INTEGER SETS
WITH IDENTICAL REPRESENTATION FUNCTIONS

YONG-GAO CHEN AND VSEVOLOD F. LEV

Abstract. We present a versatile construction allowing one to obtain pairs of integer sets with infinite symmetric difference, infinite intersection, and identical representation functions.

Let $\mathbb{N}_0$ denote the set of all non-negative integers. To every subset $A \subseteq \mathbb{N}_0$ corresponds its representation function R_A defined by

$$R_A(n) := |\{(a', a'') \in A \times A \colon n = a' + a'',\ a' < a''\}|;$$

that is, $R_A(n)$ is the number of unordered representations of the integer n as a sum of two distinct elements of A.

Answering a question of Sárközy, Dombi [D02] constructed sets $A, B \subseteq \mathbb{N}_0$ with infinite symmetric difference such that $R_A = R_B$. The result of Dombi was further extended and developed in [CW03] (where a different representation function was considered) and [L04] (a simple common proof of the results from [D02] and [CW03] using generating functions); other related results can be found in [C11, CT09, Q15, T08].

The two sets constructed by Dombi actually *partition* the ground set $\mathbb{N}_0$, which makes one wonder whether one can find $A, B \subseteq \mathbb{N}_0$ with $R_A = R_B$ so that not only the symmetric difference of A and B, but also their intersection is infinite. Tang and Yu [TY12] proved that if $A \cup B = \mathbb{N}_0$ and $R_A(n) = R_B(n)$ for all sufficiently large integer n, then at least one cannot have $A \cap B = 4\mathbb{N}_0$ (here and below $k\mathbb{N}_0$ denotes the dilate of the set $\mathbb{N}_0$ by the factor k). They further conjectured that, indeed, under the same assumptions, the intersection $A \cap B$ cannot be an infinite arithmetic progression, unless $A = B = \mathbb{N}_0$. The main goal of this note is to resolve the conjecture of Tang and Yu in the negative by constructing an infinite family of pairs of sets $A, B \subseteq \mathbb{N}_0$ with $R_A = R_B$ such that $A \cup B = \mathbb{N}_0$, while $A \cap B$ is an infinite arithmetic progression properly contained in $\mathbb{N}_0$. Our method also allows one to easily construct sets $A, B \subseteq \mathbb{N}_0$ with $R_A = R_B$ such that both their symmetric difference and intersection are infinite, while their union is arbitrarily sparse and the intersection is *not* an arithmetic progression.

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For sets $A, B \subseteq \mathbb{N}_0$ and integer m let $A - B := \{a - b\colon (a,b) \in A \times B\}$ and $m + A := \{m + a\colon a \in A\}$.

The following basic lemma is in the heart of our construction.

Lemma 1. *Suppose that $A_0, B_0 \subseteq \mathbb{N}_0$ satisfy $R_{A_0} = R_{B_0}$, and that m is a non-negative integer with $m \notin (A_0 - B_0) \cup (B_0 - A_0)$. Then, letting*

$$A_1 := A_0 \cup (m + B_0) \text{ and } B_1 := B_0 \cup (m + A_0),$$

we have $R_{A_1} = R_{B_1}$ and furthermore

i) $A_1 \cup B_1 = (A_0 \cup B_0) \cup (m + A_0 \cup B_0)$;

ii) $A_1 \cap B_1 \supseteq (A_0 \cap B_0) \cup (m + A_0 \cap B_0)$, *the union being disjoint.*

Moreover, if $m \notin (A_0 - A_0) \cup (B_0 - B_0)$, then also in i) the union is disjoint, and in ii) the inclusion is in fact an equality.

In particular, if $A_0 \cup B_0 = [0, m-1]$, then $A_1 \cup B_1 = [0, 2m-1]$, and if A_0 and B_0 indeed partition *the interval $[0, m-1]$, then A_1 and B_1 partition the interval $[0, 2m-1]$.*

Proof. Since the assumption $m \notin A_0 - B_0$ ensures that A_0 is disjoint from $m + B_0$, for any integer n we have

$$R_{A_1}(n) = R_{A_0}(n) + R_{B_0}(n - 2m) + |\{(a_0, b_0) \in A_0 \times B_0\colon a_0 + b_0 = n - m\}|.$$

Similarly,

$$R_{B_1}(n) = R_{B_0}(n) + R_{A_0}(n - 2m) + |\{(a_0, b_0) \in A_0 \times B_0\colon a_0 + b_0 = n - m\}|,$$

and in view of $R_{A_0} = R_{B_0}$, this gives $R_{A_1} = R_{B_1}$. The remaining assertions are straightforward to verify. □

Given subsets $A_0, B_0 \subseteq \mathbb{N}_0$ and a sequence $(m_i)_{i \in \mathbb{N}_0}$ with $m_i \in \mathbb{N}_0$ for each $i \in \mathbb{N}_0$, define subsequently

$$A_i := A_{i-1} \cup (m_{i-1} + B_{i-1}) \text{ and } B_i := B_{i-1} \cup (m_{i-1} + A_{i-1}), \quad i = 1, 2, \ldots \tag{1}$$

and let

$$A := \cup_{i \in \mathbb{N}_0} A_i, \ B := \cup_{i \in \mathbb{N}_0} B_i. \tag{2}$$

As an immediate corollary of Lemma 1, if $R_{A_0} = R_{B_0}$ and $m_i \notin (A_i - B_i) \cup (B_i - A_i)$ for each $i \in \mathbb{N}_0$, then $R_A = R_B$.

The special case $A_0 = \{0\}$, $B_0 = \{1\}$, $m_i = 2^{i+1}$ yields the partition of Dombi (which, we remark, was originally expressed in completely different terms). Below we analyze yet another special case obtained by fixing arbitrarily an integer $l \geq 1$ and choosing $A_0 := \{0\}$, $B_0 := \{1\}$, and

$$m_i := \begin{cases} 2^{i+1}, & 0 \le i \le 2l-2, \\ 2^{2l}-1, & i = 2l-1, \\ 2^{i+1} - 2^{i-2l}, & i \ge 2l. \end{cases} \tag{3}$$

We notice that $R_{A_0} = R_{B_0}$ in a trivial way (both functions are identically equal to 0), and that A_0 and B_0 partition the interval $[0, m_0 - 1]$. Applying Lemma 1 inductively $2l-2$ times, we conclude that in fact for each $i \le 2l-2$, the sets A_i and B_i partition the interval $[0, 2m_{i-1} - 1] = [0, m_i - 1]$, and consequently $m_i \notin (A_i - B_i) \cup (B_i - A_i)$ and $m_i \notin (A_i - A_i) \cup (B_i - B_i)$. In particular, A_{2l-2} and B_{2l-2} partition $[0, m_{2l-2} - 1]$, and therefore A_{2l-1} and B_{2l-1} partition $[0, 2m_{2l-2} - 1] = [0, m_{2l-1}]$. In addition, it is easily seen that A_{2l-1} contains both 0 and m_{2l-1}, whence $m_{2l-1} \in A_{2l-1} - A_{2l-1}$, but $m_{2l-1} \notin B_{2l-1} - B_{2l-1}$ and $m_{2l-1} \notin (A_{2l-1} - B_{2l-1}) \cup (B_{2l-1} - A_{2l-1})$. From Lemma 1 i) it follows now that $A_{2l} \cup B_{2l} = [0, 2m_{2l-1}] = [0, m_{2l} - 1]$, while

$$A_{2l} \cap B_{2l} = \Big(A_{2l-1} \cap (m_{2l-1} + A_{2l-1})\Big) \cup \Big(B_{2l-1} \cap (m_{2l-1} + B_{2l-1})\Big) = \{m_{2l-1}\}.$$

Applying again Lemma 1 we then conclude that for each $i \ge 2l$,

$$A_i \cup B_i = [0, m_i - 1]$$

(implying $m_i \notin (A_i - B_i) \cup (B_i - A_i)$) and

$$A_i \cap B_i = m_{2l-1} + \{0, m_{2l}, 2m_{2l}, \ldots, (2^{i-2l} - 1)m_{2l}\}.$$

As a result, with A and B defined by (2), we have $A \cup B = \mathbb{N}_0$ while the intersection of A and B is the infinite arithmetic progression $m_{2l-1} + m_{2l}\mathbb{N}_0$. Moreover, the condition $m_i \notin (A_i - B_i) \cup (B_i - A_i)$, which we have verified above to hold for each $i \ge 0$, results in $R_A = R_B$.

We thus have proved

Theorem 1. *Let l be a positive integer, and suppose that the sets $A, B \subseteq \mathbb{N}_0$ are obtained as in* (1)–(2) *starting from $A_0 = \{0\}$ and $B_0 = \{1\}$, with (m_i) defined by* (3)*. Then $R_A = R_B$, while $A \cup B = \mathbb{N}_0$ and $A \cap B = (2^{2l} - 1) + (2^{2l+1} - 1)\mathbb{N}_0$.*

We notice that for any fixed integers $r \ge 2^{2l} - 1$ and $m \ge 2^{2l+1} - 1$, having (3) appropriately modified (namely, setting $m_i = 2^{i-2l}m$ for $i \ge 2l$) and translating A and B, one can replace the progression $(2^{2l} - 1) + (2^{2l+1} - 1)\mathbb{N}_0$ in the statement of Theorem 1 with the progression $r + m\mathbb{N}_0$; however, the relation $A \cup B = \mathbb{N}_0$ will *not* hold true any longer unless $r = 2^{2l} - 1$ and $m = 2^{2l+1} - 1$. This suggests the following question.

Problem 1. Given that $R_A = R_B$, $A \cup B = \mathbb{N}_0$, and $A \cap B = r + m\mathbb{N}_0$ with integer $r \ge 0$ and $m \ge 2$, must there exist an integer $l \ge 1$ such that $r = 2^{2l} - 1$, $m = 2^{2l+1} - 1$, and A, B are as in Theorem 1?

The finite version of this question is as follows.

Problem 2. Given that $R_A = R_B$, $A \cup B = [0, m-1]$, and $A \cap B = \{r\}$ with integer $r \geq 0$ and $m \geq 2$, must there exist an integer $l \geq 1$ such that $r = 2^{2l} - 1$, $m = 2^{2l+1} - 1$, $A = A_{2l}$, and $B = B_{2l}$, with A_{2l} nd B_{2l} as in the proof of Theorem 1?

We conclude our note with yet another natural problem.

Problem 3. Do there exist sets $A, B \subseteq \mathbb{N}_0$ with the infinite symmetric difference and with $R_A = R_B$ which *cannot* be obtained by a repeated application of Lemma 1?

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School of Mathematical Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210023, China

E-mail address: `ygchen@njnu.edu.cn`

Department of Mathematics, The University of Haifa at Oranim, Tivon 36006, Israel

E-mail address: `seva@math.haifa.ac.il`