APPROXIMATE CONVEXITY AND AN EDGE-ISOPERIMETRIC ESTIMATE

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Abstract. We study extremal properties of the function

$$F(x) := \min\{k\|x\|^{1-1/k} \colon k \ge 1\},\ x \in [0,1],$$

where $\|x\| = \min\{x, 1-x\}$. In particular, we show that F is the pointwise largest function of the class of all real-valued functions f defined on the interval $[0,1]$, and satisfying the relaxed convexity condition

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2) + |x_2 - x_1|,\ x_1, x_2, \lambda \in [0,1]$$

and the boundary condition $\max\{f(0), f(1)\} \le 0$.

As an application, we prove that if A and S are subsets of a finite abelian group G, such that S is generating and all of its elements have order at most m, then the number of edges from A to its complement $G \setminus A$ in the directed Cayley graph induced by S on G is

$$\partial_S(A) \ge \frac{1}{m}\,|G|\,F(|A|/|G|).$$

1. Summary of results

In this section we discuss our principal results; the proofs are presented in Sections 2–4.

The central character of this paper is the function

$$F(x) := \min\{k\|x\|^{1-1/k} \colon k \ge 1\},\ x \in [0,1],$$

where k runs over positive integers, and $\|x\|$ denotes the distance from x to the nearest integer; that is, $\|x\| = \min\{x, 1-x\}$ for $x \in [0,1]$. More explicitly, letting $\beta_0 = 1/2$ and $\beta_k := (1+1/k)^{-k(k+1)}$ for integer $k \ge 1$ (so that $1/4 = \beta_1 > \beta_2 > \cdots$), we have $F(x) = kx^{1-1/k}$ whenever $\beta_k \le x \le \beta_{k-1}$, and $F(1-x) = F(x)$. The graphs of the function F and the functions $kx^{1-1/k}$ for $k \in \{1,2,3\}$ are presented in Figure 1.

Recall that for $c, p > 0$, a real-valued function f defined on a convex subset of a (real) normed vector space is called (c,p)-*convex*, or 1-*paraconvex* (cf. [R06]) if it satisfies

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2) + c\|x_2 - x_1\|^p$$

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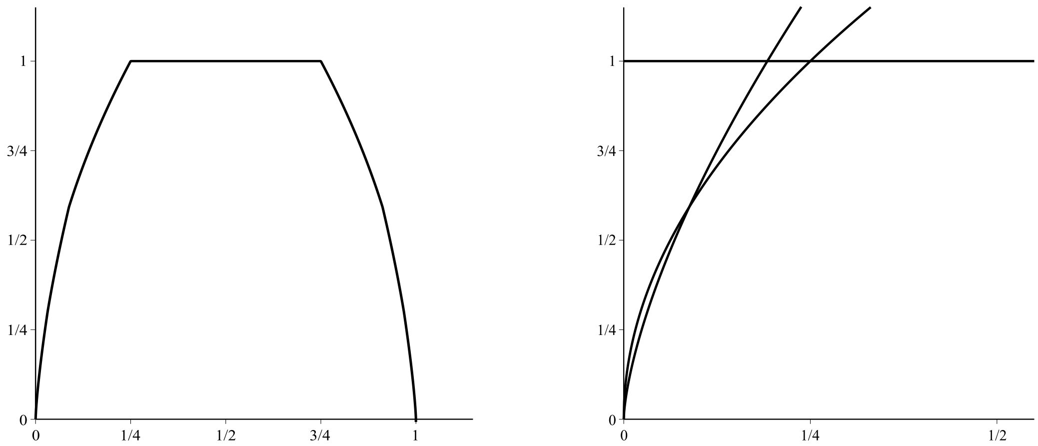

FIGURE 1. The graphs of the functions F and $kx^{1-1/k}$ for $k \in \{1,2,3\}$.

for all x_1 and x_2 in the domain of f, and all $\lambda \in [0,1]$. We denote by $\mathcal{F}$ the class of all $(1,1)$-convex functions on the interval $[0,1]$; that is, $\mathcal{F}$ consists of all real-valued functions f, defined on $[0,1]$ and satisfying

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda)f(x_2) + |x_2 - x_1|, \; x_1, x_2, \lambda \in [0,1]. \tag{1}$$

Also, let $\mathcal{F}_0$ be the subclass of all those functions $f \in \mathcal{F}$ satisfying the boundary condition

$$\max\{f(0), f(1)\} \le 0. \tag{2}$$

Since $\mathcal{F}$ is closed under translates by a linear function, any function $f \in \mathcal{F}$ can be forced into $\mathcal{F}_0$ just by adding to it an appropriate linear summand. Hence, studying these two classes is essentially equivalent.

It is immediate from the definition that the classes $\mathcal{F}$ and $\mathcal{F}_0$ are "symmetric around $x = 1/2$" in the sense that a function f belongs to the class $\mathcal{F}$ ($\mathcal{F}_0$) if and only if so does the function $x \mapsto f(1-x)$ ($x \in [0,1]$).

Our first principal result shows that the above-defined function F lies in the class $\mathcal{F}_0$ and indeed, is the (pointwise) largest function of this class.

Theorem 1. *We have $F \in \mathcal{F}_0$ and $f \le F$ for every function $f \in \mathcal{F}_0$.*

We remark that substituting $x_1 = 1$ and $x_2 = 0$ into (1) shows that all functions from the class $\mathcal{F}_0$ are uniformly bounded from above, whence the function $\sup\{f : f \in \mathcal{F}_0\}$ is well defined. It is not difficult to see that this function itself belongs to $\mathcal{F}_0$ and is pointwise bounded from above by the function F. However, proving that $F \in \mathcal{F}_0$ is more delicate.

For integer $m \geq 2$, let $\mathcal{F}_m$ denote the class of all real-valued functions, defined on the interval $[0,1]$ and satisfying the boundary condition (2) and the estimate

$$f\left(\frac{x_1+\dots+x_m}{m}\right) \leq \frac{f(x_1)+\dots+f(x_m)}{m} + (x_m - x_1), \tag{3}$$

for all $x_1,\dots,x_m \in [0,1]$ with $\min_i x_i = x_1$ and $\max_i x_i = x_m$. These classes were introduced in [L], except that the functions from the class $\mathcal{F}_2$ (up to the normalization (2)) are known as $(1,1)$-*midconvex* and under this name have been studied in a number of papers; see, for instance, [HP04, TT09b]. As shown in [L, Lemma 3], every concave function from the class $\mathcal{F}_0$ belongs to all classes $\mathcal{F}_m$. Thus, from Theorem 1 we get

Corollary 1. *We have $F \in \mathcal{F}_m$ for all $m \geq 2$.*

As an application, we establish a result from the seemingly unrelated realm of edge isoperimetry.

The edge-isoperimetric problem for a graph on the vertex set V is to find, for every non-negative integer $n \leq |V|$, the smallest possible number of edges between an n-element set of vertices and its complement in V. We refer the reader to the survey of Bezrukov [B96] and the monograph of Harper [H04] for the history, general perspective, numerous results, variations, and further references on this and related problems.

In the present paper we are concerned with the situation where the graph under consideration is a Cayley graph on a finite abelian group. We use the following notation. Given two subsets $A, S \subseteq G$ of a finite abelian group G, we consider the directed Cayley graph, induced by S on G, and we write $\partial_S(A)$ for the edge-boundary of A; that is, $\partial_S(A)$ is number of edges in this graph from an element of A to an element in its complement $G \setminus A$:

$$\partial_S(A) := |\{(a,s) \in A \times S \colon a+s \notin A\}|.$$

Equivalently, $\partial_S(A)$ is the number of edges between A and $G \setminus A$ in the *undirected* Cayley graph induced on G by the set $S \cup (-S)$.

As a consequence of Corollary 1 (thus, ultimately, of Theorem 1), we prove

Theorem 2. *Let A and S be subsets of a finite abelian group G, of which S is generating. If m is a positive integer such that the order of every element of S does not exceed m, then*

$$\partial_S(A) \geq \frac{1}{m}\,|G|F(|A|/|G|).$$

Our proof of Theorem 2 is a variation of the argument used in [L, Theorem 5] where a slightly weaker estimate is established under the stronger assumption that *all elements of* G have order at most m.

In the special particular case where G is a homocyclic group of exponent m, and $S \subseteq G$ is a standard generating subset, Theorem 2 gives a result of Bollobás and Leader [BL91, Theorem 8].

The estimate of Theorem 2, in general, fails to hold for non-abelian groups. For instance, if G is the symmetric group of order $|G| = 6$, and $S \subseteq G$ consists of two involutions, then the Cayley graph induced on G by S is a bi-directional cycle of length 6. Consequently, for a non-empty proper subset $A \subseteq G$ inducing a path in this cycle, one has

$$\partial_S(A) = 2 < \frac{1}{2}\,|G|\,F(|A|/|G|)$$

(as $F(n/6) = \sqrt{2/3}$ for $n \in \{1,5\}$ and $F(n/6) = 1$ for $n \in \{2,3,4\}$).

It would be interesting to investigate the sharpness of the estimate of Theorem 2 and to determine whether the function F in its right-hand side can be replaced with a larger function.

Back to the class $\mathcal{F}$, from Theorem 1 we derive the following result showing that, somewhat surprisingly, any function satisfying (1) must actually satisfy a stronger inequality.

Theorem 3. *For any function $f \in \mathcal{F}$, and any $x_1, x_2, \lambda \in [0,1]$, we have*

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2) + F(\lambda)|x_2 - x_1|.$$

Substituting $x_1 = 1$, $x_2 = 0$, and $f = F$ into the inequality of Theorem 3, we conclude that the factor $F(\lambda)$ in the right-hand side is optimal, and the function F cannot be replaced with a larger function.

As shown in [L, Theorem 6], for each $m \ge 2$, the functions $F_m := \sup\{f : f \in \mathcal{F}_m\}$ are well-defined and belong themselves to the classes $\mathcal{F}_m$, and [L, Theorem 5] gives a lower bound for the edge-boundary $\partial_S(A)$ in terms of these functions. In this context it is natural to investigate the infimum $\inf\{F_m : m \ge 2\}$. It is somewhat surprising that this infimum can be found explicitly, even though the individual functions F_m are known for $2 \le m \le 4$ only (see [L, Theorem 7]).

Theorem 4. *We have*

$$\inf\{F_m : m \ge 2\} = F.$$

If a function f belongs to all the classes $\mathcal{F}_m$, then $f \le \inf\{F_m : m \ge 2\} = F$ by Theorem 4 and the definition of the functions F_m; consequently, $\sup\{f : f \in \mathcal{F}_m \text{ for all } m \ge 2\} \le F$. On the other hand, $\sup\{f : f \in \mathcal{F}_m \text{ for all } m \ge 2\} \ge F$ by Corollary 1. Thus, as a direct consequence of Theorem 4 and Corollary 1, we get yet another extremal characterization of the function F, as the pointwise largest function lying in all the classes $\mathcal{F}_m$:

Corollary 2. *We have*

$$\sup\{f\colon f \in \mathcal{F}_m \text{ for all } m \geq 2\} = F.$$

As mentioned above, [L, Lemma 3] says that every concave function from the class $\mathcal{F}_0$ belongs to all the classes $\mathcal{F}_m$. For the proof of Theorem 4 we need the converse assertion, which turns out to be true even with the concavity assumption dropped.

Lemma 1. *If $f \in \mathcal{F}_m$ for all $m \geq 2$, then $f \in \mathcal{F}_0$.*

Clearly, every convex function lies in the class $\mathcal{F}$. The last result of our paper shows that all "sufficiently flat" *concave* functions also lie in $\mathcal{F}$; this complements in a sense Theorem 1 which implies that every concave function from the class $\mathcal{F}$ is "flat".

Let $\mathcal{C}$ be the class of all functions, defined and concave on the interval $[0, 1]$ and vanishing at the endpoints of this interval.

Theorem 5. *If $f \in \mathcal{C}$ and $f(x) \leq 4x(1-x)$ for all $x \in [0, 1]$, then $f \in \mathcal{F}_0$. Moreover, the function $4x(1-x)$ is best possible here in the following sense: if $h \in \mathcal{C}$ has the property that for any function $f \in \mathcal{C}$ with $f \leq h$ we have $f \in \mathcal{F}_0$, then $h(x) \leq 4x(1-x)$ for all $x \in [0, 1]$.*

We now turn to the proofs of the results discussed above. We prove Theorems 1 and 2 in Sections 2 and 3, respectively, and the proofs of Theorems 3, 4, and 5 and Lemma 1 are presented in Section 4. Concluding remarks are gathered in Section 5.

2. The proof of Theorem 1

First, we show that for any function $f \in \mathcal{F}_0$, and any real $x \in [0, 1]$ and integer $k \geq 1$, one has $f(x) \leq k\|x\|^{1-1/k}$; this will prove the second assertion of the theorem. By symmetry, $x \leq 1/2$ can be assumed without loss of generality. Applying (1) with $x_1 = \lambda^{k-1}$ and $x_2 = 0$, we get

$$f(\lambda^k) \leq \lambda f(\lambda^{k-1}) + \lambda^{k-1};$$

that is,

$$\frac{f(\lambda^k)}{\lambda^k} \leq \frac{f(\lambda^{k-1})}{\lambda^{k-1}} + \frac{1}{\lambda}.$$

Iterating, we obtain $f(\lambda^k)/\lambda^k \leq k/\lambda$ whence, substituting $\lambda = x^{1/k}$,

$$f(x) \leq k\lambda^{k-1} = kx^{1-1/k},$$

as wanted.

We now prove that F satisfies (1), and hence $F \in \mathcal{F}_0$. The proof is based on the following lemma showing that if a concave and continuous function satisfies (1) whenever $x_2 \in \{0,1\}$, then it actually satisfies (1) for all $x_1, x_2 \in [0,1]$.

Lemma 2. *Suppose that the function f is concave and continuous on the interval $[0,1]$. In order for (1) to hold for all $\lambda, x_1, x_2 \in [0,1]$, it is sufficient that it holds for all $x_1, \lambda \in [0,1]$ and $x_2 \in \{0,1\}$.*

To avoid interrupting the flow of exposition, we proceed with the proof of Theorem 1, postponing the proof of Lemma 2 to the end of this section.

As Lemma 2 shows, it suffices to establish (1) with $f = F$ and $x_2 \in \{0,1\}$. Indeed, the case $x_2 = 1$ reduces easily to that where $x_2 = 0$ using the symmetry of F around the point $1/2$. Thus, $x_2 = 0$ can be assumed, and in view of $F(0) = 0$, renaming the remaining variable, we have to prove that

$$F(\lambda x) \le \lambda F(x) + x, \ \ x, \lambda \in [0,1]. \tag{4}$$

The situation where $x \in \{0,1\}$ is immediate, and we therefore assume that $0 < x < 1$.

Addressing first the case $x \le 1/2$, we find $k \ge 1$ such that $F(x) = kx^{1-1/k}$, and notice that $F(\lambda x) \le (k+1)(\lambda x)^{1-1/(k+1)}$ by the definition of the function F; consequently, it suffices to prove that

$$(k+1)(\lambda x)^{1-1/(k+1)} \le k\lambda x^{1-1/k} + x.$$

Dividing through by x, substituting $t := \lambda^{1/(k+1)} x^{-1/(k(k+1))}$, and rearranging the terms gives this inequality the shape

$$kt^{k+1} - (k+1)t^k + 1 \ge 0,$$

and to complete the proof it remains to notice that the left-hand side factors as

$$(t-1)^2(kt^{k-1} + (k-1)t^{k-2} + \dots + 2t + 1).$$

The case $1/2 \le x < 1$ reduces to that where $0 < x \le 1/2$ as follows. Let $x' := 1 - x$, so that $0 < x' \le 1/2$. Assuming that (4) fails to hold, we get

$$F(\lambda x) > x$$

and also

$$\lambda F(x) < F(\lambda x) - x \le x', \tag{5}$$

which, in view of $F(x') = F(x)$, jointly yield

$$\frac{F(x')}{x'} < \frac{1}{\lambda} < \frac{F(\lambda x)}{\lambda x}.$$

Since $F(z)/z$ is a decreasing function of z (which is obvious for $z \in [1/2, 1]$, and follows directly from the definition of the function F for $z \in [0, 1/2]$), we conclude that $x' > \lambda x$. Thus, $\lambda x/x' < 1$, and recalling that $x' \le 1/2$, by what we have shown above,

$$F(\lambda x) = F\left(\frac{\lambda x}{x'} \cdot x'\right) \le \frac{\lambda x}{x'}\, F(x') + x'.$$

Hence, from the assumption that (4) is false,

$$\lambda F(x) + x < \frac{\lambda x}{x'}\, F(x') + x',$$

which can be written as

$$\frac{x - x'}{x'}\, \lambda F(x) > x - x',$$

contradicting (5).

It remains to prove Lemma 2. The proof uses the well-known fact that a strictly concave function is unimodal; the specific version we need here is that if f is continuous and strictly concave on the closed interval $[u, v]$, then either it is strictly monotonic on $[u, v]$, or there exists $w \in (u, v)$ such that f is strictly increasing on $[u, w]$ and strictly decreasing on $[w, v]$. As a result, the minimum of a function, strictly concave on a closed interval, is attained at one of the endpoints of the interval. We also need

Claim 1. *Suppose that the function f is defined and strictly concave on a closed interval $[u, v]$. If f is monotonically increasing on $[u, v]$, then its inverse is strictly convex. If f is monotonically decreasing on $[u, v]$, then its inverse is strictly concave.*

Proof of Lemma 2. Suppose first that f is *strictly* concave on $[0, 1]$, and assume that (1) holds for all $\lambda, x_1 \in [0, 1]$ and $x_2 \in \{0, 1\}$; hence, by symmetry, also for all $\lambda, x_2 \in [0, 1]$ and $x_1 \in \{0, 1\}$.

Given $x_1, x_2 \in [0, 1]$ with $x_1 < x_2$, let $k := (f(x_2) - f(x_1))/(x_2 - x_1)$, and consider the auxiliary function $g(x) := f(x) - kx$ (depending on x_1 and x_2). Furthermore, set $\beta := g(x_i)$ ($i \in \{1, 2\}$) and $B := \max\{g(x) \colon x \in [0, 1]\}$. Notice, that $g(x_1) = g(x_2)$, in conjunction with the strict concavity of g, implies $B > \max\{g(0), g(1)\}$. For any $\lambda \in [0, 1]$ we have then

$$\begin{aligned} &f(\lambda x_1 + (1 - \lambda)x_2) - \lambda f(x_1) - (1 - \lambda) f(x_2) \\ &\qquad\qquad = g(\lambda x_1 + (1 - \lambda)x_2) - \lambda g(x_1) - (1 - \lambda) g(x_2) \le B - \beta, \end{aligned}$$

and therefore it suffices to show that

$$x_2 - x_1 + \beta \ge B \tag{6}$$

for any $0 < x_1 < x_2 < 1$ and with $\beta = \beta(x_1, x_2)$ and $B = B(x_1, x_2)$ defined as above.

We now put the reasoning onto its head. Suppose that a real k is fixed so that, if $g(x) = f(x) - kx$ and $B = \max\{g(x)\colon x \in [0,1]\}$, then $\max\{g(0), g(1)\} < B$. Let $w \in (0,1)$ be defined by $g(w) = B$. By the intermediate value property and monotonicity of g on each of the intervals $[0,w]$ and $[w,1]$, to any given β with $\max\{g(0), g(1)\} \le \beta \le B$ there corresponds then a unique pair (x_1, x_2) with $g(x_1) = g(x_2) = \beta$ and $0 \le x_1 \le w \le x_2 \le 1$. As the above argument shows, to complete the proof (under the *strict* concavity assumption) it suffices to establish (6) with x_1 and x_2 understood as functions of the variable β ranging from $\max\{g(0), g(1)\}$ to B. Since x_1 and x_2 are actually inverses of the function g restricted to the appropriate intervals, by Claim 1, x_1 is convex and continuous, and x_2 is concave and continuous, so that $x_2 - x_1 + \beta$ is concave and continuous and consequently, (6) will follow once we obtain it for $\beta = \max\{g(0), g(1)\}$ and also for $\beta = B$. The latter case (with equality sign) is immediate from $x_2 \ge x_1$. For the former case, we notice that if $\beta = \max\{g(0), g(1)\}$, then $x_1(1-x_2) = 0$, whence, having $t \in [0,1]$ defined by $w = tx_1 + (1-t)x_2$, we get

$$\begin{aligned} x_2 - x_1 + \beta &= x_2 - x_1 + B - \big(g(tx_1 + (1-t)x_2) - tg(x_1) - (1-t)g(x_2)\big) \\ &= x_2 - x_1 + B - \big(f(tx_1 + (1-t)x_2) - tf(x_1) - (1-t)f(x_2)\big) \\ &\ge B \end{aligned}$$

by the assumption of the lemma (and the remark at the very beginning of the proof).

Finally, suppose that f is concave but, perhaps, not *strictly* concave on $[0,1]$. For $\varepsilon \in (0,1)$ let $f_\varepsilon(x) := (f(x) - \varepsilon x^2)/(1+\varepsilon)$ and define

$$\Delta(x_1, x_2, \lambda) := f(\lambda x_1 + (1-\lambda)x_2) - \lambda f(x_1) - (1-\lambda)f(x_2) - |x_2 - x_1|$$

and

$$\Delta_\varepsilon(x_1, x_2, \lambda) := f_\varepsilon(\lambda x_1 + (1-\lambda)x_2) - \lambda f_\varepsilon(x_1) - (1-\lambda)f_\varepsilon(x_2) - |x_2 - x_1|.$$

A straightforward computations confirms that

$$\Delta(x_1, x_2, \lambda) = (1+\varepsilon)\Delta_\varepsilon(x_1, x_2, \lambda) + \varepsilon|x_2 - x_1|\big(1 - \lambda(1-\lambda)|x_2 - x_1|\big). \tag{7}$$

Consequently, if $\Delta(x_1, x_2, \lambda) \le 0$ when $x_1, \lambda \in [0,1]$ and $x_2 \in \{0,1\}$, then also $\Delta_\varepsilon(x_1, x_2, \lambda) \le 0$ under the same assumptions. Since f_ε is *strictly* concave (as it is easy to verify), we conclude that $\Delta_\varepsilon(x_1, x_2, \lambda) \le 0$ actually holds for all $x_1, x_2, \lambda \in [0,1]$. Now (7) shows that $\Delta(x_1, x_2, \lambda) \le \varepsilon$ for all $x_1, x_2, \lambda \in [0,1]$, and since ε can be chosen arbitrarily small, we have indeed $\Delta(x_1, x_2, \lambda) \le 0$. □

3. The proof of Theorem 2

We use induction on $|G|$. Without loss of generality, we assume that S is a minimal (under inclusion) generating subset. Fix an element $s_0 \in S$ and write $S_0 := S \setminus \{s_0\}$.

If $S_0 = \varnothing$, then G is cyclic with $|G|$ being equal to the order of s_0, whence $|G| \le m$ and the assertion follows from $F \le 1$. Assuming now that $S_0 \ne \varnothing$, let H be the subgroup of G, generated by S_0; thus, H is proper and non-trivial. Since the quotient group G/H is cyclic, generated by $s_0 + H$, its order $l := |G/H|$ does not exceed m. For $i = 1, \dots, l$ set $A_i := A \cap (is_0 + H)$ and $x_i := |A_i|/|H|$.

Fix $i \in [1, l]$. By the induction hypothesis (as applied to the subset $(A - is_0) \cap H$ of the group H with the generating subset S_0), the number of edges from an element of A_i to an element of $(is_0 + H) \setminus A$ is at least $\frac{1}{m}|H|F(x_i)$. Furthermore, the number of edges from A_i to $((i+1)s_0 + H) \setminus A$ is at least

$$\max\{|A_i| - |A_{i+1}|, 0\} = |H| \max\{x_i - x_{i+1}, 0\} = \frac{1}{2}\,|H|\big(|x_i - x_{i+1}| + x_i - x_{i+1}\big)$$

(where x_{i+1} is to be replaced with x_1 for $i = l$). It follows that

$$\partial_S(A) \ge \frac{1}{m}\,|H|\,\big(F(x_1) + \dots + F(x_l)\big) \\ + \frac{1}{2}\,|H|\big(|x_1 - x_2| + \dots + |x_{l-1} - x_l| + |x_l - x_1|\big).$$

Choose $i, j \in [1, l]$ so that x_i is the smallest, and x_j the largest of the numbers $x_1, \dots, x_l$. From the triangle inequality,

$$|x_1 - x_2| + \dots + |x_{l-1} - x_l| + |x_l - x_1| \ge 2(x_j - x_i),$$

whence

$$\begin{aligned} \partial_S(A) &\ge \frac{1}{m}\,|G|\,\frac{F(x_1) + \dots + F(x_l)}{l} + |H|(x_j - x_i) \\ &\ge \frac{1}{m}\,|G|\,\left(\frac{F(x_1) + \dots + F(x_l)}{l} + (x_j - x_i)\right). \end{aligned}$$

Recalling that $F \in \mathcal{F}_l$ by Corollary 1, we conclude that

$$\begin{aligned} \partial_S(A) &\ge \frac{1}{m}\,|G|\,F\left(\frac{x_1 + \dots + x_l}{l}\right) \\ &= \frac{1}{m}\,|G|\,F(|A|/|G|), \end{aligned}$$

as wanted.

4. The proofs of Lemma 1 and Theorems 3, 4, and 5

Proof of Theorem 3. Fix $x_1, x_2 \in [0, 1]$ with $x_1 < x_2$ and consider the function φ defined by

$$\varphi(\lambda) := f(\lambda x_1 + (1 - \lambda)x_2) - \lambda f(x_1) - (1 - \lambda)f(x_2),\ \lambda \in [0, 1].$$

Clearly, we have $\varphi(0)=\varphi(1)=0$, and for any $u,v,t\in[0,1]$ with $u<v$, using the fact that $f\in\mathcal{F}$, we get

$$\begin{aligned}
&\varphi\big(tu+(1-t)v\big)\\
&\qquad= f\big((tu+(1-t)v)\,(x_1-x_2)+x_2\big)-(tu+(1-t)v)\,(f(x_1)-f(x_2))-f(x_2)\\
&\qquad= f\big(t(u(x_1-x_2)+x_2)+(1-t)(v(x_1-x_2)+x_2)\big)\\
&\qquad\qquad\qquad\qquad-t(uf(x_1)+(1-u)f(x_2))-(1-t)(vf(x_1)+(1-v)f(x_2))\\
&\qquad\le tf(u(x_1-x_2)+x_2)+(1-t)f(v(x_1-x_2)+x_2)+(v-u)(x_2-x_1)\\
&\qquad\qquad\qquad\qquad-t(uf(x_1)+(1-u)f(x_2))-(1-t)(vf(x_1)+(1-v)f(x_2))\\
&\qquad= t\varphi(u)+(1-t)\varphi(v)+(v-u)(x_2-x_1).
\end{aligned}$$

Hence, $(x_2-x_1)^{-1}\varphi\in\mathcal{F}_0$, and therefore $\varphi\le(x_2-x_1)F$ by Theorem 1; that is,

$$f(\lambda x_1+(1-\lambda)x_2)-\lambda f(x_1)-(1-\lambda)f(x_2)\le F(\lambda)(x_2-x_1)$$

for all $x_1,x_2,\lambda\in[0,1]$ with $x_1<x_2$. □

Proof of Lemma 1. Aiming at a contradiction, suppose that $f\in\cap_{m\ge2}\mathcal{F}_m$, but $f\notin\mathcal{F}_0$. By the latter assumption, there exist $\lambda,x_1,x_2\in[0,1]$ with $x_1\le x_2$ such that

$$f(\lambda x_1+(1-\lambda)x_2)>\lambda f(x_1)+(1-\lambda)f(x_2)+(x_2-x_1). \tag{8}$$

Clearly, we have $\lambda\in(0,1)$. This implies $x:=\lambda x_1+(1-\lambda)x_2\in(0,1)$, whence f is continuous at x by [L, Lemma 4] (which says that all functions from the classes $\mathcal{F}_m$ are continuous on $(0,1)$). It follows that there is a *rational* $\lambda\in(0,1)$ for which (8) holds true; say, $\lambda=u/m$ with integer $0<u<m$. Now (8) can be written as

$$f\left(\frac{ux_1+(m-u)x_2}{m}\right)>\frac{uf(x_1)+(m-u)f(x_2)}{m}+x_2-x_1,$$

contradicting the assumption $f\in\mathcal{F}_m$. □

Proof of Theorem 4. Let $F_0:=\inf\{F_m\colon m\ge2\}$; our goal, therefore, is to show that $F_0=F$. To begin with, we prove that

$$F_0\in\mathcal{F}_m,\ m\ge2. \tag{9}$$

To this end, we fix $\varepsilon>0$ and $x_1,\dots,x_m\in[0,1]$ with $\min_i x_i=x_1$ and $\max_i x_i=x_m$, and show that

$$F_0\left(\frac{x_1+\dots+x_m}{m}\right)\le\frac{F_0(x_1)+\dots+F_0(x_m)}{m}+(x_m-x_1)+\varepsilon. \tag{10}$$

As shown in [L], if k and l are integers with $k\mid l$, then $\mathcal{F}_l\subseteq\mathcal{F}_k$, and hence $F_l\le F_k$. It follows that

$$F_0(x)=\lim_{l\to\infty}F_{l!}(x),\quad x\in[0,1];$$

thus, there is an integer $l \ge m$ such that $F_{l!}(x_i) \le F_0(x_i) + \varepsilon$ for each $i \in [1, m]$. Since $F_{l!} \in \mathcal{F}_{l!} \subseteq \mathcal{F}_m$ in view of $m \mid l!$, we then get

$$\begin{aligned} F_0\left(\frac{x_1 + \dots + x_m}{m}\right) &\le F_{l!}\left(\frac{x_1 + \dots + x_m}{m}\right) \\ &\le \frac{F_{l!}(x_1) + \dots + F_{l!}(x_m)}{m} + (x_m - x_1) \\ &\le \frac{F_0(x_1) + \dots + F_0(x_m)}{m} + (x_m - x_1) + \varepsilon, \end{aligned}$$

establishing (10), and therefore (9).

To complete the proof we notice that (9) and Lemma 1 yield $F_0 \in \mathcal{F}_0$, whence, by Theorem 1,

$$F_0 \le F. \tag{11}$$

On the other hand, since $F \in \mathcal{F}$ is concave, by [L, Lemma 3] we have $F \in \mathcal{F}_m$ for every $m \ge 2$. It follows that $F \le F_m$ for every $m \ge 2$, implying

$$F \le F_0. \tag{12}$$

Comparing (11) and (12), we get the assertion. □

Proof of Theorem 5. For the first assertion of the theorem we show that (1) holds true, provided that $f \in \mathcal{C}$ and $f(x) \le 4x(1-x)$, $x \in [0, 1]$. Using symmetry and disposing of the trivial cases, we assume that $x_1 < x_2$ and $0 < \lambda < 1$. Furthermore, letting $x_0 := \lambda x_1 + (1-\lambda)x_2$, we rewrite the inequality to prove as

$$f(x_0) \le \frac{x_2 - x_0}{x_2 - x_1} f(x_1) + \frac{x_0 - x_1}{x_2 - x_1} f(x_2) + x_2 - x_1. \tag{13}$$

By concavity, we have

$$f(x_0) \le \frac{x_0}{x_1} f(x_1)$$

(as the point $(x_1, f(x_1))$ lies above the segment joining the points $(0, 0)$ and $(x_0, f(x_0))$), and

$$f(x_0) \le \frac{1 - x_0}{1 - x_2} f(x_2)$$

(as $(x_2, f(x_2))$ is above the segment joining $(x_0, f(x_0))$ and $(1, 0)$). Also, by the assumptions,

$$f(x_0) \le 4x_0(1 - x_0).$$

Comparing with (13), we see that it suffices to prove that

$$\begin{aligned} \min\left\{\frac{x_0}{x_1} f(x_1),\ \frac{1 - x_0}{1 - x_2} f(x_2),\ 4x_0(1 - x_0)\right\}& \\ \le \frac{x_2 - x_0}{x_2 - x_1} f(x_1) &+ \frac{x_0 - x_1}{x_2 - x_1} f(x_2) + x_2 - x_1. \end{aligned}$$

Assuming for a contradiction that this is wrong, after tedious, but routine algebraic manipulations we then derive

$$x_2 f(x_1) - x_1 f(x_2) > x_1 \frac{(x_2-x_1)^2}{x_0-x_1},$$
$$(x_2-1) f(x_1) + (1-x_1) f(x_2) > (1-x_2) \frac{(x_2-x_1)^2}{x_2-x_0},$$

and

$$(x_0-x_2) f(x_1) + (x_1-x_0) f(x_2) > (x_2-x_1)^2 - 4x_0(1-x_0)(x_2-x_1).$$

Multiplying the first of these inequalities by $1-x_0$ and the second by x_0, and adding up the resulting estimates and the third inequality, we get

$$4x_0(1-x_0)(x_2-x_1) > \left(\frac{x_1(1-x_0)}{x_0-x_1} + \frac{x_0(1-x_2)}{x_2-x_0} + 1 \right) (x_2-x_1)^2.$$

It is easily verified that this simplifies to

$$\frac{(x_2-x_1)^2}{(x_0-x_1)(x_2-x_0)} < 4$$

and further to

$$4x_0^2 - 4x_0(x_1+x_2) + (x_1+x_2)^2 < 0,$$

which cannot hold since the left-hand side is a square.

To prove the second assertion, suppose that h is a concave function on $[0,1]$ with $h(0) = h(1) = 0$ and $h(x_0) > 4x_0(1-x_0)$ for some $x_0 \in (0,1)$, and let

$$f(x) := \begin{cases} \frac{h(x_0)}{x_0}\, x & \text{if } 0 \le x \le x_0, \\ \frac{h(x_0)}{1-x_0}\, (1-x) & \text{if } x_0 \le x \le 1; \end{cases}$$

thus, f is a concave function on $[0,1]$ with $f(0) = f(1) = 0$, and $f \le h$ by concavity of h. We show that, on the other hand, $f \notin \mathcal{F}_0$, and indeed, assuming for definiteness $x_0 \le 1/2$, that

$$f(x_0) > \frac{1}{2} f(0) + \frac{1}{2} f(2x_0) + 2x_0.$$

To this end we just plug in the definition of f and rewrite this inequality as

$$2x_0 < h(x_0) - \frac{1}{2} \frac{h(x_0)}{1-x_0} (1-2x_0) = \frac{h(x_0)}{2(1-x_0)},$$

which is equivalent to the assumption $h(x_0) > 4x_0(1-x_0)$. $\square$

5. Concluding remarks

It seems natural to extend the class $\mathcal{F}$ by considering, for every finite closed interval I and every constant $c > 0$, the class $\mathcal{F}(I, c)$ of those real-valued functions f defined on I and satisfying

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda)f(x_2) + c|x_2 - x_1|$$

for all $x_1, x_2 \in I$ and $\lambda \in [0, 1]$. This, however, does not lead to any principally new results, as one has $f \in \mathcal{F}(I, c)$ if and only if $(c|I|)^{-1} f \circ \varphi \in \mathcal{F}$, where φ is a linear bijection of $[0, 1]$ onto I; that is, $\mathcal{F}(I, c)$ is obtained from $\mathcal{F}$ by a simple linear scaling. In particular, $c|I|F \circ \varphi^{-1}$ is the largest function of the subclass of all functions from $\mathcal{F}(I, c)$, non-positive at the endpoints of I. Also, as a corollary of Theorem 3, if $f \in \mathcal{F}(I, c)$, then

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda)f(x_2) + cF(\lambda)|x_2 - x_1|$$

for all $x_1, x_2 \in I$ and $\lambda \in [0, 1]$.

In contrast, it might be interesting to extend the results of this paper, and in particular Theorem 1, onto the class of all (c, p)-convex functions on a given closed interval, for every fixed $p > 0$. Normalization reduces this to studying the class $\mathcal{F}_0^{(p)}$ of all real-valued functions on $[0, 1]$, satisfying the boundary condition (2) and the inequality

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda)f(x_2) + |x_2 - x_1|^p,$$

for all $x_1, x_2, \lambda \in [0, 1]$. It is not difficult to see that the function $F^{(p)} := \sup\{f : f \in \mathcal{F}^{(p)}\}$ is well defined and lies itself in the class $\mathcal{F}^{(p)}$, and that for any $f \in \mathcal{F}^{(p)}$ we have

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda f(x_1) + (1-\lambda)f(x_2) + F^{(p)}(\lambda)|x_2 - x_1|^p,$$

for all $x_1, x_2, \lambda \in [0, 1]$; moreover, $F^{(p)}$ is the largest function with this property. In view of Theorem 1, one can expect that, perhaps, an explicit expression for the functions $F^{(p)}$ can be found.

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