# DISCRETE NORMS OF A MATRIX AND THE CONVERSE TO THE EXPANDER MIXING LEMMA

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Abstract. We define the *discrete norm* of a complex $m \times n$ matrix $A$ by

$$\|A\|_\Delta := \max_{0 \ne \xi \in \{0,1\}^n} \frac{\|A\xi\|}{\|\xi\|},$$

and show that

$$\frac{c}{\sqrt{\log h(A) + 1}} \|A\| \le \|A\|_\Delta \le \|A\|,$$

where $c > 0$ is an explicitly indicated absolute constant, $h(A) = \sqrt{\|A\|_1 \|A\|_\infty}/\|A\|$, and $\|A\|_1, \|A\|_\infty$, and $\|A\| = \|A\|_2$ are the induced operator norms of $A$. Similarly, for the *discrete Rayleigh norm*

$$\|A\|_P := \max_{\substack{0 \ne \xi \in \{0,1\}^m \\ 0 \ne \eta \in \{0,1\}^n}} \frac{|\xi^t A \eta|}{\|\xi\| \|\eta\|}$$

we prove the estimate

$$\frac{c}{\log h(A) + 1} \|A\| \le \|A\|_P \le \|A\|.$$

These estimates are shown to be essentially best possible.

As a consequence, we obtain another proof of the (slightly sharpened and generalized version of the) converse to the expander mixing lemma by Bollobás-Nikiforov and Bilu-Linial.

## 1. Summary of results

For a complex matrix $A$ with $n$ columns, we define the *discrete norm* of $A$ by

$$\|A\|_\Delta := \max_{0 \ne \xi \in \{0,1\}^n} \frac{\|A\xi\|}{\|\xi\|},$$

where the maximum is over all non-zero $n$-dimensional binary vectors $\xi$, and $\| \cdot \|$ denotes the usual Euclidean vector norm. Recalling the standard definition of the induced operator $L^2$-norm

$$\|A\| := \sup_{0 \ne x \in \mathbb{C}^n} \frac{\|Ax\|}{\|x\|},$$

we see immediately that $\|A\|_\Delta \le \|A\|$, and one can expect that, moreover, the two norms are not far from each other.

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1.1. **Norm estimates.** Our first goal is to establish a result along the lines just indicated; to state it, we introduce the notion of a *height* of a matrix.

For $p \in [1, \infty]$, let $\|A\|_p$ denote the induced operator $L^p$-norm of the matrix $A$:

$$\|A\|_p := \sup_{0 \neq x \in \mathbb{C}^n} \frac{\|Ax\|_p}{\|x\|_p},$$

where $n$ is the number of columns of $A$. We are actually interested in the following three special cases: the *column norm* $\|A\|_1$, which can be equivalently defined as the largest absolute column sum of $A$; the *row norm* $\|A\|_\infty$, which is the largest absolute row sum of $A$; and the Euclidean norm $\|A\|_2$, commonly denoted simply by $\|A\|$. These three norms are known to be related by the inequality

$$\|A\|^2 \leq \|A\|_1 \|A\|_\infty, \tag{1}$$

which can be obtained as a particular case of the Riesz-Thorin theorem, or proved directly, using basic properties of matrix norms (in particular, sub-multiplicativity of the $L^1$-norm):

$$\|A\|^2 = \|A^*A\| \leq \|A^*A\|_1 \leq \|A^*\|_1 \|A\|_1 = \|A\|_\infty \|A\|_1.$$

Also, if $A$ has $m$ rows and $n$ columns, then

$$\|A\|_1 \leq \sqrt{m}\, \|A\| \text{ and } \|A\|_\infty \leq \sqrt{n}\, \|A\|. \tag{2}$$

We now define the *height* of a non-zero complex matrix $\|A\|$ by

$$h(A) := \sqrt{\|A\|_1 \|A\|_\infty} / \|A\|;$$

thus, if $A$ is of size $m \times n$, then in view of (1) and (2),

$$1 \leq h(A) \leq \sqrt[4]{mn}. \tag{3}$$

Having defined the heights, we can state our principal results.

**Theorem 1.** *For any non-zero complex matrix $A$, we have*

$$\frac{\|A\|}{8\sqrt{2}\sqrt{\log h(A) + 2}} \leq \|A\|_\Delta \leq \|A\|.$$

In a similar vein, we define the *discrete Rayleigh norm* of a complex $m \times n$ matrix $A$ by

$$\|A\|_P := \max_{\substack{0 \neq \xi \in \{0,1\}^m \\ 0 \neq \eta \in \{0,1\}^n}} \frac{|\xi^t A \eta|}{\|\xi\| \|\eta\|}$$

(where the subscript $P$ stands for the capital Greek letter *rho*), and prove

**Theorem 2.** *For any non-zero complex matrix $A$, we have*

$$\frac{\|A\|}{32\sqrt{2}\,(\log h(A) + 4)} \leq \|A\|_P \leq \|A\|.$$

We remark that the trivial upper bounds in Theorems 1 and 2 are included solely for comparison purposes. The proofs of the theorems are presented in Section 2.

Theorem 1 to our knowledge has never appeared in the literature, while Theorem 2 extends and refines results of Bollobás and Nikiforov [BN04], and Bilu and Linial [BL06]. Specifically, somewhat hidden in the proof of [BN04, Theorem 2] is the assertion that if $A$ is Hermitian of order $n \geq 2$, then $\|A\|_P \gg \|A\|/\log n$, and [BL06, Lemma 3.3] essentially says that if $A$ is a symmetric real matrix with the diagonal entries sufficiently small in absolute value, then $\|A\|_P \gg \|A\|/\big(\log(\|A\|_\infty/\|A\|_P)+1\big)$. (The notation $X \ll Y$ will be used throughout to indicate that there is an absolute constant $C$ such that $|X| \leq C|Y|$.) The former of these results follows from Theorem 2 in view of (3); to derive the latter just observe that for $A$ symmetric,

$$h(A) = \|A\|_\infty/\|A\| \leq \|A\|_\infty/\|A\|_P.$$

It is worth pointing out that our argument is completely distinct from those used in [BN04] and [BL06].

As an application, consider the situation where $A$ is the adjacency matrix of an undirected graph; thus, $\|A\|$ is the spectral radius of the graph, and $\|A\|_1 = \|A\|_\infty$ is its maximum degree. Identifying the vectors $\xi, \eta \in \{0,1\}^n$ in the definitions of the discrete norms with the corresponding subsets of the vertex set of the graph, as an immediate consequence of Theorems 1 and 2 we get the following corollaries relating the spectral radius of a graph to its combinatorial characteristics.

**Corollary 1.** *Let $(V,E)$ be a graph with the spectral radius $\rho$ and maximum degree $\Delta$. For a vertex $v \in V$ and a subset $X \subseteq V$, denote by $N_X(v)$ the set of all neighbors of $v$ in $X$:*

$$N_X(v) := \{u \in V \colon uv \in E\}.$$

*Then for any subset $X \subseteq V$ we have*

$$\sum_{v\in V} |N_X(v)|^2 \leq \rho^2|X|,$$

*and there exists a non-empty subset $X \subseteq V$ such that*

$$\sum_{v\in V} |N_X(v)|^2 \geq \frac{\rho^2}{128\big(\log(\Delta/\rho)\, + 2\big)}\,|X|.$$

**Corollary 2.** *Let $(V,E)$ be a graph with the spectral radius $\rho$ and maximum degree $\Delta$. For subsets $X, Y \subseteq V$, denote by $e(X,Y)$ the number of edges joining a vertex from $X$ with a vertex from $Y$, those edges having both their endpoints in $X \cap Y$ being counted twice:*

$$e(X,Y) := |\{(x,y) \in X \times Y \colon xy \in E\}|.$$

*Then for any subsets $X, Y \subseteq V$ we have*

$$e(X,Y) \leq \rho\sqrt{|X||Y|},$$

*and there exist non-empty subsets* $X, Y \subseteq V$ *such that*

$$e(X,Y) \ge \frac{\rho}{32\sqrt{2}\big(\log(\Delta/\rho)\,+4\big)}\,\sqrt{|X||Y|}.$$

1.2. **Second singular value estimates.** For a complex matrix $A$, let $\sigma_2(A)$ denote its second singular value; thus, for instance, if $A$ is Hermitian of order $n$ with the eigenvalues $\lambda_1,\dots,\lambda_n$, then $\sigma_2(A)$ is the second largest among the absolute values $|\lambda_1|,\dots,|\lambda_n|$. By the second singular value of a *graph* we will mean the second singular value of its adjacency matrix.

From the singular value decomposition theorem it is easy to derive that if $D$ is a matrix of the same size as $A$ and rank at most 1, then

$$\|A-D\| \ge \sigma_2(A); \tag{4}$$

this is a particular case of the Eckart-Young-Mirsky theorem [M60] (see also [S93] for the history of this theorem which has been re-discovered a number of times). Below we choose $D$ to be the matrix all of whose elements are equal to the arithmetic mean of the elements of $A$; we denote this matrix by $\overline{A}$. It is readily verified that $\|\overline{A}\|_1 \le \|A\|_1$ and $\|\overline{A}\|_\infty \le \|A\|_\infty$, whence, in view of (4) and assuming $\operatorname{rk} A \ge 2$,

$$h(A-\overline{A}) = \sqrt{\|A-\overline{A}\|_1\|A-\overline{A}\|_\infty}/\|A-\overline{A}\| \le 2\sqrt{\|A\|_1\|A\|_\infty}/\sigma_2(A). \tag{5}$$

On the other hand, from (4) and Theorem 1 we get

$$\sigma_2(A) \le \|A-\overline{A}\| \le 8\sqrt{2}\sqrt{\log h(A-\overline{A})+2}\cdot\|A-\overline{A}\|_\Delta. \tag{6}$$

Combining (5) and (6), we obtain

**Theorem 3.** *Suppose that* $A$ *is a complex matrix of rank at least* 2*, and let* $\overline{A}$ *be the identically-sized matrix all of whose elements are equal to the arithmetic mean of the elements of* $A$*. Then, writing* $K := 2\sqrt{\|A\|_1\|A\|_\infty}/\sigma_2(A)$*, we have*

$$\|A-\overline{A}\|_\Delta \ge \frac{\sigma_2(A)}{8\sqrt{2}\sqrt{\log K+2}}.$$

Arguing the same way but using Theorem 2 instead of Theorem 1, we get

**Theorem 4.** *Suppose that* $A$ *is a complex matrix of rank at least* 2*, and let* $\overline{A}$ *be the identically-sized matrix all of whose elements are equal to the arithmetic mean of the elements of* $A$*. Then, writing* $K := 2\sqrt{\|A\|_1\|A\|_\infty}/\sigma_2(A)$*, we have*

$$\|A-\overline{A}\|_P \ge \frac{\sigma_2(A)}{32\sqrt{2}(\log K+4)}.$$

Specifying Theorems 3 and 4 to the case where $A$ is the adjacency matrix of a graph, we obtain the following corollaries (stated in terms of the second singular value of a graph which, we recall, is the second largest among the absolute values of its eigenvalues).

**Corollary 3.** *Let $(V, E)$ be a non-empty graph with the maximum degree $\Delta$, average degree $d$, and the second singular value $\sigma$. Then there exists a non-empty subset $X \subseteq V$ such that, with $N_X(v)$ as in Corollary 1, we have*

$$\sum_{v \in V} \left( |N_X(v)| - d\,\frac{|X|}{|V|} \right)^2 \geq \frac{\sigma^2}{128(\log(2\Delta/\sigma) + 2)}\,|X|.$$

Corollary 3 is a converse to a result of Alon and Spencer [AS08, Theorem 9.2.4] asserting that, under the notation of the corollary, if $(V, E)$ is $d$-regular, then

$$\sum_{v \in V} \left( |N_X(v)| - d\,\frac{|X|}{|V|} \right)^2 \leq \left( 1 - \frac{|X|}{|V|} \right) \sigma^2\,|X|$$

for any $X \subseteq V$.

It is not difficult to see that the second singular value of a non-empty graph is at least 1; thus, the ratio $2\Delta/\sigma$ in the statement of Corollary 3 (and also Corollary 4 immediately following) does not exceed $2\Delta$.

**Corollary 4.** *Let $(V, E)$ be a non-empty graph with the maximum degree $\Delta$, average degree $d$, and the second singular value $\sigma$. Then there exist non-empty subsets $X, Y \subseteq V$ such that, with $e(X, Y)$ as in Corollary 2, we have*

$$\left| e(X, Y) - d\,\frac{|X||Y|}{|V|} \right| \geq \frac{\sigma}{32\sqrt{2}\big(\log(2\Delta/\sigma)\, + 4\big)}\,\sqrt{|X||Y|}.$$

Corollary 4 is a converse to the well-known Expander Mixing Lemma (see, for instance, [AS08, Corollary 9.2.5]) which says that if $(V, E)$ is $d$-regular, then

$$\left| e(X, Y) - d\,\frac{|X||Y|}{|V|} \right| \leq \sigma \sqrt{|X||Y|}$$

for all $X, Y \subseteq V$.

We remark that Theorem 4 and Corollary 4 are rather close to [BN04, Theorem 2] and [BL06, Corollary 5.1], respectively. Namely, [BN04, Theorem 2] says that, in our notation, if $A$ is Hermitian of order $n \geq 2$, then

$$\|A - \overline{A}\|_P \gg \sigma_2(A)/\log n, \tag{7}$$

while [BL06, Corollary 5.1] essentially says that if $(V, E)$ is a $d$-regular graph with the second singular value $\sigma$ satisfying

$$\left| e(X, Y) - d\,\frac{|X||Y|}{|V|} \right| \leq \alpha \sqrt{|X||Y|} \tag{8}$$

for all $X, Y \subseteq V$, with some $0 < \alpha \leq d$, then

$$\alpha \gg \sigma/\big(\log(d/\alpha)\, + 1\big). \tag{9}$$

It is readily seen that Corollary 4 implies (9): for if $\alpha \leq \sigma$, then $\log(2d/\sigma) \leq \log(d/\alpha) + 1$, whence from Corollary 4 and (8),

$$\alpha \geq \frac{\sigma}{32\sqrt{2}\big(\log(2d/\sigma)\, + 4\big)} \gg \frac{\sigma}{\log(d/\alpha) + 1}.$$

As to (7), it cannot be formally derived from Theorem 4, but follows easily from Theorem 2 and the estimates (3) and (4):

$$\|A - \overline{A}\|_P \gg \frac{\|A - \overline{A}\|}{\log h(A - \overline{A}) + 1} \geq \frac{\sigma_2(A)}{\log n}.$$

1.3. **Sharpness.** Theorems 1 and 2 are sharp in the sense that the logarithmic function in their lower bounds cannot be replaced with any slower growing function. To see this, for integer $n \geq 4$ consider the vector $x = (1, 1/\sqrt{2}, \dots, 1/\sqrt{n})^t$, and let $A = xx^t$; thus, $A$ is a symmetric real matrix of order $n$ with the entries $1/\sqrt{ij}$ $(i, j \in [1, n])$. It is readily verified that $\|A\|_1 = \|A\|_\infty < 2\sqrt{n}$ and $Az = \langle x, z\rangle x$, whence $\|A\| = \|x\|^2 > \log n$ and therefore $h(A) < 2\sqrt{n}/\log n$. Consequently, for every non-zero vector $\xi \in \{0, 1\}^n$, writing $k := \|\xi\|^2$, we have

$$\|A\xi\| = \langle x, \xi\rangle \|x\| \leq \left(1 + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}}\right) \frac{1}{\|x\|} \|A\|$$
$$< \frac{2}{\sqrt{\log n}} \|A\| \sqrt{k} < \frac{2}{\sqrt{\log h(A)}} \|A\| \|\xi\|,$$

implying

$$\|A\|_\Delta < \frac{2}{\sqrt{\log h(A)}} \|A\|.$$

Similarly, for all non-zero $\xi, \eta \in \{0, 1\}^n$, writing $k := \|\xi\|^2$ and $l := \|\eta\|^2$, we have

$$|\xi^t A\eta| = \langle \xi, x\rangle \langle \eta, x\rangle < 2\sqrt{k} \cdot 2\sqrt{l} < \frac{4}{\log n} \|A\| \|\xi\| \|\eta\| < \frac{4}{\log h(A)} \|A\| \|\xi\| \|\eta\|$$

whence

$$\|A\|_P < \frac{4}{\log h(A)} \|A\|.$$

Furthermore, Bollobás and Nikiforov [BN04, Section 3] construct regular graphs $(V, E)$ of arbitrarily large even order $n := |V|$ and degree $n/2$ such that, denoting by $A$ the adjacency matrix of $(V, E)$, and by $\overline{A}$ the square matrix of order $n$ with all elements equal to $1/2$ (which is the average of the elements of $A$), one has $\|A - \overline{A}\|_P \ll \sigma_2(A)/\log n$; this shows that the logarithmic factors in Theorem 4 and Corollary 4 cannot be replaced with sub-logarithmic ones. Another example of this sort is given by Bilu and Linial [BL06, Theorem 5.1]. Although we have not checked carefully the details, we believe that the constructions of Bollobás-Nikiforov and Bilu-Linial can also be used to show that Theorem 3 and Corollary 3 are tight.

An interesting question not addressed by these observations is whether Corollaries 1 and 2 are sharp; that is, whether one can improve Theorems 1 and 2 under the extra assumption that the matrix $A$ under consideration is zero-one and symmetric. Notice that if $A$ corresponds to a *regular* graph, then the norm $\|A\|$ is equal to the degree of the graph, and taking the vectors $\xi$ and $\eta$ in the definitions of discrete norms to be the all-1 vectors, we see that in this case $\|A\| = \|A\|_\Delta = \|A\|_P$. Consequently, any example showing that the logarithmic factors in Corollaries 1 and 2 cannot be dropped should involve highly non-regular graphs. In this direction we prove the following result, giving at least a partial solution to the problem.

**Theorem 5.** *For integer $m \ge 1$, let $\Gamma_m$ be the graph on the set $\{0,1\}^m$ of all binary vectors of length $m$, with two vectors adjacent if and only if they have disjoint supports. Then, denoting by $A_m$ the adjacency matrix of $\Gamma_m$, we have $\|A_m\|_\Delta \ll \|A_m\|/\sqrt[4]{m}$ and $\|A_m\|_P \ll \|A_m\|/\sqrt{m}$, with absolute implicit constants.*

The graph of Theorem 5 is similar to the well-known Kneser graphs; however, unlike the "standard" Kneser graphs, the vertex set of our graph is not restricted to vectors of fixed weight. The graph is simple, except for the loop attached to the zero vector; clearly, removing this loop will not affect significantly any of the norms in question.

To complete this introduction we mention a very interesting paper by Nikiforov [N], of which we learned when our own paper has already been written. The methods employed in the two papers are distinct, and the results do not overlap; however, some of the results are rather close.

We now turn to the proofs. Theorems 1 and 2 are proved in the next section; as we have explained above, Theorems 3 and 4, as well as Corollaries 1–4, are their direct consequences, and will not be addressed any more. Theorem 5 is proved in Section 3.

## 2. Proofs of Theorems 1 and 2

Both proofs share the same toolbox: Lemma 1 showing that for any complex matrix $A$, there exists a vector $z$ with $\|Az\|/\|z\|$ close to $\|A\|$ and the ratios of its non-zero coordinates bounded in terms of the height $h(A)$, and Lemmas 2–5 showing that a low-height vector cannot be approximately orthogonal to all binary vectors simultaneously.

For a non-zero vector $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we define the *logarithmic diameter* of $z$ by

$$\ell(z) := \frac{\max\{|z_i| \colon i \in [n]\}}{\min\{|z_i| \colon i \in [n],\ z_i \ne 0\}}.$$

**Lemma 1.** *Let $n \ge 1$ be an integer and $K \ge 1$ a real number. For any non-zero complex matrix $A$ with $n$ columns of height $h(A) \le K$, there exists a vector $z \in \mathbb{C}^n$ such that $\|Az\| > \frac{1}{2}\|A\|\|z\|$ and $\ell(z) < 8K^2 + 1$.*

*Proof.* Fix a unit-length vector $x = (x_1, \dots, x_n)^t \in \mathbb{C}^n$ with $\|Ax\| = \|A\|$ and let $M := 8K^2 + 1$. Consider the decomposition

$$x = \sum_{k=-\infty}^{\infty} x^{(k)},$$

where for every integer $k$, the vector $x^{(k)} = (x_1^{(k)}, \dots, x_n^{(k)})^t$ is defined by

$$x_i^{(k)} := \begin{cases} x_i & \text{if } M^k \le |x_i| < M^{k+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Notice, that $\ell(x^{(k)}) < M$ whenever $x^{(k)} \ne 0$. We have

$$\|A\|^2 = |\langle Ax, Ax\rangle| \le \sum_{k,l=-\infty}^{\infty} |\langle Ax^{(k)}, Ax^{(l)}\rangle| \tag{10}$$

and, since the vectors $x^{(k)}$ are pairwise orthogonal,

$$\sum_{k=-\infty}^{\infty} \|x^{(k)}\|^2 = \|x\|^2 = 1.$$

Since $h(A) \le K$ implies

$$|\langle Au, Av\rangle| \le \|Au\|_\infty \|Av\|_1 \le (\|A\|_\infty \|A\|_1)\|u\|_\infty \|v\|_1 \le K^2\|A\|^2 \|u\|_\infty \|v\|_1$$

for all $u, v \in \mathbb{C}^n$, the contribution to the right-hand side of (10) of the summands with $l \ge k+2$ can be estimated as follows:

$$\begin{aligned} \sum_{k,l\colon\, l\ge k+2} |\langle Ax^{(k)}, Ax^{(l)}\rangle| &\le K^2\|A\|^2 \sum_{k,l\colon\, l\ge k+2} M^{k+1} \sum_{i\in[n]\colon\, M^l\le|x_i|<M^{l+1}} |x_i| \\ &= K^2\|A\|^2 \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{l-2} M^{k+1} \sum_{i\in[n]\colon\, M^l\le|x_i|<M^{l+1}} |x_i| \\ &\le \frac{K^2}{M-1}\,\|A\|^2 \sum_{l=-\infty}^{\infty} \sum_{i\in[n]\colon\, M^l\le|x_i|<M^{l+1}} |x_i|^2 \\ &= \frac{1}{8}\,\|A\|^2\|x\|^2 \\ &= \frac{1}{8}\,\|A\|^2. \end{aligned}$$

By symmetry,

$$\sum_{k,l\colon\, |k-l|\ge 2} |\langle Ax^{(k)}, Ax^{(l)}\rangle| \le \frac{1}{4}\,\|A\|^2. \tag{11}$$

Assuming that the assertion of the lemma fails to hold, we have $\|Ax^{(k)}\| \le \frac{1}{2}\,\|A\|\|x^{(k)}\|$ for every integer $k$. Hence, under this assumption, for any fixed integer $d$,

$$\begin{aligned} \sum_{k,l\colon\, k-l=d} |\langle Ax^{(k)}, Ax^{(l)}\rangle| &\le \frac{1}{4}\,\|A\|^2 \sum_{l=-\infty}^{\infty} \|x^{(l)}\|\|x^{(l+d)}\| \\ &\le \frac{1}{4}\,\|A\|^2 \sum_{l=-\infty}^{\infty} \|x^{(l)}\|^2 \\ &= \frac{1}{4}\,\|A\|^2, \end{aligned}$$

the second inequality being strict unless $d = 0$. It follows that

$$\sum_{k,l\colon |k-l|\le 1} |\langle Ax^{(k)}, Ax^{(l)}\rangle| < \frac{3}{4}\,\|A\|^2;$$

along with (11) this yields

$$\sum_{k,l=-\infty}^{\infty} |\langle Ax^{(k)}, Ax^{(l)}\rangle| < \|A\|^2,$$

contradicting (10). □

For non-zero vectors $u, v \in \mathbb{C}^n$, we write $\cos(u,v) := \langle u, v\rangle / \|u\|\|v\|$.

**Lemma 2.** *Let $n \ge 1$ be an integer and $K \ge 1$ a real number. If $z \in \mathbb{R}^n$ is a vector with non-negative coordinates and logarithmic diameter $\ell(z) \le K$, then there exists a binary vector $\xi \in \{0,1\}^n$ such that* $\cos(z,\xi) \ge 1/\sqrt{\log K + 1}$.

*Proof.* Passing to the appropriate coordinate subspace and scaling the vector $z$, we assume that all its coordinates are between 1 and $K$. For $t \ge 0$, denote by $\Phi(t)$ the number of those coordinates which are greater than or equal to $t$, and let $\xi_t \in \{0,1\}^n$ be the characteristic vector of this set of coordinates; thus $\|\xi_t\|^2 = \Phi(t)$. Also, straightforward verification shows that

$$\int_t^K \Phi(\tau)\,d\tau = \langle z, \xi_t\rangle - t\Phi(t), \quad t \in [0,K] \tag{12}$$

and

$$\int_1^K 2\tau\Phi(\tau)\,d\tau = \|z\|^2 - n. \tag{13}$$

Let $\kappa := 1/\sqrt{\log K + 1}$. From (12) we get $\langle z, \xi_t\rangle \ge t\Phi(t) = t\|\xi_t\|^2$; consequently, if the assertion of the lemma were wrong, for each $t > 0$ we would have

$$\langle z, \xi_t\rangle^2 \le \kappa^2\|z\|^2\|\xi_t\|^2 \le \kappa^2\|z\|^2 \cdot \frac{1}{t}\langle z, \xi_t\rangle;$$

hence

$$\langle z, \xi_t\rangle \le \frac{1}{t}\,\kappa^2\|z\|^2, \quad t > 0.$$

Substituting this estimate into (12), integrating over $t$ in the range $[1,K]$, using (12) and (13), and taking into account that $\Phi(1) = n$ and $\langle \xi_1, z\rangle = \|z\|_1$, we obtain

$$\begin{aligned}
\kappa^2\|z\|^2\log K &\ge \int_1^K \left(\int_t^K \Phi(\tau)\,d\tau\right) dt + \int_1^K t\Phi(t)\,dt \\
&= \int_1^K (\tau-1)\Phi(\tau)\,d\tau + \int_1^K t\Phi(t)\,dt \\
&= \int_1^K 2\tau\Phi(\tau)\,d\tau - \int_1^K \Phi(\tau)\,d\tau \\
&= \|z\|^2 - \|z\|_1. 
\end{aligned} \tag{14}$$

From the assumption that the assertion of the lemma in wrong we get

$$\langle z, \xi_1 \rangle < \kappa \|z\| \|\xi_1\|$$

(for otherwise the assertion would hold true with $\xi = \xi_1$). As a result,

$$\|z\|_1^2 = \langle z, \xi_1 \rangle^2 < \kappa^2 \|z\|^2 \|\xi_1\|^2 = \kappa^2 \|z\|^2 n \le \kappa^2 \|z\|^2 \|z\|_1,$$

whence

$$\|z\|_1 < \kappa^2 \|z\|^2.$$

Substituting into (14) we get

$$\kappa^2 \|z\|^2 \log K > \|z\|^2 - \kappa^2 \|z\|^2,$$

in a contradiction with our choice of $\kappa$. □

Lemma 2 is easy to extend onto *arbitrary* real vectors (which may have some of their coordinates negative).

**Lemma 3.** *Let $n \ge 1$ be an integer and $K \ge 1$ a real number. If $z \in \mathbb{R}^n$ is a vector with the logarithmic diameter $\ell(z) \le K$, then there exists $\xi \in \{0,1\}^n$ such that $|\cos(z, \xi)| \ge 1/\sqrt{2(\log K + 1)}$.*

*Proof.* Write $z = z^+ - z^-$, where $z^+$ and $z^-$ have non-negative coordinates and disjoint supports. Observing that $\|z^+\|^2 + \|z^-\|^2 = \|z\|^2$, choose $z' \in \{z^+, z^-\}$ with $\|z'\| \ge \|z\|/\sqrt{2}$. Clearly, we have $\ell(z') \le \ell(z) \le K$; therefore, by Lemma 2, there exists $\xi \in \{0,1\}^n$ with

$$\cos(z', \xi) \ge \frac{1}{\sqrt{\log K + 1}}.$$

Assuming without loss of generality that for any vanishing coordinate of $z'$, the corresponding coordinate of $\xi$ also vanishes, we then get

$$|\langle z, \xi \rangle| = \langle z', \xi \rangle \ge \frac{1}{\sqrt{\log K + 1}} \|z'\| \|\xi\| \ge \frac{1}{\sqrt{2(\log K + 1)}} \|z\| \|\xi\|,$$

proving the assertion. □

For the remainder of this section, we extend the notion of height of a matrix (introduced in Section 1) onto vectors by identifying them with one-column or one-row matrices; that is, the height of a non-zero complex vector $z$ is

$$h(z) := \sqrt{\|z\|_1 \|z\|_\infty} / \|z\|.$$

We now prove a version of Lemma 3 which applies to a wider class of vectors; namely, real vectors of bounded height (instead of the bounded logarithmic diameter).

**Lemma 4.** *Let $n \ge 1$ be an integer and $K \ge 1$ a real number. If $z \in \mathbb{R}^n$ is a vector of height $h(z) \le K$, then there exists $\xi \in \{0,1\}^n$ such that $|\cos(z, \xi)| \ge 1/(2\sqrt{\log(2K^2) + 1})$.*

*Proof.* Let $M := \|z\|^2/\|z\|_1$. Writing $z = (z_1, \dots, z_n)^t$, we have

$$\sum_{i\colon |z_i|<M/2} z_i^2 \le \frac{1}{2}\, M\|z\|_1 = \frac{1}{2}\|z\|^2,$$

whence

$$\sum_{i\colon |z_i|\ge M/2} z_i^2 \ge \frac{1}{2}\, \|z\|^2. \tag{15}$$

Consider the vector $z' = (z'_1, \dots, z'_n)^t$ defined by

$$z'_i = \begin{cases} z_i & \text{if } |z_i| \ge M/2, \\ 0 & \text{if } |z_i| < M/2, \end{cases}$$

for each $i \in [n]$. Since $\ell(z') \le \|z\|_\infty/(M/2) = 2h^2(z) \le 2K^2$, by Lemma 3 there exists $\xi \in \{0,1\}^n$ with

$$|\cos(z',\xi)| \ge \frac{1}{\sqrt{2(\log(2K^2)+1)}}.$$

To complete the proof we notice that $\|z'\| \ge \|z\|/\sqrt{2}$ by (15), and that if $\xi$ is supported on the set of those $i \in [n]$ with $|z_i| \ge M/2$ (as we can safely assume), then $\langle z', \xi\rangle = \langle z, \xi\rangle$. □

Finally, we extend Lemma 4 onto vectors with *complex* coordinates.

**Lemma 5.** *Let $n \ge 1$ be an integer and $K \ge 1$ a real number. If $z \in \mathbb{C}^n$ is a vector of height $h(z) \le K$, then there exists $\xi \in \{0,1\}^n$ such that $|\cos(z,\xi)| \ge 1/(2\sqrt{4\log(2K)+2})$.*

*Proof.* Write $z = x + iy$, where $x, y \in \mathbb{R}^n$ and $i$ is the imaginary unit. Assume for definiteness that $\|x\| \ge \|y\|$, so that $\|x\| \ge \|z\|/\sqrt{2}$ in view of $\|z\|^2 = \|x\|^2 + \|y\|^2$. Since

$$h(x) = \frac{\sqrt{\|x\|_1\|x\|_\infty}}{\|x\|} \le \frac{\sqrt{\|z\|_1\|z\|_\infty}}{\|z\|/\sqrt{2}} = \sqrt{2}h(z) \le \sqrt{2}K,$$

by Lemma 4 there exists a non-zero $\xi \in \{0,1\}^n$ with

$$|\langle x,\xi\rangle| \ge \frac{1}{2\sqrt{\log(4K^2)+1}}\,\|x\|\|\xi\| \ge \frac{1}{2\sqrt{4\log(2K)+2}}\,\|z\|\|\xi\|.$$

The assertion now follows in view of $|\langle x,\xi\rangle| \le |\langle z,\xi\rangle|$. □

We are eventually ready to prove Theorems 1 and 2.

*Proof of Theorem 1.* Suppose that $A$ is a complex matrix with $m$ rows and $n$ columns, and set $K := h(A)$. Since $h(A^*) = h(A)$, by Lemma 1, there exists $z \in \mathbb{C}^m$ such that $\|A^*z\| > \frac{1}{2}\|A^*\|\|z\|$ and $\ell(z) < 9K^2$. Write $z = (z_1, \dots, z_m)^t$ and choose $j \in [m]$ so that $|z_j| = \min\{|z_i|\colon i \in [1,n], z_i \ne 0\}$. In view of

$$h^2(z) = \frac{\|z\|_1\|z\|_\infty}{\|z\|^2} = \frac{\|z\|_1|z_j|}{\|z\|^2}\,\ell(z) \le \ell(z) < 9K^2$$

we then get $h(z) < 3K$, whence

$$\begin{aligned} h(A^*z) &= \frac{\sqrt{\|A^*z\|_1\|A^*z\|_\infty}}{\|A^*z\|} \\ &< \frac{\sqrt{\|A^*\|_1\|z\|_1 \cdot \|A^*\|_\infty\|z\|_\infty}}{\|A^*\|\|z\|/2} \\ &= 2h(A^*)h(z) \\ &< 6K^2, \end{aligned}$$

and by Lemma 5, there exists $0 \neq \xi \in \{0,1\}^n$ with

$$|\langle A^*z, \xi\rangle| > \frac{1}{2\sqrt{4\log(12K^2)+2}}\,\|A^*z\|\|\xi\|.$$

As a result,

$$|\langle z, A\xi\rangle| = |\langle A^*z, \xi\rangle| > \frac{1}{4\sqrt{4\log(12K^2)+2}}\,\|A\|\|z\|\|\xi\|,$$

implying

$$\begin{aligned} \|A\xi\| &> \frac{1}{4\sqrt{4\log(12K^2)+2}}\,\|A\|\|\xi\| \qquad (16) \\ &> \frac{1}{8\sqrt{2}\sqrt{\log K+2}}\,\|A\|\|\xi\|. \end{aligned}$$

□

*Proof of Theorem 2.* Observing that the assumptions of Theorems 1 and 2 are identical, we re-use the proof of the former theorem, including the notation $K = h(A)$ and the conclusion that there exists a vector $\xi \in \{0,1\}^n$ satisfying (16). For brevity, denote the denominator of the fraction in the right-hand side of (16) by $f(K)$. Similarly to the computation in the proof of Theorem 1, and taking into account that $h(\xi) = 1$ (as $\xi$ is a binary vector), we obtain

$$\begin{aligned} h(A\xi) &= \frac{\sqrt{\|A\xi\|_1\|A\xi\|_\infty}}{\|A\xi\|} \\ &< \frac{\sqrt{\|A\|_1\|\xi\|_1 \cdot \|A\|_\infty\|\xi\|_\infty}}{\|A\|\|\xi\|/f(K)} \\ &= Kf(K). \end{aligned}$$

Applying Lemma 5 to the vector $A\xi$, we now find a binary vector $\eta \in \{0,1\}^m$ with

$$\begin{aligned} |\langle \eta, A\xi\rangle| &> \frac{1}{2\sqrt{4\log(2Kf(K))+2}}\,\|A\xi\|\|\eta\| \\ &> \frac{1}{2f(K)\sqrt{4\log(2Kf(K))+2}}\,\|A\|\|\xi\|\|\eta\|. \end{aligned}$$

Finally, it is not difficult to verify that for any $K \geq 1$, the denominator in the right-hand side is smaller than $32\sqrt{2}(\log K + 4)$, and result follows. □

## 3. Proof of Theorem 5

Since

$$A_m := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{\otimes m},$$

and since the eigenvalues of the matrix $A_1$ are $\varphi := (1+\sqrt{5})/2$ and $1-\varphi = (1-\sqrt{5})/2$, we have $\|A_m\| = \varphi^m$.

We split Theorem 5 into two theorems stated in the language and notation of Corollaries 1 and 2. These two theorems will then be given separate proofs.

**Theorem 5′.** *For any integer $m \ge 1$ and subset $X \subseteq \{0,1\}^m$, writing $N_X(v)$ for the set of neighbors of a vertex $v \in \{0,1\}^m$ in $X$ (in the graph $\Gamma_m$), we have*

$$\sum_{v\in\{0,1\}^m} |N_X(v)|^2 \ll \frac{\varphi^{2m}}{\sqrt{m}}\,|X|,$$

*with an absolute implicit constant.*

**Theorem 5″.** *For any integer $m \ge 1$ and subsets $X, Y \subseteq \{0,1\}^m$, writing $e(X,Y)$ for the number of edges in $\Gamma_m$ joining a vertex from $X$ with a vertex from $Y$, we have*

$$e(X,Y) \ll \frac{\varphi^m}{\sqrt{m}}\sqrt{|X||Y|},$$

*with an absolute implicit constant.*

We now prepare the technical ground for the proofs of both theorems.

Recall, that the entropy function is defined by

$$H(x) := -x\ln x - (1-x)\ln(1-x),\ 0 < x < 1,$$

extended by continuity onto the endpoints: $H(0) = H(1) = 0$.

Let

$$\Omega := \{(x,y) \in \mathbb{R}^2 \colon x \ge 0,\ y \ge 0,\ x+y \le 1\},$$

and consider the function

$$f(x,y) := (1-x)H\left(\frac{y}{1-x}\right) + (1-y)H\left(\frac{x}{1-y}\right), \quad (x,y) \in \Omega$$

(again, extended by continuity to vanish at the vertex points $(0,0)$, $(0,1)$, and $(1,0)$). Investigating the partial derivatives

$$\frac{\partial f}{\partial x} = \ln\frac{(1-x-y)^2}{x(1-x)}$$

and

$$\frac{\partial^2 f}{\partial x^2} = -\frac{1-x-y+2xy}{x(1-x)(1-x-y)} < 0, \tag{17}$$

with similar expressions for the derivatives with respect to $y$, we conclude that $f$ is concave on $\Omega$, and that it is a unimodal function of $x$ for any fixed $y \in [0,1]$, and a unimodal function of $y$ for any fixed $x \in [0,1]$. Consequently, the maximum of $f$ on $\Omega$

is attained in the unique point $(x_0, y_0) \in \Omega$ where both partial derivatives $\partial f/\partial x$ and $\partial f/\partial y$ vanish; that is,

$$\frac{(1-x-y)^2}{x(1-x)} = \frac{(1-x-y)^2}{y(1-y)} = 1.$$

The solution of this system is easily found to be $x_0 = y_0 = (5-\sqrt{5})/10 \approx 0.276$, and a simple computation confirms that the corresponding maximum value is

$$f(x_0, y_0) = 2\ln\varphi.$$

We will also need well-known estimates for the binomial coefficients which can be easily derived, for instance, from [McWS77, Ch. 10, §11, Lemmas 7 and 8]:

$$\frac{1}{\sqrt{2m}}\, e^{mH(k/m)} \le \binom{m}{k} \le \sum_{i=0}^{k} \binom{m}{i} \le e^{mH(k/m)}, \quad 0 \le k \le m/2, \tag{18}$$

and

$$\sum_{i=0}^{k} \binom{m}{i} \ll_\varepsilon \frac{1}{\sqrt{m}}\, e^{mH(k/m)}, \quad 1 \le k \le (1-\varepsilon)m/2, \tag{19}$$

for any $\varepsilon > 0$ (with the implicit constant depending on $\varepsilon$).

The following lemma is used in the proof of Theorem 5′.

**Lemma 6.** *For integer $m \ge 0$ and $j \in [0, m]$, let*

$$\tau_m(j) := \sum_{i=0}^{m-j} \binom{m-i}{j}\binom{m-j}{i}.$$

*Then*

$$\max\{\tau_m(j) \colon j \in [0, m]\} \ll \frac{\varphi^{2m}}{\sqrt{m}},$$

*with an absolute implicit constant.*

*Proof.* We use the notation introduced at the beginning of this section; thus, for instance, in view of (18),

$$\binom{m-i}{j}\binom{m-j}{i} \le e^{(m-i)H(j/(m-i))+(m-j)H(i/(m-j))} = e^{mf(i/m,j/m)}. \tag{20}$$

Let $I := (0.2, 0.3)$. Since $x_0 = y_0 \in I$, we have $\max_{\Omega\setminus(I\times I)} f < 2\ln\varphi$; therefore, by (20), we can fix $B < \varphi^2$ so that

$$\tau_m(j) = O(mB^m), \quad j/m \notin I, \tag{21}$$

and also

$$\tau_m(j) = \sum_{\substack{0\le i\le m-j\\ i/m\in I}} \binom{m-i}{j}\binom{m-j}{i} + O(mB^m), \quad j/m \in I. \tag{22}$$

For every pair $(i, j)$ with $(i/m, j/m) \in I \times I$, we have

$$\frac{1}{4} = \frac{0.2m}{m-0.2m} < \frac{i}{m-j} < \frac{0.3m}{m-0.3m} = \frac{3}{7},$$

and by symmetry, the resulting estimate holds true also for the ratio $j/(m-i)$; consequently, in view of (22) and (19), if $j/m \in I$, then

$$\begin{aligned}\tau_m(j) &\le \sum_{\substack{0\le i\le m-j\\ i/m\in I}} \frac{1}{\sqrt{m}}\, e^{(m-i)H(j/(m-i))} \cdot \frac{1}{\sqrt{m}}\, e^{(m-j)H(i/(m-j))} + O(mB^m) \\ &\le \frac{1}{m}\sum_{i=0}^{m-j} e^{mf(i/m,j/m)} + O(mB^m). \end{aligned} \tag{23}$$

Since $f(x, j/m)$ is a concave function of $x$ for any fixed $j \in [0, m]$, on each interval of the form $[i/m, (i+1)/m]$ it attains its minimum value at one of the endpoints of the interval, and so does the function $e^{mf(x,j/m)}$. Hence,

$$\begin{aligned}\int_{i/m}^{(i+1)/m} e^{mf(x,j/m)}\,dx &\ge \frac{1}{m}\,\min\{e^{mf(x,j/m)} \colon i/m \le x \le (i+1)/m\} \\ &= \frac{1}{m}\,\min\{e^{mf(i/m,j/m)}, e^{mf((i+1)/m,j/m)}\}; \quad 0 \le i \le m-j-1.\end{aligned}$$

Similarly, unimodality of $f(x, j/m)$ on the interval $x \in [0, 1-j/m]$ implies that of $e^{mf(x,j/m)}$; as a result, adding up for all $i \in [0, m-1-j]$ the estimate just obtained, we get

$$\begin{aligned}\frac{1}{m}\sum_{i=0}^{m-j} e^{mf(i/m,j/m)} &\le \int_0^{1-j/m} e^{mf(x,j/m)}\,dx + \frac{1}{m}\,\max\{e^{mf(i/m,j/m)} \colon 0 \le i \le m-j\} \\ &\le \int_0^{1-j/m} e^{mf(x,j/m)}\,dx + \frac{\varphi^{2m}}{m}.\end{aligned} \tag{24}$$

We now use the second-order polynomial approximation to show that

$$f(x,y) \le 2\ln\varphi - \frac{2}{3}(x-x_0)^2, \quad (x,y) \in \Omega; \tag{25}$$

substituting this estimate into (24) will eventually allow us to compete the proof of the lemma.

Let $z_0 := x_0/(1-x_0)$. A simple computation confirms that

$$z_0 = 2 - \varphi \approx 0.382,$$
$$H'(z_0) = \ln(z_0^{-1} - 1) = \ln\varphi,$$

and

$$H''(z) = -\frac{1}{z(1-z)} \le -4, \quad z \in (0,1);$$

consequently, by Taylor's formula,

$$H(z) \le H(z_0) + (z-z_0)\ln\varphi - 2(z-z_0)^2, \quad z \in (0,1).$$

Applying this estimate with $z = y/(1-x)$ and multiplying the result by $1-x$, in view of $1/(1-x) \ge 1$ we get

$$(1-x)H\left(\frac{y}{1-x}\right) \le (1-x)H(z_0) + (y-(1-x)z_0)\ln\varphi - 2(y-(1-x)z_0)^2.$$

Interchanging $x$ and $y$ and adding the resulting estimate to the one just obtained yields

$$f(x,y) \le L(x,y) - 2Q(x,y), \quad (x,y) \in \Omega, \tag{26}$$

where

$$L(x,y) = (2-x-y)H(z_0) + (x+y-(2-x-y)z_0)\ln\varphi$$

and

$$Q(x,y) = (x-(1-y)z_0)^2 + (y-(1-x)z_0)^2.$$

One easily verifies that $H(z_0) = (z_0+1)\ln\varphi$ and, as a result, the linear part is actually constant:

$$L(x,y) = 2\ln\varphi. \tag{27}$$

To estimate the quadratic part we set $\xi := x - x_0$ and $\eta := y - y_0$; with this notation, and taking into account that $x_0 = (1-y_0)z_0$ and $y_0 = (1-x_0)z_0$, we have

$$\begin{aligned} Q(x,y) &= (\xi + z_0\eta)^2 + (\eta + z_0\xi)^2 \\ &= (z_0^2+1)(\xi^2+\eta^2) + 4z_0\xi\eta \\ &\ge (z_0-1)^2(\xi^2+\eta^2) \\ &= z_0(\xi^2+\eta^2) \\ &\ge \frac{1}{3}(x-x_0)^2. \end{aligned} \tag{28}$$

From (26), (27), and (28) we get the desired estimate (25). Substituting it into (24) and recalling (23), we obtain

$$\begin{aligned} \tau_m(j) &\le \varphi^{2m} \int_0^{1-j/m} e^{-(2/3)m(x-x_0)^2}\,dx + O(\varphi^{2m}/m) \\ &< \varphi^{2m} \int_{-\infty}^{\infty} e^{-(2/3)m(x-x_0)^2}\,dx + O(\varphi^{2m}/m) \\ &= O(\varphi^{2m}/\sqrt{m}), \quad j/m \in I; \end{aligned}$$

along with (21), this proves the lemma. □

We are now ready for the proofs of Theorems 5′ and 5″.

*Proof of Theorem 5′.* Writing for brevity

$$\sigma(X) := \sum_{v\in\{0,1\}^m} |N_X(v)|^2,$$

we want to prove that

$$\sigma(X) \ll \frac{\varphi^{2m}}{\sqrt{m}}\,|X| \tag{29}$$

for every subset $X \subseteq \{0,1\}^m$.

For a vector $v \in \mathbb{R}^m$, let $|v|$ denote the number of non-zero coordinates of $v$; thus, for instance, if $v \in \{0,1\}^m$, then $|v| = \|v\|^2$. Since, for any $v \in \{0,1\}^m$, the total number of neighbors of $v$ in $\Gamma_m$ is $2^{m-|v|}$, we have

$$\sigma(X) \le \sum_{v\in\{0,1\}^m} |N_X(v)|^2 \le \sum_{v\in\{0,1\}^m} 4^{m-|v|} = 5^m < \varphi^{2m} \cdot 1.91^m,$$

establishing (29) in the case where $|X| \ge 1.92^m$. On the other hand, $|N_X(v)| \le |X|$ implies

$$\frac{\sigma(X)}{|X|} \le \sum_{v\in\{0,1\}^m} |N_X(v)| \le 2^m|X|,$$

and if $|X| \le 1.3^m$, then the right-hand side does not exceed $2.6^m$, whereas $\varphi^2 > 2.61$. With these observations in mind, for the rest of the proof we assume that

$$1.3^m < |X| < 1.92^m. \tag{30}$$

For $r \in [0,m]$, write $B_r := \{v \in \{0,1\}^m \colon |v| \le r\}$; thus,

$$|B_r| = \sum_{i=0}^{r} \binom{m}{i}.$$

Let $q \in [1, m-1]$ be defined by

$$|B_{q-1}| < |X| \le |B_q|.$$

In view of (30) and (18), this implies

$$cm < q < Cm \tag{31}$$

with some absolute constants $0 < c < C < 1/2$; consequently,

$$\frac{|B_q|}{|B_{q-1}|} \le 1 + \binom{m}{q} \Big/ \binom{m}{q-1} = 1 + \frac{m-q+1}{q} = O(1).$$

It follows that for any set $Y \subseteq \{0,1\}^m$ with $X \subseteq Y$ and $|Y| = |B_q|$ we have

$$\sigma(X)/|X| \le (\sigma(Y)/|Y|) \cdot (|Y|/|X|) \ll \sigma(Y)/|Y|,$$

showing that it suffices to prove (29) under the assumption $|X| = |B_q|$.

Using partial summation, we get

$$\begin{aligned}
\sigma(X) &= \sum_{x,y\in X} |\{v \in \{0,1\}^m \colon \langle x+y, v\rangle = 0\}| \\
&= \sum_{x,y\in X} 2^{m-|x+y|} \\
&= \sum_{k=0}^{m} 2^{m-k} |\{(x,y) \in X \times X \colon |x+y| = k\}| \\
&= \sum_{k=0}^{m-1} 2^{m-1-k} |\{(x,y) \in X \times X \colon |x+y| \le k\}| \\
&\qquad\qquad\qquad + |\{(x,y) \in X \times X \colon |x+y| \le m\}|,
\end{aligned}$$

and we now apply a result of Bollobás and Leader [BL03, Corollary 4] which says (in a dual form, and in the language of set families) that if $q \in [0, m]$, and $X$ is a set of $m$-dimensional binary vectors with $|X| = |B_q|$, then for any integer $k \in [0, m]$, the number of pairs $(x, y) \in X \times X$ with $|x + y| \le k$ is maximized when $X = B_q$. As a result, we can replace our present assumption $|X| = |B_q|$ with the stronger assumption $X = B_q$.

For $r \in [0, m]$, write $S_r := \{v \in \{0,1\}^m \colon |v| = r\}$; thus, $B_q = S_0 \cup \ldots \cup S_q$, and, in view of (31),

$$\frac{|S_{r-1}|}{|S_r|} = \binom{m}{r-1} \Big/ \binom{m}{r} = \frac{r}{m-r+1} < \frac{C}{1-C} < 1, \quad 1 \le r \le q,$$

implying

$$\sum_{r=0}^{q} (q+1-r)^2 |S_r| \le \sum_{r=0}^{q} \left( \frac{C}{1-C} \right)^{q-r} (q+1-r)^2 |S_q| \ll |S_q| \le |B_q|. \tag{32}$$

We now claim that to prove (29) with $X = B_q$, it suffices to prove it in the case where $X = S_r$, for all $r \in [0, m]$. To see this, we notice that if (29) is established in this special case, then, by the Cauchy-Schwartz inequality and (32),

$$\begin{aligned}
\sigma(B_q) &= \sum_{v \in \{0,1\}^m} \left( \sum_{r=0}^{q} (q+1-r)\sqrt{|S_r|} \cdot \frac{|N_{S_r}(v)|}{(q+1-r)\sqrt{|S_r|}} \right)^2 \\
&\le \sum_{v \in \{0,1\}^m} \left( \sum_{r=0}^{q} (q+1-r)^2 |S_r| \right) \sum_{r=0}^{q} \frac{1}{(q+1-r)^2} \frac{|N_{S_r}(v)|^2}{|S_r|} \\
&\ll |B_q| \sum_{r=0}^{q} \frac{1}{(q+1-r)^2} \frac{1}{|S_r|} \sum_{v \in \{0,1\}^m} |N_{S_r}(v)|^2 \\
&\ll \frac{\varphi^{2m}}{\sqrt{m}} |B_q| \sum_{r=0}^{q} \frac{1}{(q+1-r)^2} \\
&\ll \frac{\varphi^{2m}}{\sqrt{m}} |B_q|.
\end{aligned}$$

We thus can assume that $X = S_r$ for some $r \in [0, m]$. Therefore,

$$|N_X(v)| = \begin{cases} \binom{m-|v|}{r} & \text{if } |v| \le m - r, \\ 0 & \text{if } |v| > m - r. \end{cases}$$

Consequently,

$$\sigma(X)/|X| = \sum_{i=0}^{m-r} \binom{m}{i} \binom{m-i}{r}^2 \Big/ \binom{m}{r} = \sum_{i=0}^{m-r} \binom{m-i}{r} \binom{m-r}{i},$$

and the result now follows from Lemma 6. □

*Proof of Theorem 5′.* Suppose that $m \ge 1$ and $\varnothing \ne X, Y \subseteq \{0,1\}^m$; we want to show that $e(X, Y) \ll (\varphi^m/\sqrt{m})\sqrt{|X||Y|}$.

We start with the observation that if there is a vertex $x \in X$ with $|N_Y(x)| < e(X,Y)/(2|X|)$, then, letting $X' := X \setminus \{x\}$, we have $X' \neq \varnothing$ and

$$\frac{e(X',Y)}{\sqrt{|X'||Y|}} \geq \frac{e(X,Y)}{\sqrt{|X||Y|}}:$$

this follows readily from $e(X',Y) = e(X,Y) - |N_Y(x)|$ and $|X'| = |X| - 1$. A similar remark applies to the vertices $y \in Y$ having "too few" neighbors in $X$. Repeating this procedure, we ensure that $|N_Y(x)| \geq e(X,Y)/(2|X|)$ for every vertex $x \in X$, and that $|N_X(y)| \geq e(X,Y)/(2|Y|)$ for every vertex $y \in Y$.

We keep using the notation $|v|$ for the number of non-zero coordinates of a vector $v \in \mathbb{R}^m$. Let $m_1 := \max\{|x| : x \in X\}$, and choose arbitrarily a vertex $x \in X$ with $|x| = m_1$. Similarly, let $m_2 := \max\{|y| : y \in Y\}$ and choose $y \in Y$ with $|y| = m_2$. We have

$$\frac{e(X,Y)}{2|X|} \leq |N_Y(x)| \leq \sum_{k=0}^{m_2} \binom{m-m_1}{k}$$

and

$$\frac{e(X,Y)}{2|Y|} \leq |N_X(y)| \leq \sum_{k=0}^{m_1} \binom{m-m_2}{k}.$$

To complete the proof, we now show that

$$P := \sum_{k=0}^{m_2} \binom{m-m_1}{k} \cdot \sum_{k=0}^{m_1} \binom{m-m_2}{k} \ll \frac{\varphi^{2m}}{m}$$

uniformly in $m_1, m_2 \in [0, m]$.

Assume for definiteness that $m_1 \leq m_2$. If $m_2 > (m-m_1)/2$, then replacing $m_2$ with $\lfloor (m-m_1)/2 \rfloor$ enlarges the second factor in the definition of $P$, whereas the first factor can get at most twice smaller. As a result, we can assume that

$$m_2 \leq (m-m_1)/2, \tag{33}$$

and (in view of $m_1 \leq m_2$) also that

$$m_1 \leq (m-m_2)/2; \tag{34}$$

consequently,

$$0 \leq m_1, m_2 \leq m/2.$$

Write $\mu_i := m_i/m$ ($i \in \{1,2\}$). Taking into account (33) and (34), by (18) we get

$$P \leq e^{f(\mu_1,\mu_2)m},$$

and if both $\mu_2/(1-\mu_1)$ and $\mu_1/(1-\mu_2)$ are bounded away from $1/2$, then indeed

$$P \ll \frac{1}{m}\, e^{f(\mu_1,\mu_2)m} \tag{35}$$

by (19).

Let $\Omega_0 := [0, 0.3)^2$, and write $M := \max_{\Omega \setminus \Omega_0} f$. Since the maximum of $f$ on $\Omega$ is attained at the unique point $(x_0, y_0) \in \Omega_0$ (as explained at the beginning of this section), we have $M < f(x_0, y_0) = 2\ln\varphi$; hence, $P \le e^{mM} = o(\varphi^{2m}/m)$ for $(\mu_1, \mu_2) \notin \Omega_0$. On the other hand, if $(\mu_1, \mu_2) \in \Omega_0$, then

$$\frac{\mu_1}{1-\mu_2} \le \frac{3}{7} \quad \text{and} \quad \frac{\mu_2}{1-\mu_1} \le \frac{3}{7},$$

which in view of (35) gives

$$P \ll \frac{1}{m}\, e^{f(x_0,y_0)m} = \frac{\varphi^{2m}}{m}.$$

This completes the proof of Theorem 5″. □

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