geq |A|^{n-1};$$

combining this estimate with the arithmetic mean-geometric mean inequality yields

$$\sum_{i=1}^n |\pi_i(A)| \geq n |A|^{1-1/n}. \quad (1)$$

The Loomis–Whitney inequality is known to be sharp, turning into an exact equality when the set under consideration is an axis-aligned rectangular parallelepiped (and, in the convex set settings, only in this case, as it follows from the argument of [5]). In contrast, the estimate (1) is not sharp; say, it shows that for a three-dimensional, five-point set, the sum of the projection sizes is at least as large as $3 \cdot 5^{2/3} \approx 8.77$, while it is not difficult to see that, in fact, this sum cannot be smaller than 10. This leads naturally to the following question: exactly how small can the sum in the left-hand side of (1) be for a finite set $A \subset \mathbb{R}^n$ of given size? Loosely speaking, we want to know how much the points of an n -dimensional set of given size can hide behind each other.

The answer to a tightly related question is due to Bollobás and Leader [1, Thm. 15], where it was cast as the edge-isoperimetric problem for the n -dimensional grid graph; see also Harper [4, Thm. 7.1] and historical comments [4, p. 142], as well as the references contained therein. Our goal here is to give a direct, independent, and self-contained solution of a discrete-geometric flavour, avoiding references to graph theory and making the underlying rearrangement procedure maximally algorithmic. This enables us to prove a stability result showing that, in certain cases, the set with the smallest sum of the projection sizes is, essentially, unique. Furthermore, we establish some algebraic properties of the smallest possible value of this sum as a function of the size of the set being projected. Finally, in Appendix we discuss and solve a similar problem for the one-dimensional projections.

2 Summary of Results

If $|A| = K^n$ with an integer $K \geq 1$, then (1) implies

$$\sum_{i=1}^n |\pi_i(A)| \geq nK^{n-1},$$

which is attained for the discrete n -dimensional cube $A = [0, K - 1]^n$. The situation where $|A|$ is *not* a perfect n th power is much subtler and requires some preparation to discuss.

Denote by \mathbb{N}_0 the set of all non-negative integers. We define the *cube order* on \mathbb{N}_0^n inductively, saying that $x = (x_1, \dots, x_n)$ precedes $y = (y_1, \dots, y_n)$, and writing $x \preceq y$, if one of the following holds:

- (i) The largest coordinate of x is smaller than that of y ;
- (ii) the largest coordinates of x and y are equal to the same common value, say μ , and the largest index $i \in [1, n]$ with $x_i = \mu$ is smaller than the largest index $i \in [1, n]$ with $y_i = \mu$;
- (iii) the largest coordinates of x and y are equal to the same common value μ , the largest index $i \in [1, n]$ with $x_i = \mu$ is equal to the largest index $i \in [1, n]$ with $y_i = \mu$, and for this index i we have $\pi_i(x) \preceq \pi_i(y)$ in the cube order on \mathbb{N}_0^{n-1} .

More formally, x precedes y if either there exists $l_0 \in \mathbb{N}_0$ and $j \in [1, n]$ such that $\{i \in [1, n]: x_i = l\} = \{i \in [1, n]: y_i = l\}$ for each $l > l_0$, and also $\{i \in [j + 1, n]: x_i = l_0\} = \{i \in [j + 1, n]: y_i = l_0\}$, while $x_j < y_j = l_0$.

For integer $m \geq 0$, by $\mathcal{I}_n(m)$ we denote the length- m initial segment of \mathbb{N}_0^n with respect to the cube order; thus, for instance, $\mathcal{I}_n(0) = \emptyset$, $\mathcal{I}_1(m) = [0, m - 1]$,

$$\mathcal{I}_2(10) = \{(0, 0), (1, 0), (0, 1), (1, 1), (2, 0), (2, 1), (0, 2), (1, 2), (2, 2), (3, 0)\},$$

and

$$\begin{aligned} \mathcal{I}_3(20) = \{ & (0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0), (0, 0, 1), (1, 0, 1), (0, 1, 1), \\ & (1, 1, 1), (2, 0, 0), (2, 1, 0), (2, 0, 1), (2, 1, 1), (0, 2, 0), (1, 2, 0), \\ & (0, 2, 1), (1, 2, 1), (2, 2, 0), (2, 2, 1), (0, 0, 2), (1, 0, 2)\}. \end{aligned}$$

Informally, initial segments fill in \mathbb{N}_0^n cube-wise: once the cube $[0, K - 1]^n$ has been completed, for some integer $K \geq 1$, the n faces

$$0 \leq x_1, \dots, x_i \leq K, x_{i+1} = K, 0 \leq x_{i+2}, \dots, x_n \leq K - 1, \quad i \in [0, n - 1],$$

are filled in one by one to get a covering of the next cube $[0, K]^n$, etc.

Speaking about initial segments we will always mean finite initial segments of \mathbb{N}_0^n with respect to the cube order, with the value of n determined by the context. We notice that the cube order is identical to the order introduced in [1, Sect. 3], and similar to the order introduced in [4, Sect. 7.1.1] (the latter order is defined on the whole grid \mathbb{Z}^n). We say that the initial segment I_1 is *shorter* than the initial segment I_2 if $|I_1| < |I_2|$; equivalently, if $I_1 \subset I_2$. If $m = (K + 1)^i K^{n-i}$ with some $i \in [0, n - 1]$, then

$$\mathcal{I}_n(m) = \{(x_1, \dots, x_n) \in \mathbb{N}_0^n : 0 \leq x_1, \dots, x_i \leq K, 0 \leq x_{i+1}, \dots, x_n \leq K - 1\} \tag{2}$$

is an axis-aligned rectangular parallelepiped; in this case we say that the segment $\mathcal{I}_n(m)$ is *closed* (the intuition behind this term will be clear from the next section). The *edges* of a closed initial segment are its orthogonal projections onto the coordinate

axes; thus, for instance, the two-element initial segment has one edge of size 2, all other edges being of size 1.

In general, for any integer $m \geq 1$ there are uniquely defined integers $K \geq 1$ and $i \in [0, n - 1]$ such that

$$K^n \leq m < (K + 1)^n \tag{3}$$

and, indeed,

$$(K + 1)^i K^{n-i} \leq m < (K + 1)^{i+1} K^{n-i-1}; \tag{4}$$

writing then

$$m = (K + 1)^i K^{n-i} + R, \quad 0 \leq R < (K + 1)^i K^{n-1-i}, \tag{5}$$

the initial segment $\mathcal{I}_n(m)$ is the disjoint union of the closed $\mathcal{I}_n((K + 1)^i K^{n-i})$, which is the parallelepiped in the right-hand side of (2), and a translate of the $(n - 1)$ -dimensional initial segment $\mathcal{I}_{n-1}(R)$, contained in the hyperplane $x_{i+1} = K$.

We can now state our first principal result.

Theorem 2.1 *For every integer $n \geq 1$ and every finite set $A \subset \mathbb{R}^n$, letting $m := |A|$, we have*

$$\sum_{i=1}^n |\pi_i(A)| \geq \sum_{i=1}^n |\pi_i(\mathcal{I}_n(m))|. \tag{6}$$

In Sect. 5 we prove Theorem 2.1 in the particular case where $A \subset \mathbb{N}_0^n$; the general case then follows readily by observing that if $A \subseteq S^n$ with a finite set $S \subseteq \mathbb{R}$, then for any injective mapping $\varphi: S \rightarrow \mathbb{N}_0$, writing $B := \varphi^{\otimes n}(A) \subset \mathbb{N}_0^n$ we have $|B| = |A|$ and $|\pi_i(A)| = |\pi_i(B)|$ for each $i \in [1, n]$. Indeed, this observation shows that the estimate of Theorem 2.1 remains valid when $A \subseteq S^n$ with a set S of any nature, not necessarily contained in the set of real numbers (although in this case the projections π_i must be redefined appropriately).

We denote the left-hand side of (6) by $\sigma_n(A)$ and, with some abuse of notation, its right-hand side by $\sigma_n(m)$; that is, $\sigma_n(A)$ is the sum of the sizes of the projections of the finite set $A \subset \mathbb{R}^n$ onto the coordinate hyperplanes, and $\sigma_n(m)$ is this sum in the special case where the set in question is the length- m initial segment. Thus, for instance,

$$\sigma_n(0) = 0, \quad \sigma_1(m) = 1 \text{ if } m > 0, \tag{7}$$

and, as one can easily verify,

$$\sigma_2(m) = \begin{cases} 2K + 1 & \text{if } K^2 < m \leq K(K + 1), \\ 2K + 2 & \text{if } K(K + 1) < m \leq (K + 1)^2 \end{cases} \tag{8}$$

for any integer $K \geq 0$. Also, it follows from the explanation above that for K, i , and R defined by (3)–(5), we have

$$\begin{aligned} \sigma_n(m) &= \sigma_n((K + 1)^i K^{n-i}) + \sigma_{n-1}(R) \\ &= (nK + n - i)(K + 1)^{i-1} K^{n-i-1} + \sigma_{n-1}(R); \end{aligned} \tag{9}$$

along with (7), this relation gives a recursive, completely algebraic definition of the quantities $\sigma_n(m)$.

Addressing the corresponding stability problem, we say that a finite set $A \subset \mathbb{R}^n$ is a *minimiser* if its projection sum $\sigma_n(A)$ is smallest possible among all sets in \mathbb{R}^n of size $|A|$. Thus, Theorem 2.1 says that every initial segment of \mathbb{N}_0^n is a minimiser, but it is not true in general that any minimiser is an initial segment, or even is “similar” to an initial segment in some reasonable sense (see Sect. 6 for a rigorous definition of similarity). Say, for integer $K \geq C \geq 1$, the set $A := [0, K - C] \times [0, K + C] \subset \mathbb{N}_0^2$ is not an initial segment, while $|A| = (K + 1)^2 - C^2$ and therefore $\sigma_2(A) = 2K + 2 = \sigma_2(|A|)$ (cf. (8)), showing in view of Theorem 2.1 that A is a minimiser. In Sect. 6 we prove, however, that every *closed* initial segment is (up to similarity) a unique minimiser.

Theorem 2.2 *Suppose that $n \geq 1$ is an integer. If $m = (K + 1)^i K^{n-i}$ with integers $K \geq 1$ and $i \in [0, n - 1]$, then every minimiser in \mathbb{R}^n of size m is a Cartesian product of i real sets of size $K + 1$, and $n - i$ real sets of size K .*

The following lemma, proved in Sect. 4, is an important ingredient of the proof of Theorem 2.1.

Lemma 2.3 *Let $n \geq 1$ be an integer.*

- (i) *Suppose that $I_1, I_2, J_1, J_2 \subset \mathbb{N}_0^n$ are initial segments such that $|I_1| + |I_2| = |J_1| + |J_2|$, $|J_1| \leq |I_1| \leq |I_2| \leq |J_2|$, and J_2 is closed. Then*

$$\sigma_n(J_1) + \sigma_n(J_2) \leq \sigma_n(I_1) + \sigma_n(I_2).$$

- (ii) *If $I, I_1, I_2 \subset \mathbb{N}_0^n$ are non-empty initial segments such that $|I| = |I_1| + |I_2|$, then*

$$\sigma_n(I) < \sigma_n(I_1) + \sigma_n(I_2).$$

- (iii) *If $n \geq 2$ and $I_{n-1} \subset \mathbb{N}_0^{n-1}$, $I_n \subset \mathbb{N}_0^n$ are non-empty initial segments such that $|I_{n-1}| = |I_n|$, then*

$$\sigma_n(I_n) > \sigma_{n-1}(I_{n-1}).$$

An “algebraic restatement” of Lemma 2.3 may be of interest.

Lemma 2.3’ *Let $n \geq 1$ be an integer.*

- (i) *Suppose that $l_1 \leq m_1 \leq m_2 \leq l_2$ are non-negative integers such that $m_1 + m_2 = l_1 + l_2$. If $\mathcal{I}_n(l_2)$ is closed, then*

$$\sigma_n(l_1) + \sigma_n(l_2) \leq \sigma_n(m_1) + \sigma_n(m_2).$$

(ii) If $m_1, m_2 \geq 1$ are integers, then

$$\sigma_n(m_1 + m_2) < \sigma_n(m_1) + \sigma_n(m_2).$$

(iii) If $n \geq 2$ and $m \geq 1$ is an integer, then

$$\sigma_n(m) > \sigma_{n-1}(m).$$

We remark that subadditivity established by Lemmas 2.3(ii) and 2.3'(ii) can be viewed as a combinatorial analogue of the well-known physical fact that merging two spherical droplets into one reduces the total surface area.

Our last result establishes yet another interesting algebraic property of the functions σ_n .

Theorem 2.4 For any integers $n, s \geq 1$ and $m_1, \dots, m_s \geq 0$, we have

$$\sigma_n(m_1 + \dots + m_s) \leq \sigma_{n-1}(m_1) + \dots + \sigma_{n-1}(m_s) + \max\{m_1, \dots, m_s\}. \quad (10)$$

In Sect. 7 we derive Theorem 2.4 from Theorem 2.1 and, indeed, show that, somewhat unexpectedly, the two theorems are equivalent in the sense that each of them follows easily from the other one.

In the next section we introduce important notation and terminology used throughout. Having finished with this, we prove Lemma 2.3 in Sect. 4, and Theorems 2.1 and 2.2 in Sects. 5 and 6, respectively. The equivalence of the former theorem and Theorem 2.4 is established in Sect. 7. Finally, in the Appendix we develop a similar theory for the one-dimensional projections, and in particular show that their sum is also minimised when the set under consideration is an initial segment.

The proofs are purely combinatorial, based on point rearrangements.

3 Notation and Terminology

For integer $1 \leq i \leq n$, we denote by $\{e_1, \dots, e_n\}$ the standard basis of \mathbb{R}^n , and write

$$\mathcal{X}_i := \text{Sp}\{e_i\}$$

for the coordinate axes, and

$$\mathcal{L}_i := \text{Sp}\{e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n\}$$

for the coordinate hyperplanes. The axis \mathcal{X}_n will be referred to as *vertical*, and the corresponding hyperplane \mathcal{L}_n and its translates, as well as the projection π_n , as *horizontal*.

The reader may find helpful to think of points $x \in \mathbb{Z}^n$ as unit cubes, with the base vertex at x , and visualise sets $A \subseteq \mathbb{Z}^n$ as built of such cubes.

The intersections of a set $A \subseteq \mathbb{Z}^n$ with horizontal hyperplanes will be called the *slabs* of A . If $n = 1$, then the slabs are zero-dimensional; hence, either empty, or singletons.

Recall that an initial segment of \mathbb{N}_0^n is *closed* if it is an axis-aligned rectangular parallelepiped whose edges differ by at most 1, and for all $1 \leq i < j \leq n$, the edge along \mathcal{X}_j is not longer than the edge along \mathcal{X}_i (cf. (2)). Notice that closed initial segments in \mathbb{N}_0^n are stable under permuting non-empty slabs.

Given an initial segment $I \subset \mathbb{N}_0^n$, we define its (strict) *interior* to be the longest closed initial segment (strictly) contained in I , and we denote the interior and the strict interior of I by $\text{int}(I)$ and $\text{int}^*(I)$, respectively. Similarly we define the (strict) *closure* of I to be the shortest closed initial segment (strictly) containing I , and denote the closure and the strict closure of I by $\text{cl}(I)$ and $\text{cl}^*(I)$, respectively. The boundary of I is defined by $\partial I := I \setminus \text{int}(I)$, and its strict boundary by $\partial^* I := I \setminus \text{int}^*(I)$; thus, the boundary is empty if and only if I is closed, while the strict boundary is always non-empty whenever $I \neq \emptyset$. Boundaries can be treated either as $(n - 1)$ -dimensional sets embedded in \mathbb{N}_0^n , or as initial segments in \mathbb{N}_0^{n-1} . We remark that for an initial segment $I \neq \emptyset$, any of the three conditions $\text{cl}^*(I) = \text{cl}(I)$, $\text{int}^*(I) = \text{int}(I)$, and $\partial^* I = \partial I$ is equivalent to I not being closed.

As a version of (9), for any initial segment $I \subset \mathbb{N}_0^n$ with $|I| > 1$, we have

$$\sigma_n(I) = \sigma_n(\text{int}^*(I)) + \sigma_{n-1}(\partial^* I). \tag{11}$$

(If $|I| = 1$, then $\text{int}^*(I) = \emptyset$ and the left-hand side of (11) exceeds by 1 its right-hand side.) This basic, but important identity allows us to argue inductively in the forthcoming proofs. It becomes evident upon observing that the strict boundary $\partial^* I$ is an $(n - 1)$ -dimensional set, attached to and not larger than a face of $\text{int}^*(I)$ which we visualise as a rectangular parallelepiped; hence $\partial^* I$ does not contribute to the projection along the axis, normal to the face under consideration. Notation-wise, we will occasionally use (11) in the form

$$\sigma_n(|I|) = \sigma_n(|\text{int}^*(I)|) + \sigma_{n-1}(|\partial^* I|),$$

the equivalence following from the fact that $\text{int}^*(I)$ and $\partial^* I$ are initial segments (in \mathbb{N}_0^n and \mathbb{N}_0^{n-1} , respectively).

4 Proof of Lemma 2.3

We use induction on n . The base case $n = 1$ is easy to verify, and we proceed assuming that $n \geq 2$. We first prove (iii), then (i), and, finally, (ii).

Addressing (iii), we use (the inner) induction on the common size m of the initial segments I_{n-1} and I_n . For $m = 1$ the estimate follows from

$$\sigma_n(I_n) = n = \sigma_{n-1}(I_{n-1}) + 1.$$

For $m \geq 2$, by (11) and the induction hypothesis, we have

$$\sigma_n(I_n) = \sigma_n(|\text{int}^*(I_n)|) + \sigma_{n-1}(|\partial^* I_n|) > \sigma_{n-1}(|\text{int}^*(I_n)|) + \sigma_{n-1}(|\partial^* I_n|).$$

Applying (ii) inductively, we see that the right-hand side is larger than

$$\sigma_{n-1}(|\text{int}^*(I_n)| + |\partial^* I_n|) = \sigma_{n-1}(|I_n|) = \sigma_{n-1}(|I_{n-1}|) = \sigma_{n-1}(I_{n-1}).$$

This completes the proof of (iii), and we now turn to (i).

If $J_1 = \emptyset$, then the assertion follows from (ii), and we thus assume that $J_1 \neq \emptyset$, implying $I_1 \neq \emptyset$. Our plan is to shorten I_1 and lengthen I_2 , while keeping the sum $|I_1| + |I_2|$ intact, to get, after a number of iterations, to the situation where $I_2 = J_2$. Formally, we act as follows. Let

$$\delta := \min \{|\partial^* I_1|, |\text{cl}^*(I_2) \setminus I_2|\} \tag{12}$$

(the number of elements to be transferred from I_1 to I_2), and define I'_1 and I'_2 to be the initial segments of \mathbb{N}_0^n of sizes $|I'_1| = |I_1| - \delta$ and $|I'_2| = |I_2| + \delta$. Notice that $\delta > 0$, and since J_2 is closed, if $I_2 \neq J_2$, then we have $\text{cl}^*(I_2) \subseteq J_2$, implying $\delta \leq |J_2 \setminus I_2|$. Consequently, $|I'_2| = |I_2| + \delta \leq |J_2|$, whence $|I'_1| = |J_1| + |J_2| - |I'_2| \geq |J_1|$.

We now prove that

$$\sigma_n(I'_1) + \sigma_n(I'_2) \leq \sigma_n(I_1) + \sigma_n(I_2). \tag{13}$$

If $I'_2 \neq J_2$ (equivalently, if $I'_1 \neq J_1$), then we iterate the procedure, until eventually we replace the initial segments I_1 and I_2 with J_1 and J_2 , respectively, and the assertion will then follow from (13). Thus, to complete the proof of (i) it remains to establish (13). To this end, we have to distinguish two cases.

Suppose first that $|\partial^* I_1| > |\text{cl}^*(I_2) \setminus I_2|$, so that I_2 is not closed in view of $|I_1| \leq |I_2|$; consequently, $|\partial^* I_2| = |\partial^* I_2| > 0$ and $\text{int}(I_2) = \text{int}^*(I_2)$. In this case we have $\delta = |\text{cl}^*(I_2) \setminus I_2|$ whence $|I'_2| = |\text{cl}^*(I_2)|$ and therefore $I'_2 = \text{cl}^*(I_2)$ and $\text{int}^*(I'_2) = \text{int}^*(I_2)$; also, $|I'_1| > |I_1| - |\partial^* I_1| = |\text{int}^*(I_1)|$, implying $\text{int}^*(I'_1) = \text{int}^*(I_1)$. As a result, using (11), we get

$$\begin{aligned} \sigma_n(I_1) + \sigma_n(I_2) &= \sigma_n(\text{int}^*(I_1)) + \sigma_{n-1}(\partial^* I_1) + \sigma_n(\text{int}^*(I_2)) + \sigma_{n-1}(\partial^* I_2) \\ &= \sigma_n(\text{int}^*(I'_1)) + \sigma_n(\text{int}^*(I'_2)) + (\sigma_{n-1}(\partial^* I_1) + \sigma_{n-1}(\partial^* I_2)). \end{aligned} \tag{14}$$

We now notice that $\partial^* I'_2$ is a closed initial segment in \mathbb{N}_0^{n-1} , and that

$$|\partial^* I'_1| + |\partial^* I'_2| = (|\partial^* I_1| - \delta) + (|\partial^* I_2| + \delta) = |\partial^* I_1| + |\partial^* I_2|.$$

Also,

$$|\partial^* I'_2| = |I'_2| - |\text{int}^*(I'_2)| = |\text{cl}^*(I_2)| - |\text{int}^*(I_2)| > |I_2| - |\text{int}^*(I_2)| = |\partial^* I_2|.$$

Therefore, an inductive application of (i) in dimension $n - 1$ yields

$$\sigma_{n-1}(\partial^* I_1) + \sigma_{n-1}(\partial^* I_2) \geq \sigma_{n-1}(\partial^* I'_1) + \sigma_{n-1}(\partial^* I'_2).$$

Combining this with (14), and using (11) once again, we obtain

$$\begin{aligned} \sigma_n(I_1) + \sigma_n(I_2) &\geq \sigma_n(\text{int}^*(I'_1)) + \sigma_n(\text{int}^*(I'_2)) + \sigma_{n-1}(\partial^* I'_1) + \sigma_{n-1}(\partial^* I'_2) \\ &= \sigma_n(I'_1) + \sigma_n(I'_2), \end{aligned}$$

which is the desired estimate (13).

Now suppose that $|\partial^* I_1| \leq |\text{cl}^*(I_2) \setminus I_2|$. In this case $\delta = |\partial^* I_1|$, $I'_1 = \text{int}^*(I_1)$, $\text{int}^*(I'_2) = \text{int}(I_2)$, and

$$|\partial^* I'_2| = |I'_2 \setminus \text{int}(I_2)| = |\partial I_2| + \delta = |\partial^* I_1| + |\partial I_2|. \tag{15}$$

There are two further sub-cases.

If I_2 is closed, then (15) gives $|\partial^* I'_2| = |\partial^* I_1|$; as a result, using (11) we get

$$\begin{aligned} \sigma_n(I_1) + \sigma_n(I_2) &\geq (\sigma_n(\text{int}^*(I_1)) + \sigma_{n-1}(\partial^* I_1)) + \sigma_n(I_2) \tag{16} \\ &= \sigma_n(I'_1) + (\sigma_{n-1}(\partial^* I_1) + \sigma_n(\text{int}(I_2))) \\ &= \sigma_n(I'_1) + (\sigma_{n-1}(\partial^* I'_2) + \sigma_n(\text{int}^*(I'_2))) \\ &= \sigma_n(I'_1) + \sigma_n(I'_2), \end{aligned}$$

which is 13.

Remark 4.1 The inequality in (16) is strict if and only if I_1 is a singleton; this fact will be used in the forthcoming proof of (ii).

If I_2 is *not* closed, then $\partial I_2 = \partial^* I_2$ and $\text{int}^*(I_2) = \text{int}(I_2) = \text{int}^*(I'_2)$. Recalling (15), in this case we apply (ii) inductively in dimension $n - 1$ to get

$$\sigma_{n-1}(\partial^* I_1) + \sigma_{n-1}(\partial^* I_2) > \sigma_{n-1}(\partial^* I'_2)$$

whence, by (11),

$$\begin{aligned} \sigma_n(I_1) + \sigma_n(I_2) &\geq \sigma_n(\text{int}^*(I_1)) + (\sigma_n(\text{int}^*(I_2)) + \sigma_{n-1}(\partial^* I_1) + \sigma_{n-1}(\partial^* I_2)) \\ &> \sigma_n(I'_1) + (\sigma_n(\text{int}^*(I'_2)) + \sigma_{n-1}(\partial^* I'_2)) \tag{17} \\ &= \sigma_n(I'_1) + \sigma_n(I'_2). \end{aligned}$$

This establishes (13), and therefore (i).

Finally, we prove (ii). Without loss of generality, we assume $|I_1| \leq |I_2|$. If

$$|\text{cl}^*(I_2) \setminus I_2| \leq |I_1|, \tag{18}$$

then we re-use the above argument for (i) with $J_2 := \text{cl}^*(I_2)$ and $J_1 := \mathcal{I}_n(|I_1| + |I_2| - |J_2|)$, defining δ by (12) and then letting $I'_1 := \mathcal{I}_n(|I_1| - \delta)$ and $I'_2 := \mathcal{I}_n(|I_2| + \delta)$,

to have the estimate (13). We notice that if $|I_1| > 1$, then $\delta \leq |\partial^* I_1| < |I_1|$, implying $|I_1''| \geq 1$; moreover, if $|I_1| = 1$, then the inequality in (13) is strict, as it follows from (17) and Remark 4.1 above, concerning (16). (This reflects the geometrically obvious fact that if I_1 consists of one single point, then moving this point to I_2 reduces the total sum of the hyperplane projections by at least 1.)

Continuing in this way, we find initial segments $I_1'' \subseteq I_2''$ satisfying $|I_1''| + |I_2''| = |I_1| + |I_2|$ and

$$\sigma_n(I_1'') + \sigma_n(I_2'') \leq \sigma_n(I_1) + \sigma_n(I_2) \tag{19}$$

such that either $I_1'' = \emptyset$, $I_2'' = I$, and (19) holds actually as a strict inequality, or $I_1'' \neq \emptyset$ and $|\text{cl}^*(I_2'') \setminus I_2''| > |I_1''|$, cf. (18). In the former case (ii) follows readily. In the latter case, recalling that $I = \mathcal{I}_n(|I_1''| + |I_2''|)$, we have $\text{int}^*(I) = \text{int}(I_2'')$, whence

$$|\partial^* I| = (|I_1''| + |I_2''|) - |\text{int}(I_2'')| = |\partial I_2''| + |I_1''|;$$

consequently, using (11), and then applying (ii) inductively,

$$\begin{aligned} \sigma_n(I) &= \sigma_n(\text{int}^*(I)) + \sigma_{n-1}(\partial^* I) \\ &= \sigma_n(\text{int}(I_2'')) + \sigma_{n-1}(|\partial I_2''| + |I_1''|) \\ &\leq \sigma_n(\text{int}(I_2'')) + \sigma_{n-1}(\partial I_2'') + \sigma_{n-1}(I_1''). \end{aligned} \tag{20}$$

However, as a version of (11) (essentially equivalent to (9)), we have

$$\sigma_n(\text{int}(I_2'')) + \sigma_{n-1}(\partial I_2'') = \sigma_n(I_2''), \tag{21}$$

and by (iii),

$$\sigma_{n-1}(I_1'') < \sigma_n(I_1''). \tag{22}$$

Combining (20)–(22) and (19), we get

$$\sigma_n(I) < \sigma_n(I_1'') + \sigma_n(I_2'') \leq \sigma_n(I_1) + \sigma_n(I_2).$$

This completes the proof of (ii), and hence of the whole Lemma 2.3.

5 Proof of Theorem 2.1

As explained in the introduction, it suffices to show that for any finite set $A \subset \mathbb{N}_0^n$, writing $m := |A|$, we have $\sigma_n(A) \geq \sigma_n(m)$. The proof goes by induction on n , the base case $n = 1$ being trivial as the zero-dimensional projection of any non-empty set has, by convention, size 1. The assertion is readily verified for $m \in \{0, 1\}$, too. Suppose thus that $\min\{n, m\} \geq 2$.

Our strategy is to start out with any minimiser $A \subset \mathbb{N}_0^n$ and modify it, in a finite number of rearrangements not increasing the projection sum, to get the initial segment

$\mathcal{I}_n(|A|)$. We achieve this in several steps, some of which may need to be iterated, as explained below.

Step 1 Let H (for “height”) denote this number of non-empty slabs of A . Step 1 consists of three sub-procedures to be quoted later in the proof of Theorem 2.2, we therefore point them out here individually.

Step 1.1 We permute the slabs so that the number of elements of A in any higher slab does not exceed the number of elements in a lower slab; that is, letting $A^{(k)} := A \cap (ke_n + \mathcal{L}_n)$, we have $|A^{(0)}| \geq \dots \geq |A^{(H-1)}| > 0$ and $|A^{(k)}| = 0$ for $k \notin [0, H-1]$. This does not affect the projection sum $\sigma_n(A)$ as each non-horizontal projection of A is a disjoint union of the corresponding projections of the slabs:

$$\pi_i(A) = \bigcup_{k \geq 0} \pi_i(A^{(k)}), \quad i \in [1, n-1]. \tag{23}$$

To simplify the notation, we keep denoting by A the resulting set. Clearly, the number of non-empty slabs of A remains equal to H .

Step 1.2 We now replace each slab $A^{(k)}$ with the initial segment $\mathcal{I}_{n-1}(|A^{(k)}|)$, without enlarging the projection sum. (It is readily seen that the horizontal projection does not increase, and the sum of the side projections does not increase in view of (23) and by the induction hypothesis.) We use the same notation A for the new set obtained in this way, but from now on we assume that each slab of A is an $(n-1)$ -dimensional initial segment. Since the sequence $(|A^{(k)}|)_{k \geq 0}$ is non-increasing, this implies $A^{(k+1)} \subseteq e_n + A^{(k)}$ for each $k \geq 0$.

Step 1.3 Let K denote the largest edge size of the closed $(n-1)$ -dimensional initial segment $\text{cl}(A^{(0)})$; that is, the size of the projection of $A^{(0)}$ onto the coordinate axis \mathcal{X}_1 . If $K < H$, then we swap the coordinate axes \mathcal{X}_1 and \mathcal{X}_n so that the number of non-empty slabs decreases to K , and repeat the whole procedure from the beginning of Step 1.

We keep permuting the slabs and swapping the axes until A gets rearranged so that the number H of non-empty slabs does not exceed the largest edge size K of the closure $\text{cl}(A^{(0)})$ of the lowest slab, and each slab is an $(n-1)$ -dimensional initial segment.

Step 2 A repeated application of this step will ensure that all non-empty slabs of A , with the possible exception of the highest slab $A^{(H-1)}$, have the same interior. Assuming this is not the case, there are integers $k \in [1, H-2]$ with $|A^{(k)}| < |\text{int}(A^{(0)})|$. Let k be the smallest such integer. If, indeed, we had $|A^{(k)}| + |A^{(H-1)}| \leq |\text{int}(A^{(0)})|$, then we would be able to remove $A^{(H-1)}$ from A and replace $A^{(k)}$ with the (appropriate vertical translate of the) initial segment $\mathcal{I}_{n-1}(|A^{(k)}| + |A^{(H-1)}|)$, without changing the horizontal projection $\pi_n(A)$. By Lemma 2.3 (ii), this would result in the strict decrease of the sum of the non-horizontal projections, contradicting the assumption that A is a minimiser. Thus, we have $|A^{(k)}| + |A^{(H-1)}| > |\text{int}(A^{(0)})|$, and we replace the slab $A^{(k)}$ with $\text{int}(A^{(0)})$, and the upper slab $A^{(H-1)}$ with the initial segment $\mathcal{I}_{n-1}(|A^{(H-1)}| + |A^{(k)}| - |\text{int}(A^{(0)})|)$; by Lemma 2.3 (i), applied with $I_1 = A^{(H-1)}$, $I_2 = A^{(k)}$, and $J_2 = \text{int}(A^{(0)})$, this does not increase the sum of non-horizontal projections of A , and it is clear that the horizontal projection $\pi_n(A)$ remains unchanged.

We emphasise that Step 2 affects neither the number H of the slabs of A , nor the lower slab $A^{(0)}$, and that if this step ever gets applied, then the resulting set satisfies $|A^{(0)}| > |A^{(H-1)}|$.

Iterating Step 2, we ensure that all non-empty slabs of A , excepting perhaps $A^{(H-1)}$, have their interiors identical to that of $A^{(0)}$, which we assume to hold from now on.

Step 3 Recall that by K we denote the largest edge size of the closed $(n - 1)$ -dimensional segment $\text{cl}(A^{(0)})$. As a result of the rearrangements of Step 1, we have $K \geq H$, and the present Step 3 is to be repeated as long as the strict inequality $K > H$ holds, or until A gets rearranged as in initial segment.

If A can be cast as an initial segment just by relabelling the coordinate axes, this will complete the proof; this scenario will be referred to as a *trivial exit*.

Otherwise, we are going to cut from A a “vertical slab” resting on the strict boundary $\partial^*A^{(0)}$, and place it as a new horizontal slab (as a result of which H will grow by 1). Given that only side projections of both slabs contribute to the projection sum $\sigma_n(A)$, it will not be affected by this rearrangement. Formally, we need to consider two cases.

The first case is $|A^{(H-1)}| \geq |\text{int}^*(A^{(0)})|$ (covering, in particular, the situation where $H = 1$). In this case each slab of A contains a vertical translate of the strict interior $\text{int}^*(A^{(0)})$, and we define A' to be the union of all these H translates, and let $A'' := A \setminus A'$. Notice that A'' is a non-empty subset of a hyperplane parallel to one of the (non-horizontal) coordinate hyperplanes. Considering A'' as an $(n - 1)$ -dimensional set, we have

$$\sigma_n(A) = \sigma_n(A') + \sigma_{n-1}(A''). \tag{24}$$

Observe that if we replace the set A'' with the initial segment $\mathcal{I}_{n-1}(|A''|)$, by the induction hypothesis and the assumption that A is a minimiser, this will not affect the projection sum $\sigma_{n-1}(A'')$.

If $H = K - 1$, then A' is the union of $K - 1$ vertical translates of the closed initial segment $\text{int}^*(A^{(0)})$, which is the n -dimensional axis-aligned rectangular parallelepiped with the edge sizes $K - 1$ and (possibly) K . The set A'' is strictly smaller than the face of A' it is attached to, for otherwise the original set A would be a closed initial segment, up to relabelling the coordinate axes, and we would have the trivial exit scenario. Hence A'' can be replaced with the initial segment $\mathcal{I}_{n-1}(|A''|)$ and re-attached to the appropriate face of A' to get a set which, up to a coordinate axes relabelling, is an initial segment, completing the proof (for the present subcase $H = K - 1$). Observe, for the forthcoming proof of Theorem 2.2, that the fact that A'' is strictly smaller than the face of A' it is attached to, precludes the output initial segment $\mathcal{I}_n(|A|)$ from being closed.

Assuming now that $H < K - 1$, let $K_1 \geq \dots \geq K_{n-1}$ be the edge sizes of $\text{int}^*(A^{(0)})$, so that $K \geq K_1$ and $K_{n-1} \geq K - 1$. We have then

$$\begin{aligned} |A''| &\leq |\partial^*A^{(0)}| \cdot H < K_1 \dots K_{n-2} \cdot (K - 1) \leq K_1 \dots K_{n-2} \cdot K_{n-1} \\ &= |\text{int}^*(A^{(0)})|. \end{aligned} \tag{25}$$

It follows that we can detach A'' from A , replace it with the initial segment $\mathcal{I}_{n-1}(|A''|)$, and re-attach this initial segment as the upper slab, thus increasing H by 1 and changing

the lower slab of A from the original $A^{(0)}$ to its strict interior $\text{int}^*(A^{(0)})$. Notice that, as a consequence of (25), the resulting set has fewer elements in its upper slab than in the lower slab.

For the second case $|A^{(H-1)}| < |\text{int}^*(A^{(0)})|$, we define A' to be the union of $H - 1$ (rather than H as in the first case) vertical translates of the set $\text{int}^*(A^{(0)})$, with the coordinate along the vertical axis ranging from 0 to $H - 2$, and we let $A'' := A \setminus (A' \cup A^{(H-1)})$. Thus A' is an axis-aligned rectangular parallelepiped, attached to two faces of which are the $(n - 1)$ -dimensional sets A'' and $A^{(H-1)}$; hence,

$$\sigma_n(A) = \sigma_n(A') + \sigma_{n-1}(A'') + \sigma_{n-1}(A^{(H-1)}).$$

Keeping the notation for the edges of $\text{int}^*(A^{(0)})$, similarly to (25) we have

$$\begin{aligned} |A''| &\leq |\partial^* A^{(0)}| \cdot (H - 1) \leq K_1 \dots K_{n-2} \cdot (H - 1) \\ &< K_1 \dots K_{n-2} \cdot K_{n-1} = |\text{int}^*(A^{(0)})|; \end{aligned} \tag{26}$$

Therefore, A'' can be rearranged as an $(n - 1)$ -dimensional initial segment, and then detached from A and re-attached as either the upper, or the second-from-the-top slab, retaining the non-increasing order of slab sizes. As above, this rearrangement makes H larger by 1, and changes the lower slab of A from the original $A^{(0)}$ to its strict interior $\text{int}^*(A^{(0)})$. Furthermore, the size of the horizontal projection of A becomes $|\text{int}^*(A^{(0)})|$.

Finally (just for the second-from-the-top slab) we invoke the rearrangement of Step 2 to ensure that all, but the upper slab of A have the same interior; hence, are actually identical closed initial segments since $A^{(0)}$ is now a closed initial segment.

Observe that, unless we have achieved our goal of rearranging A as an initial segment (as in the trivial exit scenario or the case where $|A^{(H-1)}| \geq |\text{int}^*(A^{(0)})|$ and $H = K - 1$), the procedure introduced in Step 3 results in H growing by 1, with the strict inequality $|A^{(H-1)}| < |A^{(0)}|$ for the rearranged A , and with $A^{(0)}$ being a closed $(n - 1)$ -dimensional initial segment. In addition, the parameter K is either unchanged, or decreases by 1, the latter happening if and only if the new “base slab” $A^{(0)}$ is an $(n - 1)$ -dimensional cube. Therefore, if the new parameters satisfy $K < H$ (that is, $H = K + 1$), then the set A got rearranged into a cube with the edge size K , with an $(n - 1)$ -dimensional initial segment attached to its upper face as boundary; that is, into a (non-closed) n -dimensional initial segment.

Let us put down the latter observation as a remark to be used in the proof of Theorem 2.2 in the sequel.

Remark 5.1 When Step 3 is applied with a non-trivial exit, the output set has strictly fewer elements in its upper slab than in the lower one; in particular, this set cannot be converted into a closed initial segment by relabelling the axes.

We thus see that, applying Step 3 repeatedly, we will either rearrange A as in initial segment, or arrive at the special situation where $K = H$, dealt with at Step 4 below.

Step 4 For this last step of our algorithm we assume that $K = H$ where, we recall, H is the number of non-empty slabs of A , and K is the largest size of a projection of

the lower slab $A^{(0)}$ onto a non-vertical coordinate axis. This step is not to be iterated; it is applied at most once and after completing it, A will be rearranged as an initial segment.

If A can be rearranged as an initial segment by merely relabelling the coordinate axes, then the algorithm stops and proof is completed; as in Step 3, this situation will be referred to as the *trivial exit*. Assuming that we cannot exit trivially, we consider two cases, as in the above Step 3.

The first one is a straightforward modification of the corresponding case of Step 3, with $\text{int}^*(A^{(0)})$ replaced by $\text{int}(A^{(0)})$ and the equality $H = K - 1$ replaced by $H = K$. Namely, if $|A^{(H-1)}| \geq |\text{int}(A^{(0)})|$, then we define A' to be the union of $H = K$ vertical translates of $\text{int}(A^{(0)})$, one on top of the other, and let $A'' = A \setminus A'$. The set A' is an n -dimensional axis-aligned rectangular parallelepiped with the maximum edge size K and minimum edge size at least $K - 1$, and A'' is attached to a maximal-sized face of A' . Hence, having A'' replaced by the $(n - 1)$ -dimensional initial segment $\mathcal{I}_{n-1}(|A''|)$, the set A can be cast as the initial segment by relabelling the coordinate axes.

Preparing the ground for the proof of Theorem 2.2 in the next section, we notice that A'' is non-empty, and is strictly smaller than the face of A' it is attached to, for otherwise A can be rearranged as an initial segment by relabelling the coordinate axes, which would lead to the trivial exit scenario. It follows that the output initial segment $\mathcal{I}_n(|A|)$ is not closed.

Moving on to the second case, for the rest of Step 4 we assume that $|A^{(H-1)}| < |\text{int}(A^{(0)})|$. The set $A \setminus A^{(H-1)}$ consists of $H - 1 = K - 1$ slabs, each one being an $(n - 1)$ -dimensional initial segment with the same interior $\text{int}(A^{(0)})$. All projections onto the non-vertical axes of the closed $(n - 1)$ -dimensional initial segment $\text{int}(A^{(0)})$ have size K or $K - 1$; therefore, the stack of $K - 1$ vertical translates of $\text{int}^*(A^{(0)})$, which we denote as A' , is a closed n -dimensional initial segment. Furthermore, the set $A'' := A \setminus (A' \cup A^{(H-1)})$ is non-empty, for otherwise A could be rearranged as the initial segment by relabelling the coordinate axes, which is ruled out by the assumption that we have not exited Step 4 trivially.

Thus, attached to the upper horizontal face of A' is the slab $A^{(H-1)}$, and to some of its “vertical” faces—the “vertical slab” A'' (which, by the construction, is strictly smaller than the face of A' it is attached to). Our plan is to replace A'' by the same-sized $(n - 1)$ -dimensional initial segment, and then apply Lemma 2.3 to transfer elements from $A^{(H-1)}$ and A'' to A' , to augment this latter set to its strict closure $\text{cl}^*(A')$.

Since the horizontal projections of $A^{(H-1)}$ and A'' are disjoint, we have

$$\sigma_n(A) = \sigma_n(A') + \sigma_{n-1}(A'') + \sigma_{n-1}(A^{(H-1)}).$$

By the induction hypothesis, replacing A'' with the initial segment $\mathcal{I}_{n-1}(|A''|)$ does not increase the summand $\sigma_{n-1}(A'')$. As usual, we do not change the notation, but assume below that A'' is an $(n - 1)$ -dimensional initial segment.

Let I_1 and I_2 be the smallest and the largest, respectively among the initial segments A'' and $A^{(H-1)}$, and let $J_2 := \text{cl}^*(A') \setminus A'$; that is, J_2 is the face to be added to the closed initial segment A' in order to obtain the “next” closed initial segment $\text{cl}^*(A')$. Notice that, by the virtue of the cube order, $|J_2|$ is the size of the *largest* face of A' , whence $|J_2| > |I_2|$. Also notice that I_1 and I_2 are non-empty as so are A'' and

$A^{(H-1)}$. If we had $|J_2| \geq |I_1| + |I_2|$ then, applying Lemma 2.3 (ii), we could have replaced A'' and $A^{(H-1)}$ with one single $(n - 1)$ -dimensional initial segment of size $|A''| + |A^{(H-1)}|$ attached to the largest face of A , decreasing the sum of the projections of A ; this would contradict the assumptions that A is a minimiser. Therefore we have $|I_1| \leq |I_2| \leq |J_2| < |I_1| + |I_2|$, we set $J_1 := \mathcal{I}_{n-1}(|I_1| + |I_2| - |J_2|)$ and replace A'' and $A^{(H-1)}$ with the initial segments J_2 attached to the appropriate face of A' to convert it to $\text{cl}^*(A')$, and J_1 attached as a boundary to $\text{cl}^*(A')$ (which can be done in view of $|J_1| \leq |J_2|$). This rearranges A as an initial segment. Observe that for this last scenario, the output set $\mathcal{I}_n(|A|)$ cannot be closed in view of the estimate $0 < |J_1| < |J_2|$ resulting from

$$|J_1| = |I_1| + |I_2| - |J_2| \leq 2|I_2| - |J_2| < |J_2|.$$

In conclusion, we put down a remark adding to the claim or Remark 5.1, to be used shortly in the proof of Theorem 2.2 that comes in the next section.

Remark 5.2 The only way that Steps 3 and 4 can yield a closed initial segment is that they are exited trivially; in particular, the input set must be an axis-aligned rectangular parallelepiped.

This ends the proof of Theorem 2.1.

6 Proof of Theorem 2.2

Applying the argument presented after the statement of Theorem 2.1 in Sect. 2, we assume without loss of generality that $A \subset \mathbb{N}_0^n$.

We define *similarities* to be bijective transformations of the set \mathbb{Z}^n involving (finitely many) permutations of the horizontal hyperplanes and axes relabellings. Thus, two sets in \mathbb{Z}^n are *similar* if they can be obtained from each other by a finite series of permutations of the slabs and relabellings of the coordinate axes. For $n = 1$, any two sets of the same size are similar, proving the assertion in this case. For $n \geq 1$, if $A_1, A_2 \subset \mathbb{Z}^n$ are similar, finite sets, then $\sigma_n(A_1) = \sigma_n(A_2)$.

Our argument uses induction by n and is based on a careful examination of the proof of Theorem 2.1 in the previous section; in fact, it has been prepared by the observations made there, and particularly at the key Steps 3 and 4. Clearly, it suffices to show that if $A \subset \mathbb{N}_0^n$ is a minimiser such that $\mathcal{I}_n(|A|)$ is closed, then all rearrangements made in the course of the proof are, in fact, similarities; that is, involve only permuting the slabs and relabelling the coordinate axes.

Suppose thus that $A \subseteq \mathbb{N}_0^n$ is a minimiser with $\mathcal{I}_n(|A|)$ closed. Going back to the proof of Theorem 2.1 in Sect. 5, we recall Remarks 5.1 and 5.2, which state that once a Step 3 or 4 gets invoked *non-trivially* in the rearrangement procedure constituting the proof of Theorem 2.1, the output of the procedure cannot be a closed segment.

It follows that after completing Steps 1 and 2, the set A may have only required an axes relabelling to get transformed into a closed initial segment. In fact, no application of Step 2 would have been possible either, for any such application results in a set with its upper slab strictly smaller than the lower one.

Thus, rearranging A to $\mathcal{I}_n(|A|)$ has only required Step 1 followed, possibly, by an axes relabelling. We recall that Step 1 consists of a number of iterations of the procedure that involves Steps 1.1–1.3, consisting in, respectively, permuting slabs, replacing each slab with the equal-sized $(n - 1)$ -dimensional initial segment, and swapping the coordinate axes.

Consider the last iteration of Step 1 (or without loss of generality assume that Step 1 is applied only once). Specifically, consider its part Step 1.2, which consists in replacing each slab with the same-sized $(n - 1)$ -dimensional initial segment. Let A' and A'' denote the corresponding input and output sets, respectively. Thus, A'' is an axis-aligned rectangular parallelepiped, with all of its edges differing in size by 1 at most. It follows that the common size of all slabs of A'' is the cardinality of a closed $(n - 1)$ -dimensional initial segment, and we invoke the induction hypothesis to conclude that on the last iteration, replacing each slab of A' with an $(n - 1)$ -dimensional initial segment is induced by an $(n - 1)$ -dimensional similarity transformation.

On the other hand, we note that the horizontal projection of A' is the union of its slabs, viewed as $(n - 1)$ -dimensional sets (which have the same common size, as so do the slabs of A''). Since we are working with minimisers, this implies that the size of this union is equal to the size of each individual slab of A' , and therefore all the slabs of A' are actually identical. As a result, the same similarity transformation that converts, say, the lower slab of A' into a closed $(n - 1)$ -dimensional initial segment, will also work for all other slabs of A' converting them into (identical) closed initial segments. Extending this transformation to act as an identity on the last coordinate, we obtain an n -dimensional similarity transformation that replaces all slabs of A' with the $(n - 1)$ -dimensional initial segments.

We conclude that rearranging A' into A'' , and hence the whole last iteration of Step 1, can be achieved using a similarity transformation. Making our way backwards, the same is true for all the preceding iterations. Consequently, the whole Step 1 acted on A as a similarity, and the assertion follows.

This completes the proof of Theorem 2.2.

7 Proof of Equivalence of Theorems 2.1 and 2.4

Given integers $n, s \geq 1$ and $m_1, \dots, m_s \geq 0$, consider a set $A \subset \mathbb{N}_0^n$ with s non-empty slabs which are the $(n - 1)$ -dimensional initial segments of lengths m_1, \dots, m_s . Since the side projections of all these slabs are pairwise disjoint, while the horizontal projections are all contained in the largest of them, we have

$$\sigma_n(A) = \sigma_{n-1}(m_1) + \dots + \sigma_{n-1}(m_s) + \max \{m_1, \dots, m_s\}.$$

However, the left-hand side is at least as large as $\sigma_n(|A|) = \sigma_n(m_1 + \dots + m_s)$ by Theorem 2.1. Hence Theorem 2.1 implies Theorem 2.4.

Conversely, assuming Theorem 2.4, one can prove Theorem 2.1 by induction on n , as follows. Given a finite set $A \subset \mathbb{N}_0^n$, consider the slab decomposition $A = A^{(1)} \cup \dots \cup A^{(s)}$, with the $A^{(k)}$ listing all non-empty slabs of A . For each $k \in [1, s]$, let $m_k := |A^{(k)}|$. Trivially, we have $|\pi_n(A)| \geq \max \{m_1, \dots, m_s\}$. Also, disjointness of the side projections yields

$$\sum_{i=1}^{n-1} |\pi_i(A)| = \sum_{k=1}^s \sum_{i=1}^{n-1} |\pi_i(A^{(k)})|.$$

By the induction hypothesis, the double sum in the right-hand side is at least as large as $\sigma_{n-1}(m_1) + \dots + \sigma_{n-1}(m_s)$. Therefore, by Theorem 2.4,

$$\begin{aligned} \sum_{i=1}^n |\pi_i(A)| &\geq \sigma_{n-1}(m_1) + \dots + \sigma_{n-1}(m_s) + \max \{m_1, \dots, m_s\} \\ &\geq \sigma_n(m_1 + \dots + m_s) = \sigma_n(|A|), \end{aligned}$$

as claimed by Theorem 2.1.

Appendix: One-Dimensional Projections

It would be interesting to extend our results onto k -dimensional projections for all integers $k \in [1, n - 2]$. Below we consider the case $k = 1$, establishing the analogs of Theorems 2.1, 2.2, and 2.4 for the one-dimensional projections. In particular, we show that the sum of the sizes of these projections is also minimised on the initial segments of \mathbb{N}_0^n with respect to the cube order, and prove the corresponding stability result.

For each $i \in [1, n]$, denote by ρ_i the orthogonal projections of \mathbb{R}^n onto the coordinate axis \mathcal{X}_i , and given a finite set $A \subset \mathbb{R}^n$, let

$$\lambda_n(A) := \sum_{i=1}^n |\rho_i(A)|;$$

also, for integer $m \geq 0$ let $\lambda_n(m) := \lambda_n(\mathcal{I}_n(m))$. Thus, for instance, $\lambda_1(m) = m$, $\lambda_2(m) = \sigma_2(m)$, and if $|A| = K^n$ with an integer $K \geq 1$, then by the arithmetic-geometric mean inequality,

$$\lambda_n(A) \geq n \left(\prod_{i=1}^n |\rho_i(A)| \right)^{1/n} \geq n|A|^{1/n} = nK,$$

with equality attained for the discrete cube $A = [0, K - 1]^n$. The key to understanding the quantity λ_n is the equality

$$\lambda_n(m + 1) = \begin{cases} \lambda_n(m) + 1 & \text{if } \mathcal{I}_n(m) \text{ is closed,} \\ \lambda_n(m) & \text{otherwise.} \end{cases} \tag{27}$$

An immediate corollary is that if $K^n \leq m \leq (K + 1)^n$, with a positive integer K , and $i \in [0, n]$ is the smallest integer such that $m \leq (K + 1)^i K^{n-i}$, then (cf. (9))

$$\lambda_n(m) = \lambda_n((K + 1)^i K^{n-i}) = nK + i.$$

As an analog of Theorems 2.1 and 2.2, we now have

Theorem 8.1 *For every integer $n \geq 1$ and every finite set $A \subset \mathbb{R}^n$, letting $m := |A|$, we have $\lambda_n(A) \geq \lambda_n(m)$. Moreover, if $m = (K + 1)^i K^{n-i}$ with integers $K \geq 1$ and $i \in [0, n]$, then equality is attained if and only if A is the Cartesian product of i real sets of size $K + 1$, and $n - i$ real sets of size K .*

Although it is possible to prove Theorem 8.1 adapting the proofs of Theorems 2.1 and 2.2 to our present settings, somewhat surprisingly, one can get away with a much easier, non-inductive argument.

Proof of Theorem 8.1 The case where $m = 0$ is trivial, and we assume that $m > 0$; that is, A is non-empty. For each $j \in [1, n]$, let $m_j := |\rho_j(m)|$; thus, $m \leq m_1 \dots m_n$ and $\lambda_n(A) = m_1 + \dots + m_n$. If the largest of the numbers m_j exceeds the smallest of them by at least 2, then we decrease by 1 the largest, and simultaneously increase by 1 the smallest; clearly, this operation does not affect the sum of the numbers, and their product will only get larger. Iterating, we will eventually find n positive integers, the largest of them exceeding the smallest one by at most 1, so that their product is at least m , and their sum is $\lambda_n(A)$. Denoting by I the closed initial segment whose edges are determined by these resulting integers, we then have $m \leq |I|$ and $\lambda_n(A) = \lambda_n(I)$. The former relation gives $\mathcal{I}_n(m) \subseteq I$, and then the latter yields $\lambda_n(A) = \lambda_n(I) \geq \lambda_n(\mathcal{I}_n(m)) = \lambda_n(m)$. This proves the first assertion of the theorem.

For the second assertion, assume that $\lambda_n(A) = \lambda_n(m) = nK + i$. We also assume without loss of generality that $i \leq n - 1$ (if $i = n$, then we can replace K with $K + 1$). Analysing the argument above, we conclude that if the smallest among the projections $|\rho_j(A)|$ differed from the largest by at least 2, then the size of the above-defined closed initial segment I would satisfy the strict inequality $m < |I|$, implying $\lambda_n(A) = \lambda_n(I) > \lambda_n(m)$ in view of (27), a contradiction. It follows that the largest of the projections $|\rho_j(A)|$ differs from the smallest one by at most 1. Let L denote the smallest of these projections, and let $k \in [0, n - 1]$ be the number of indices $j \in [1, n]$ with $|\rho_j(A)| = L + 1$ (so that there are $n - k$ those indices $j \in [1, n]$ with $|\rho_j(A)| = L$). From $nK + i = \lambda_n(A) = (L + 1)k + L(n - k) = nL + k$ and $k, i \in [0, n - 1]$ we then conclude that $L = K$ and $k = i$. Thus, A is contained in the Cartesian product of i sets of size $K + 1$ and $n - i$ sets of size K , and it is in fact equal to this product since $|A| = (K + 1)^i K^{n-i}$. \square

In conclusion, we remark that the one-dimensional analog of Theorem 2.4 is the estimate

$$\lambda_n(sm) \leq \lambda_{n-1}(m) + s, \quad m, s \geq 0.$$

The interested reader will easily verify that this estimate follows from the first assertion of Theorem 8.1 and, in fact, is equivalent to it.

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