Extrapolation and Local Acceleration of an Iterative Process for Common Fixed Point Problems

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February 16, 2012. Revised: April 15, 2012

Abstract

We consider sequential iterative processes for the common fixed point problem of families of *cutter operators* on a Hilbert space. These are operators that have the property that, for any point $x \in \mathcal{H}$, the hyperplane through Tx whose normal is $x - Tx$ always "cuts" the space into two half-spaces one of which contains the point x while the other contains the (assumed nonempty) fixed point set of T. We define and study generalized relaxations and extrapolation of cutter operators and construct extrapolated cyclic cutter operators. In this framework we investigate the Dos Santos local acceleration method in a unified manner and adopt it to a composition of cutters. For these we conduct convergence analysis of successive iteration algorithms.

Keywords: Common fixed point, cyclic projection method, cutter operator, quasi-nonexpansive operators, Dos Santos local acceleration.

AMS 2010 Subject Classification: 46B45, 37C25, 65K15, 90C25

1 Introduction

Our point of departure that motivates us in this work is a local acceleration technique of Cimmino's [Cim38] well-known simultaneous projection method for linear equations. This technique is referred to in the literature as the *Dos Santos* (DS) method, see Dos Santos [DSa87] and Bauschke and Borwein [BB96, Section 7], although Dos Santos attributes it, in the linear case, to De Pierro's Ph.D. Thesis [DPi81]. The method essentially uses the line through each pair of consecutive Cimmino iterates and chooses the point on this line which is closest to the solution x^* of the linear system $Ax = b$. The nice thing about it is that existence of the solution of the linear system must be assumed, but the method does not need the solution point x^* in order to proceed with the locally accelerated DS iterative process. This approach was also used by Appleby and Smolarski [AS05]. On the other hand, while trying to be as close as possible to the solution point x^* in each iteration, the method is not yet known to guarantee overall acceleration of the process. Therefore, we call it a *local acceleration* technique. In all the above references the DS method works for *simultaneous projection methods* and our first question was whether it can also work with *sequential projection methods*? Once we discovered that this is possible, the next natural question for sequential locally accelerated DS iterative process, is how far can the principle of the DS method be upgraded from the linear equations model? Can it work for closed and convex sets feasibility problems? I.e., can the locally accelerated DS method be preserved if orthogonal projections onto hyperplanes are replaced by metric projections onto closed and convex sets? Furthermore, can the latter be replaced by subgradient projectors onto closed and convex sets in a valid locally accelerated DS method? Finally, can the theory be extended to handle common fixed point problems? If so, for which classes of operators?

In this study we answer these questions by focusing on the class of operators $T : \mathcal{H} \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space, that have the property that, for any $x \in \mathcal{H}$, the hyperplane through Tx whose normal is $x - Tx$ always "cuts" the space into two half-spaces one of which contains the point x while the other contains the (assumed nonempty) fixed point set of T. This explains the name *cutter operators* or *cutters* that we introduce here. These operators, introduced and investigated by Bauschke and Combettes [BC01, Definition 2.2] and by Combettes [Com01], play an important role in optimization and feasibility theory since many commonly used operators are actually cutters. We define generalized relaxations and extrapolation of cutter operators and construct *extrapolated cyclic cutter operators.* For this cyclic extrapolated cutters we present convergence results of successive iteration processes for

common fixed point problems.

Finally we show that these iterative algorithmic frameworks can handle sequential locally accelerated DS iterative processes, thus, cover some of the earlier results about such methods and present some new ones.

The paper is organized as follows. In Section 2 we give the definition of cutter operators and bring some of their properties that will be used here. Section 3 presents the main convergence results. Applications to specific convex feasibility problems, which show how the locally accelerated DS iterative processes follow from our general convergence results, are furnished in Section 4.

2   Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and with a norm $\| \cdot \|$. Given $x, y \in \mathcal{H}$ we denote

$$H(x, y) := \{ u \in \mathcal{H} \mid \langle u - y, x - y \rangle \leq 0 \}. \tag{1}$$

Definition 1   An operator $T : \mathcal{H} \to \mathcal{H}$ is called a *cutter operator* or, in short, a *cutter*

$$\text{if } \operatorname{Fix} T \subseteq H(x, Tx) \quad \text{for all } x \in \mathcal{H}, \tag{2}$$

where $\operatorname{Fix} T$ is the fixed point set of T, equivalently,

$$q \in \operatorname{Fix} T \text{ implies that } \langle Tx - x, Tx - q \rangle \leq 0 \quad \text{for all } x \in \mathcal{H}. \tag{3}$$

The inequality in (3) can be written equivalently in the form

$$\langle Tx - x, q - x \rangle \geq \| Tx - x \|^2. \tag{4}$$

The class of cutter operators is denoted by $\mathcal{T}$, i.e.,

$$\mathcal{T} := \{ T : \mathcal{H} \to \mathcal{H} \mid \operatorname{Fix} T \subseteq H(x, Tx) \quad \text{for all } x \in \mathcal{H} \}. \tag{5}$$

The class $\mathcal{T}$ of operators was introduced and investigated by Bauschke and Combettes in [BC01, Definition 2.2] and by Combettes in [Com01]. Yamada and Ogura [YO04] and Mainge [Mai10] named the cutters *firmly quasi-nonexpansive* operators. These operators were named *directed operators* in Zaknoon [Zak03] and further employed under this name by Segal [Seg08] and Censor and Segal [CS08, CS08a, CS09]. Cegielski [Ceg10, Definition 2.1] named and studied these operators as *separating operators*. Since both *directed* and *separating* are key words of other, widely-used, mathematical

entities we decided in [CC11] to use the term *cutter operators*. This name can be justified by the fact that the bounding hyperplane of $H(x, Tx)$ "cuts" the space into two half-spaces, one which contains the point x while the other contains the set $\operatorname{Fix} T$. We recall definitions and results on cutter operators and their properties as they appear in [BC01, Proposition 2.4] and [Com01], which are also sources for further references.

Bauschke and Combettes [BC01] showed the following:

(i) The set of all fixed points of a cutter operator with nonempty $\operatorname{Fix} T$ is a closed and convex subset of $\mathcal{H}$, because $\operatorname{Fix} T = \cap_{x \in \mathcal{H}} H(x, Tx)$.

Denoting by I the identity operator,

$$\text{if } T \in \mathcal{T} \text{ then } I + \lambda(T - I) \in \mathcal{T} \quad \text{for all } \lambda \in [0, 1]. \tag{6}$$

This class of operators is fundamental because many common types of operators arising in convex optimization belong to the class and because it allows a complete characterization of Fejér-monotonicity [BC01, Proposition 2.7]. The localization of fixed points is discussed by Goebel and Reich in [GR84, pp. 43–44]. In particular, it is shown there that a firmly nonexpansive operator, namely, an operator $T : \mathcal{H} \to \mathcal{H}$ that fulfills

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle \text{ for all } x, y \in \mathcal{H}, \tag{7}$$

and has a fixed point, satisfies (3) and is, therefore, a cutter operator. The class of cutter operators, includes additionally, according to [BC01, Proposition 2.3], among others, the resolvents of a maximal monotone operators, the orthogonal projections and the subgradient projectors. Another family of cutters appeared recently in Censor and Segal [CS08a, Definition 2.7]. Note that every cutter operator belongs to the class of operators $\mathcal{F}^0$, defined by Crombez [Cro05, p. 161],

$$\mathcal{F}^0 := \{T : \mathcal{H} \to \mathcal{H} \mid \|Tx - q\| \leq \|x - q\| \text{ for all } q \in \operatorname{Fix} T \text{ and } x \in \mathcal{H}\}, \tag{8}$$

whose elements are called elsewhere quasi-nonexpansive or paracontracting operators. An example of a quasi-nonexpansive operator $T : \mathcal{H} \to \mathcal{H}$ is a nonexpansive one, i.e., an operator satisfying $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in \mathcal{H}$, with $\operatorname{Fix} T \neq \emptyset$.

Definition 2 Let $T : \mathcal{H} \to \mathcal{H}$ and let $\lambda \in (0, 2)$. We call the operator $T_\lambda := I + (1 - \lambda)T$ a *relaxation* of T.

Definition 3 We say that an operator $T : \mathcal{H} \to \mathcal{H}$ with $\operatorname{Fix} T \neq \emptyset$ is *strictly quasi-nonexpansive* if

$$\|Tx - z\| < \|x - z\| \tag{9}$$

for all $x \notin \operatorname{Fix} T$ and for all $z \in \operatorname{Fix} T$. We say that T is α-*strongly quasi-nonexpansive*, where $\alpha > 0$, or, in short, *strongly quasi-nonexpansive* if

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \alpha \|Tx - x\|^2 \tag{10}$$

for all $x \in \mathcal{H}$ and for all $z \in \operatorname{Fix} T$.

It is well-known that a relaxation of a cutter operator is strongly quasi-nonexpansive (see [Com01, Proposition 2.3(ii)]). Since the converse implication is also true we have the following result.

Lemma 4 *Let* $T : \mathcal{H} \to \mathcal{H}$ *be an operator which has a fixed point and let* $\lambda \in (0, 2)$. *Then* T *is a cutter if and only if its relaxation* T_λ *is* $(2 - \lambda)/\lambda$-*strongly quasi-nonexpansive.*

Proof. Since

$$T_\lambda x - x = \lambda(Tx - x), \tag{11}$$

we have, by the properties of the inner product,

$$\begin{aligned}
\|T_\lambda x - z\|^2 &- \|x - z\|^2 + \frac{2 - \lambda}{\lambda}\|T_\lambda x - x\|^2 \\
&= \|x - z + \lambda(Tx - x)\|^2 - \|x - z\|^2 + \lambda(2 - \lambda)\|Tx - x\|^2 \\
&= 2\lambda(\|Tx - x\|^2 - \langle z - x, Tx - x \rangle) = 2\lambda\langle z - Tx, x - Tx \rangle, \tag{12}
\end{aligned}$$

for all $x \in \mathcal{H}$ and for all $z \in \operatorname{Fix} T$, from which the required result follows. ∎

Definition 5 We say that an operator $T : \mathcal{H} \to \mathcal{H}$ is *demiclosed* at 0 if for any weakly converging sequence $\{x^k\}_{k=0}^\infty$, $x^k \rightharpoonup y \in \mathcal{H}$ as $k \to \infty$, with $Tx^k \to 0$ as $k \to \infty$, we have $Ty = 0$.

It is well-known that for a nonexpansive operator $T : \mathcal{H} \to \mathcal{H}$, the operator $T - I$ is demiclosed at 0, see Opial [Opi67, Lemma 2].

Definition 6 We say that an operator $T : \mathcal{H} \to \mathcal{H}$ is *asymptotically regular* if

$$\|T^{k+1}x - T^k x\| \to 0, \text{ as } k \to \infty, \tag{13}$$

for all $x \in \mathcal{H}$.

It is well-known that if T is a nonexpansive and asymptotically regular operator with $\operatorname{Fix} T \neq \emptyset$ then, for any $x \in \mathcal{H}$, the sequence $\{T^k x\}_{k=0}^\infty$ converges weakly to a fixed point of T, see [Opi67, Theorem 1].

3 Main results

We deal in this paper with a finite family of cutter operators $U_i : \mathcal{H} \to \mathcal{H}$, $i = 1, 2, \ldots, m$, with $\bigcap_{i=1}^{m} \operatorname{Fix} U_i \neq \emptyset$ and with compositions of U_i, $i = 1, 2, \ldots, m$. We propose local acceleration techniques for algorithms which apply this operation. For an operator $U : \mathcal{H} \to \mathcal{H}$ we define the operator $U_{\sigma,\lambda} : \mathcal{H} \to \mathcal{H}$ by

$$U_{\sigma,\lambda} x := x + \lambda \sigma(x)(Ux - x), \tag{14}$$

where $\lambda \in (0, 2)$ is a *relaxation parameter* and $\sigma : \mathcal{H} \to (0, +\infty)$ is a *step size function*. We call the operator $U_{\sigma,\lambda}$ the *generalized relaxation* of U (cf. [Ceg10, Section 1] and [CC11, Definition 9.16]). A generalized relaxation $U_{\sigma,\lambda}$ of U with $\lambda\sigma(x) \geq 1$ for all $x \in \mathcal{H}$ is called an *extrapolation* of U. Of course, for $x \in \operatorname{Fix} U$, we can set $\sigma(x)$ arbitrarily, e.g., $\sigma(x) = 1$. Denoting $U_\sigma := U_{\sigma,1}$, it is clear that

$$U_{\sigma,\lambda} x = x + \lambda(U_\sigma x - x), \tag{15}$$

and that $\operatorname{Fix} U_{\sigma,\lambda} = \operatorname{Fix} U_\sigma = \operatorname{Fix} U$ (note that $\sigma(x) > 0$ for all $x \in \mathcal{H}$).

Let $U_i : \mathcal{H} \to \mathcal{H}$ be a cutter operator, $i = 1, 2, \ldots, m$, with $\bigcap_{i=1}^{m} \operatorname{Fix} U_i \neq \emptyset$. Define the operator $U : \mathcal{H} \to \mathcal{H}$ as the composition

$$U := U_m U_{m-1} \cdots U_1. \tag{16}$$

Since any cutter operator is strongly quasi-nonexpansive, (see Lemma 4), $\operatorname{Fix} U = \bigcap_{i=1}^{m} \operatorname{Fix} U_i$ and U is strongly quasi-nonexpansive, see [BB96, Proposition 2.10]. Consequently, U is asymptotically regular, see [BB96, Corollary 2.8], and, if U is nonexpansive then any sequence $\{x^k\}_{k=0}^{\infty}$, generated by the recurrence (Piccard iteration)

$$x^0 \in \mathcal{H} \text{ is arbitrary, and } x^{k+1} = Ux^k, \text{ for all } k \geq 0, \tag{17}$$

converges weakly to a fixed point of U, see [Opi67, Theorem 1]. We extend this convergence result to the generalized relaxation of U, defined by (14). We call an operator U of the form (16) a *cyclic cutter*. Contrary to the simultaneous cutter, a cyclic cutter needs not to be a cutter. This "contradiction of terms" resembles the situation with the, so-called, "subgradient projection" onto a convex set which needs not to be a projection onto the set because it needs not be an element of the set.

In order to prove our convergence result for the generalized relaxation of U, defined by (14), let $S_0 := I$ and $S_i := U_i U_{i-1} \cdots U_1$, $i = 1, 2, \ldots, m$. Of course, $U = S_m$. Further, denote

$$u^0 = x, \ u^i = U_i u^{i-1} \ \text{ and } \ y^i = u^i - u^{i-1}, \ \text{ for all } \ i = 1, 2, \ldots, m. \tag{18}$$

6

According to this notation, used throughout this paper, we have $u^i = S_i x$ and $\sum_{j=i}^{m} y^j = Ux - S_{i-1}x$, $i = 1, 2, \ldots, m$, in particular, $\sum_{j=1}^{m} y^j = Ux - x$.

Lemma 7 *If $U_i : \mathcal{H} \to \mathcal{H}$, $i = 1, 2, \ldots, m$, are cutter operators such that $\bigcap_{i=1}^{m} \operatorname{Fix} U_i \neq \emptyset$, then for any $z \in \bigcap_{i=1}^{m} \operatorname{Fix} U_i$ the following inequalities hold*

$$\langle Ux - x, z - x \rangle \geq \sum_{i=1}^{m} \langle y^i + y^{i+1} + \cdots + y^m, y^i \rangle \geq \frac{1}{2} \sum_{i=1}^{m} \|y^i\|^2 \geq \frac{1}{2m} \left\| \sum_{i=1}^{m} y^i \right\|^2, \tag{19}$$

for all $x \in \mathcal{H}$.

Proof. We prove the first inequality in (19) by induction on m. For $m = 1$ it follows directly from Definition 1. Suppose that the first inequality in (19) is true for some $m = t$. Define $V_1 := I$, $V_i := U_i U_{i-1} \cdots U_2$ for $i = 2, 3, \ldots, t+1$, $v^1 := h$, where h is an arbitrary element of $\mathcal{H}$, $v^i := U_i v^{i-1}$ and $z^i := v^i - v^{i-1}$, $i = 2, 3, \ldots, t+1$. If we set $h = U_1 x$ then, of course, $S_i x = V_i h$, $u^i = v^i$ and $y^i = z^i$, $i = 2, 3, \ldots, t+1$. It follows from the induction hypothesis that

$$\langle V_{t+1} h - h, z - h \rangle \geq \sum_{i=2}^{t+1} \langle z^i + z^{i+1} + \cdots + z^{t+1}, z^i \rangle \tag{20}$$

for all $h \in \mathcal{H}$ and $z \in \cap_{i=2}^{t+1} \operatorname{Fix} U_i$. Thus, if $h = U_1 x$ then, for all $x \in \mathcal{H}$ and $z \in \cap_{i=1}^{t+1} \operatorname{Fix} U_i$, we obtain

$$\begin{aligned}
\langle S_{t+1} x - x, z - x \rangle &= \langle V_{t+1} h - x, z - x \rangle \\
&= \langle V_{t+1} h - h, z - x \rangle + \langle U_1 x - x, z - x \rangle \\
&\geq \langle V_{t+1} h - h, z - h \rangle + \langle V_{t+1} h - h, h - x \rangle + \|y^1\|^2 \\
&\geq \sum_{i=2}^{t+1} \langle z^i + z^{i+1} + \cdots + z^{t+1}, z^i \rangle + \left\langle \sum_{i=2}^{t+1} z^i, h - x \right\rangle + \|y^1\|^2 \\
&= \sum_{i=2}^{t+1} \langle y^i + y^{i+1} + \cdots + y^{t+1}, y^i \rangle + \langle y^2 + y^3 + \cdots + y^{t+1}, y^1 \rangle + \|y^1\|^2 \\
&= \sum_{i=1}^{t+1} \langle y^i + y^{i+1} + \cdots + y^{t+1}, y^i \rangle. \tag{21}
\end{aligned}$$

Therefore, the first inequality in (19) is true for $m = t+1$, and the induction is complete. The second inequality follows from

$$\sum_{i=1}^{m} \langle y^i + y^{i+1} + \cdots + y^m, y^i \rangle - \frac{1}{2} \sum_{i=1}^{m} \|y^i\|^2 = \frac{1}{2} \| \sum_{i=1}^{m} y^i \|^2. \tag{22}$$

The third inequality in (19) follows from the convexity of the function $\| \cdot \|^2$.
∎

Define the step size function $\sigma_{\max} : \mathcal{H} \to (0, +\infty)$ by

$$\sigma_{\max}(x) := \begin{cases} \dfrac{\sum_{i=1}^{m} \langle Ux - S_{i-1}x, S_i x - S_{i-1}x \rangle}{\|Ux - x\|^2}, & \text{for } x \notin \operatorname{Fix} U, \\ 1, & \text{for } x \in \operatorname{Fix} U. \end{cases} \tag{23}$$

If we set $y^i = S_i x - S_{i-1}x$, $i = 1, 2, \ldots, m$, (compare with (18)) in Lemma 7, then we obtain for $x \notin \operatorname{Fix} U$

$$\sigma_{\max}(x) \geq \frac{\frac{1}{2} \sum_{i=1}^{m} \|S_i x - S_{i-1}x\|^2}{\|Ux - x\|^2} \geq \frac{1}{2m}. \tag{24}$$

Lemma 8 *Let $U_i : \mathcal{H} \to \mathcal{H}$ be cutter operators, $i = 1, 2, \ldots, m$, with $\bigcap_{i=1}^{m} \operatorname{Fix} U_i \neq \emptyset$. The operator $U_\sigma := U_{\sigma,1}$, defined by (14), where the step size function $\sigma := \sigma_{\max}$ is given by (23), is a cutter.*

Proof. Taking $z \in \bigcap_{i=1}^{m} \operatorname{Fix} U_i$, the first inequality in (19) with $y^i = S_i x - S_{i-1}x$, $i = 1, 2, \ldots, m$, can be rewritten, for all $x \in \mathcal{H}$, in the form

$$\|Ux - x\|^2 \sigma(x) \leq \langle Ux - x, z - x \rangle. \tag{25}$$

Therefore, for the operator $U_\sigma x$, defined by (14) with the step size $\sigma(x)$ as in (23), we have

$$\langle U_\sigma x - x, z - x \rangle = \sigma(x)\langle Ux - x, z - x \rangle \geq \sigma^2(x)\|Ux - x\|^2 = \|U_\sigma x - x\|^2, \tag{26}$$

and the proof is complete. ∎

Theorem 9 *Let $U_i : \mathcal{H} \to \mathcal{H}$, $i = 1, 2, \ldots, m$, be a cutter operator with $\bigcap_{i=1}^{m} \operatorname{Fix} U_i \neq \emptyset$. Let the sequence $\{x^k\}_{k=0}^{\infty}$ be defined by*

$$x^{k+1} = U_{\sigma,\lambda_k} x^k, \tag{27}$$

where $U_{\sigma,\lambda}$ is given by (14), $\lambda_k \in [\varepsilon, 2-\varepsilon]$ for an arbitrary constant $\varepsilon \in (0,1)$, and $\sigma := \sigma_{\max}$ is given by (23). Then

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - \frac{\lambda_k(2 - \lambda_k)}{4} \frac{\sum_{i=1}^{m} \|S_i x^k - S_{i-1}x^k\|^2}{\|Ux^k - x^k\|^2}$$

$$\leq \|x^k - z\|^2 - \frac{\lambda_k(2 - \lambda_k)}{4m^2} \|Ux^k - x^k\|^2, \tag{28}$$

for all $x^k \notin \operatorname{Fix} U$ and for all $z \in \operatorname{Fix} U$. Consequently, $x^k \rightharpoonup x^* \in \operatorname{Fix} U$ as $k \to \infty$, if one of the following conditions is satisfied:

(i) $U - I$ is demiclosed at 0, or

(ii) $U_i - I$ are demiclosed at 0, for all $i = 1, 2, \ldots, m$.

Proof. Let $z \in \operatorname{Fix} U$. By inequality (24), we have

$$
\begin{aligned}
\|x^{k+1} - z\|^2 = \|U_{\sigma, \lambda_k} x^k - z\|^2 &= \|x^k - z + \lambda_k (U_\sigma x^k - x^k)\|^2 \\
&= \|x^k - z\|^2 + \lambda_k^2 \|U_\sigma x^k - x^k\|^2 - 2\lambda_k \langle U_\sigma x^k - x^k, z - x^k \rangle \\
&\leq \|x^k - z\|^2 + \lambda_k^2 \|U_\sigma x^k - x^k\|^2 - 2\lambda_k \|U_\sigma x^k - x^k\|^2 \\
&= \|x^k - z\|^2 - \lambda_k (2 - \lambda_k) \|U_\sigma x^k - x^k\|^2 \\
&= \|x^k - z\|^2 - \lambda_k (2 - \lambda_k) \sigma^2(x^k) \|U x^k - x^k\|^2 \\
&\leq \|x^k - z\|^2 - \frac{\lambda_k (2 - \lambda_k)}{4} \frac{(\sum_{i=1}^m \|S_i x^k - S_{i-1} x^k\|^2)^2}{\|U x^k - x^k\|^2} \\
&\leq \|x^k - z\|^2 - \frac{\lambda_k (2 - \lambda_k)}{4m^2} \|U x^k - x^k\|^2, \tag{29}
\end{aligned}
$$

for all $x^k \notin \operatorname{Fix} U$. Therefore, $\{\|x^k - z\|\}_{k=0}^\infty$ is decreasing, $\{x^k\}_{k=0}^\infty$ is bounded, and as $k \to \infty$, we have

$$
\|U x^k - x^k\| \to 0, \tag{30}
$$

and

$$
\|S_i x^k - S_{i-1} x^k\| \to 0, \tag{31}
$$

for all $i = 1, 2, \ldots, m$. Let $x^* \in \mathcal{H}$ be a weak cluster point of $\{x^k\}_{k=0}^\infty$ and $\{x^{n_k}\}_{k=0}^\infty \subset \{x^k\}_{k=0}^\infty$ be a subsequence which converges weakly to x^*.

(i) Suppose that $U - I$ is demiclosed at 0. Condition (30) yields that $x^* \in \operatorname{Fix} U$. The convergence of the whole sequence $\{x^k\}_{k=0}^\infty$ to x^* follows now from [BB96, Theorem 2.16 (ii)].

(ii) Suppose that $U_i - I$ are demiclosed at 0, for all $i = 1, 2, \ldots, m$. Condition (31) for $i = 1$ yields

$$
\|U_1 x^{n_k} - x^{n_k}\| = \|S_1 x^{n_k} - S_0 x^{n_k}\| \to 0 \text{ as } k \to \infty. \tag{32}
$$

Due to demiclosedness of $U_1 - I$ at 0, we have that $U_1 x^* = x^*$. Since

$$
\|(U_1 x^{n_k} - U_1 x^*) - (x^{n_k} - x^*)\| = \|U_1 x^{n_k} - x^{n_k}\| \to 0 \text{ as } k \to \infty, \tag{33}
$$

and $x^{n_k} \rightharpoonup x^*$, as $k \to \infty$, we have that $U_1 x^{n_k} \rightharpoonup U_1 x^* = x^*$ as $k \to \infty$. Since $U_2 - I$ is demiclosed at 0, condition (31) for $i = 2$ implies that $U_2 x^* = x^*$. In a similar way we obtain that $U_i x^* = x^*$ for $i = 3, 4, \ldots, m$. Therefore,

$$
U x^* = S_m x^* = U_m U_{m-1} \cdots U_1 x^* = x^*. \tag{34}
$$

We conclude that the subsequence $\{x^{n_k}\}_{k=0}^\infty$ converges weakly to a fixed point of the operator U. The weak convergence of the whole sequence $\{x^k\}_{k=0}^\infty$ to $x^* \in \operatorname{Fix} U$ follows now from [BB96, Theorem 2.16 (ii)]. ∎

Remark 10 Note that

$$\sum_{i=1}^{m}\langle y^i + y^{i+1} + \cdots + y^m, y^i\rangle = \sum_{i=1}^{m}\langle y^1 + y^2 + \cdots + y^i, y^i\rangle. \qquad (35)$$

Therefore, the step size $\sigma_{\max}(x)$ for $x \notin \operatorname{Fix} U$, given by (23), can be equivalently written as

$$\sigma_{\max}(x) = \frac{\sum_{i=1}^{m}\langle S_i x - x, S_i x - S_{i-1}x\rangle}{\|Ux - x\|^2}. \qquad (36)$$

Furthermore, by (22), we have

$$\sigma_{\max}(x) = \frac{\|Ux - x\|^2 + \sum_{i=1}^{m}\|S_i x - S_{i-1}x\|^2}{2\|Ux - x\|^2}. \qquad (37)$$

Remark 11 Theorem 9 remains true if we suppose that the step size function σ is an arbitrary function satisfying the inequalities

$$\alpha\frac{\sum_{i=1}^{m}\|S_i x - S_{i-1}x\|^2}{\|Ux - x\|^2} \le \sigma(x) \le \sigma_{\max}(x), \qquad (38)$$

where $\alpha \in (0, 1/2]$ and $x \notin \operatorname{Fix} U$. The existence of such a function follows from Lemma 7.

Remark 12 Even if we take $\lambda = 2$, the generalized relaxation $U_{\sigma_{\max},\lambda}$ needs not to be an extrapolation of U because the inequality $\sigma_{\max}(x) \ge 1/2$ is not guaranteed. We only know that $\sigma_{\max}(x) \ge 1/(2m)$ (see (24)). It is known, however, that U is $1/m$-SQNE (see [YO04, Proposition 1(d)(iii)]), consequently, it is a $2m/(m + 1)$-relaxed cutter (see Lemma 4). Therefore, $U_{\frac{1+m}{2m}}$ is a cutter. Since U_σ is a cutter (see Lemma 8), one can easily check that U_σ is also a cutter, where

$$\sigma = \max((1 + m)/(2m), \sigma_{\max}). \qquad (39)$$

Therefore, $U_{\sigma,\lambda}$ with $\lambda \in (2m/(m + 1), 2]$ is an extrapolation of U and one can expect that an application of $U_{\sigma,\lambda}$ leads in practice to a local acceleration of the convergence of sequences generated by the recurrence $x^{k+1} = U_{\sigma,\lambda_k}x^k$, in comparison to the classical cyclic cutter method $x^{k+1} = Ux^k$. Note, that if we apply the step size σ given by (39), then Theorem 9 remains true, because $\|U_{\sigma_{\max}}x - x\| \le \|U_\sigma x - x\|$.

4 Applications

In this section we show how our general results unify, generalize and extend several existing local acceleration schemes. We consider two kinds of operators U_i satisfying the assumptions of Section 3: (i) U_i is the metric projection onto a closed and convex set $C_i \subset \mathcal{H}$, and (ii) U_i is a subgradient projection onto a set $C_i = \{x \in \mathcal{H} \mid c_i(x) \leq 0\}$, where $c_i : \mathcal{H} \to \mathbb{R}$ is a continuous and convex function, $i = 1, 2, \ldots, m$. Recall that $\operatorname{Fix} P_{C_i} = C_i$ and that P_{C_i} is a firmly nonexpansive operator (see, e.g., Zarantonello [Zar71, Lemma 1.2]), consequently, P_{C_i} is both nonexpansive (by the Cauchy–Schwarz inequality) and a cutter operator (see [BB96, Lemma 2.4 (ii)]). Furthermore, Opial's demiclosedness principle yields that the operator $P_{C_i} - I$ is demiclosed at 0 (see [Opi67, Lemma 2]).

First we consider the case $U_i = P_{C_i}$. If $C_i \subseteq \mathcal{H}$ is a general closed and convex subset, one can apply Theorem 9 directly. If C_i is a hyperplane, i.e., $C_i = H(a^i, b_i) := \{x \in \mathcal{H} \mid \langle a^i, x \rangle = b_i\}$, where $a^i \in \mathcal{H}$, $a^i \neq 0$ and $b_i \in \mathbb{R}$, then

$$U_i := P_{C_i} x = x - \frac{\langle a^i, x \rangle - b_i}{\|a^i\|^2} a^i. \tag{40}$$

In this case an application of Theorem 9 will be presented in Subsection 4.1. If C_i is a half-space, $C_i = H_-(a^i, b_i) := \{x \in \mathcal{H} \mid \langle a^i, x \rangle \leq b_i\}$, then

$$U_i := P_{C_i} x = x - \frac{(\langle a^i, x \rangle - b_i)_+}{\|a^i\|^2} a^i, \tag{41}$$

where $\alpha_+ := \max(0, \alpha)$ for any real number α. In this case an application of Theorem 9 will be presented in Subsection 4.2.

Now consider the case, where U_i is a subgradient projection onto $C_i := \{x \in \mathcal{H} \mid c_i(x) \leq 0\}$, where $c_i : \mathcal{H} \to \mathbb{R}$ is a continuous and convex function. Then we have

$$U_i x = P_{c_i} x := \begin{cases} x - \dfrac{(c_i(x))_+}{\|g_i(x)\|^2} g_i(x), & \text{if } g_i(x) \neq 0, \\ x, & \text{if } g_i(x) = 0, \end{cases} \tag{42}$$

where $g_i(x) \in \partial c_i(x) := \{g \in \mathcal{H} \mid \langle g, y - x \rangle \geq c_i(y) - c_i(x), \text{ for all } y \in \mathcal{H}\}$ is a subgradient of the function c_i at the point x. It follows from the definition of the subgradient that U_i is a cutter, $i = 1, 2, \ldots, m$. Note that $\operatorname{Fix} U_i = C_i$, consequently,

$$\bigcap_{i=1}^m \operatorname{Fix} U_i = C \neq \emptyset. \tag{43}$$

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Suppose that the subgradients g_i are bounded on bounded subsets, $i \in I$ (this holds if, e.g., $\mathcal{H} = \mathbb{R}^n$, see [BB96, Corollary 7.9]). Then the operator $U_i - \mathrm{Id}$ is demiclosed at 0, $i \in I$. Indeed, let $x^k \rightharpoonup x^*$ and $\lim_{k\to\infty} \|U_i x^k - x^k\| = 0$. Then we have

$$\lim_{k\to\infty} \|U_i x^k - x^k\| = \lim_{k\to\infty} \frac{(c_i(x^k))_+}{\|g_i(x^k)\|} = 0. \tag{44}$$

The sequence $\{x^k\}_{k=0}^\infty$ is bounded due to its weak convergence. Since a continuous and convex function is locally Lipschitz-continuous, the subgradients sequence $\{g_i(x^k)\}_{k=0}^\infty$ is also bounded. Equality (44) implies now the convergence $\lim_{k\to\infty} c_i(x^k)_+ = 0$. Since c_i is weakly lower semi-continuous, we have $c_i(x^*) = 0$, i.e., $U_i - \mathrm{Id}$ is demiclosed at 0.

The local acceleration schemes of the cyclic projection and the cyclic subgradient projection methods, referred to in Subsections 4.1, 4.2 and 4.3 below, follow the same basic principle of the Dos Santos (DS) local acceleration principle referred to in Section 1. Namely, consider the line through two consecutive iterates of the cyclic method applied to a linear system of equations and find on it a point closest to the nonempty intersection of the feasibility problem sets.

4.1 Local acceleration of the sequential Kaczmarz method for linear equations

Consider a consistent system of linear equations

$$\langle a^i, x \rangle = b_i, \quad i = 1, 2, \ldots, m, \tag{45}$$

where $a^i \in \mathcal{H}$, $a^i \neq 0$ and $b_i \in \mathbb{R}$, for all $i = 1, 2, \ldots, m$. Let $C_i = H(a^i, b_i) := \{x \in \mathcal{H} \mid \langle a^i, x \rangle = b_i\}$, $U_i = P_{C_i}$ be defined by (40), $i = 1, 2, \ldots, m$, and assume that $C := \bigcap_{i=1}^m C_i \neq \emptyset$. Denote $U := U_m U_{m-1} \cdots U_1$. The operator U is nonexpansive as a composition of nonexpansive operators. The *Kaczmarz method* for solving a system of equations (45) has the form

$$x^{k+1} = U x^k, \tag{46}$$

where $x^0 \in \mathcal{H}$ is arbitrary. The method was introduced by Kaczmarz in 1937 for a square nonsingular system of linear equations in $\mathbb{R}^n$ (see [Kac37]). It is well-known that for any starting point $x^0 \in \mathcal{H}$ any sequence generated by the recurrence (46) converges strongly to a fixed point of the operator U (see [GPR67, Theorem 1]). The local acceleration scheme for this method (46) which we propose here is a special case of the iterative procedure $x^{k+1} =$

$U_{\sigma,\lambda_k}x^k$, where the operator $U_{\sigma,\lambda}$ is defined by (14), the step size function $\sigma : \mathcal{H} \to (0, +\infty)$ is defined by (36) and the relaxation parameter is $\lambda \in (0, 2)$. Since

$$
\begin{aligned}
\sum_{i=1}^{m} \langle S_i x - x, S_i x - S_{i-1} x \rangle &= \sum_{i=1}^{m} \langle S_i x - x, U_i u^{i-1} - u^{i-1} \rangle \\
&= \sum_{i=1}^{m} \langle S_i x - x, \frac{b_i - \langle a^i, u^{i-1} \rangle}{\|a^i\|^2} a^i \rangle \\
&= \sum_{i=1}^{m} \frac{b_i - \langle a^i, u^{i-1} \rangle}{\|a^i\|^2} (\langle S_i x, a^i \rangle - \langle x, a^i \rangle) \\
&= \sum_{i=1}^{m} \frac{b_i - \langle a^i, u^{i-1} \rangle}{\|a^i\|^2} (b_i - \langle a^i, x \rangle), \qquad (47)
\end{aligned}
$$

we obtain the following form for the step size

$$
\sigma_{\max}(x) = \frac{\displaystyle\sum_{i=1}^{m} (b_i - \langle a^i, x \rangle) \frac{b_i - \langle a^i, u^{i-1} \rangle}{\|a^i\|^2}}{\|Ux - x\|^2}, \qquad (48)
$$

where $x \notin \operatorname{Fix} U$. This step size is equivalent to those in (23), (36) and (37).

Corollary 13 *Let $U := U_m U_{m-1} \cdots U_1$, where U_i is given by (40), $i = 1, 2, \ldots, m$, the sequence $\{x^k\}_{k=0}^{\infty} \subset \mathcal{H}$ be defined by the recurrence*

$$
x^{k+1} = U_{\sigma,\lambda_k} x^k = x^k + \lambda_k \sigma(x^k)(Ux^k - x^k) \qquad (49)
$$

for all $x^k \notin \operatorname{Fix} U$, where $x^0 \in \mathcal{H}$ is arbitrary, $\sigma := \sigma_{\max}$ is defined by (48) and $\lambda_k \in [\varepsilon, 2 - \varepsilon]$ for an arbitrary constant $\varepsilon \in (0, 1)$. Then $\{x^k\}_{k=0}^{\infty}$ converges weakly to a solution of the system (45).

One can prove even the strong convegence of $\{x^k\}_{k=0}^{\infty}$ in Corollary 13. The proof will be presented elsewhere. As mentioned in Remark 12, Corollary 13 remains true, is we set $\sigma := \max((m + 1)/(2m), \sigma_{\max})$. In this case, the method is an extrapolation of the Kaczmarz method if $\lambda_k \in (2m/(m+1), 2]$.

Remark 14 When C_i are hyperplanes and $U_i = P_{C_i}$, $i = 1, 2, \ldots, m$, then, for any $u \in \mathcal{H}$ and for any $z \in C_i$, we have

$$
\langle U_i u - u, z - u \rangle = \|U_i u - u\|^2, \qquad (50)
$$

$i = 1, 2, \ldots, m$. Therefore, it follows from the proof of Lemma 7 that the first inequality in (19) is, actually, an equality. Consequently, we have equality in (25) and the operator U_σ defined by (14) with the step size $\sigma := \sigma_{\max}$ given by (48) has the property

$$\langle U_\sigma x - x, z - U_\sigma x \rangle = 0, \tag{51}$$

for all $z \in C$, or, equivalently,

$$\langle U_\sigma x - x, z - x \rangle = \|U_\sigma x - x\|^2. \tag{52}$$

This yields the following nice property of the operator U_σ

$$\|U_\sigma x - z\| = \min\{\|x + \alpha(Ux - x) - z\| \mid \alpha \in \mathbb{R}\}, \tag{53}$$

for all $z \in C$. We can expect that this property leads in practice to a local acceleration of the convergence to a solution of the system (45), of sequences generated by the recurrence $x^{k+1} = U_{\sigma,\lambda_k} x^k$, where the operator $U_{\sigma,\lambda}$ is defined by (14) with the step size given by (48) and $\lambda_k \in [\varepsilon, 2 - \varepsilon]$ for an arbitrary constant $\varepsilon \in (0, 1)$.

4.2 Local acceleration of the sequential cyclic projection method for linear inequalities

Given a consistent system of linear inequalities

$$\langle a^i, x \rangle \le b_i, \quad i = 1, 2, \ldots, m, \tag{54}$$

where $a^i \in \mathcal{H}$, $a^i \ne 0$ and $b_i \in \mathbb{R}$. Let $C_i = H_-(a^i, b_i) := \{x \in \mathcal{H} \mid \langle a^i, x \rangle \le b_i\}$ and $U_i = P_{C_i}$, $i = 1, 2, \ldots, m$, be defined by (41). Assume that $C = \bigcap_{i=1}^m C_i \ne \emptyset$. Denoting $S_0 := I$, $S_i := U_i U_{i-1} \cdots U_1$, $i = 1, 2, \ldots, m$, we have $U = S_m$. By the nonexpansiveness of U, Theorem 9(i) guarantees the weak convergence of sequences generated by the recurrence $x^{k+1} = U_{\sigma,\lambda_k} x^k$, where the starting point $x^0 \in \mathcal{H}$ is arbitrary, the step size function is given by (36) and the relaxation parameter $\lambda_k \in [\varepsilon, 2 - \varepsilon]$ for an arbitrary constant $\varepsilon \in (0, 1)$. Since U is nonexpansive, we would rather prefer to apply the step size $\sigma(x)$ given by (37), and, as explained in Remark 11, the convergence also holds in this case. Note that property (53) does not hold in general for a system of linear inequalities. An application of the step size $\sigma(x)$ given by (36) ensures that U_σ is a cutter but does not guarantee that $\|U_\sigma x - z\| \le \|Ux - z\|$ for any $z \in \bigcap_{i=1}^m C_i$, unless $\sigma(x) \ge 1$.

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The step size $\sigma(x)$ can be also presented, equivalently, in the following way. Denote, as before, $u^i = S_i x$, i.e., $u^0 = x$, $u^i = U_i u^{i-1}$, $i = 1, 2, \ldots, m$. Of course, $u^m = Ux$ thus,

$$Ux = u^0 + \sum_{i=1}^{m}(u^i - u^{i-1}) = x - \sum_{i=1}^{m} \frac{(\langle a^i, u^{i-1}\rangle - b_i)_+}{\|a^i\|^2} a^i. \qquad (55)$$

For any $z \in C$ we have,

$$\langle Ux - x, z - x\rangle = -\left\langle \sum_{i=1}^{m} \frac{(\langle a^i, u^{i-1}\rangle - \beta_i)_+}{\|a^i\|^2} a^i, z - x \right\rangle$$

$$= \sum_{i=1}^{m}(\langle a^i, x\rangle - \langle a^i, z\rangle)\frac{(\langle a^i, u^{i-1}\rangle - \beta_i)_+}{\|a^i\|^2}$$

$$\geq \sum_{i=1}^{m}(\langle a^i, x\rangle - b_i)\frac{(\langle a^i, u^{i-1}\rangle - b_i)_+}{\|a^i\|^2}, \qquad (56)$$

i.e.,

$$\langle Ux - x, z - x\rangle \geq \sum_{i=1}^{m}(\langle a^i, x\rangle - b_i)\frac{(\langle a^i, u^{i-1}\rangle - b_i)_+}{\|a^i\|^2}. \qquad (57)$$

The same inequality can be obtained by an application of the first inequality in (19). We can apply the above inequality to define the following step size

$$\sigma_{\max}(x) = \frac{\displaystyle\sum_{i=1}^{m}(\langle a^i, x\rangle - \beta_i)\frac{(\langle a^i, u^{i-1}\rangle - b_i)_+}{\|a^i\|^2}}{\|Ux - x\|^2}, \qquad (58)$$

where $x \notin \operatorname{Fix} U$. As mentioned in Remark 12, an application of the following step size

$$\sigma(x) = \max\left(\frac{m+1}{2m}, \sigma_{\max}(x)\right), \qquad (59)$$

where $x \notin \operatorname{Fix} U$ (compare with Remark 12), seems reasonable.

Corollary 15 *Let $U := U_m U_{m-1} \cdots U_1$, where U_i is given by (41), $i = 1, 2, \ldots, m$, the sequence $\{x^k\}_{k=0}^{\infty} \subset \mathcal{H}$ be defined by the recurrence*

$$x^{k+1} = x^k + \lambda_k \sigma(x^k)(Ux^k - x^k), \qquad (60)$$

for all $x^k \notin \operatorname{Fix} U$, where $x^0 \in \mathcal{H}$ is arbitrary, $\sigma := \sigma_{\max}$ is given by (58) and $\lambda_k \in [\varepsilon, 2 - \varepsilon]$ for an arbitrary constant $\varepsilon \in (0, 1)$. Then $\{x^k\}_{k=0}^{\infty}$ converges weakly to a solution of the system (54).

One can prove even the strong convegence of $\{x^k\}_{k=0}^\infty$ in Corollary 15. The proof will be presented elsewhere. As mentioned in Remark 12, Corollary 15 remains true, is we set $\sigma := \max((m+1)/(2m), \sigma_{\max})$.

4.3 Local acceleration of the sequential cyclic subgradient projection method

Let $c_i : \mathcal{H} \to \mathbb{R}$ be continuous and convex functions, $i = 1, 2, \ldots, m$ and consider the following system of convex inequalities

$$c_i(x) \le 0, \, i = 1, 2, \ldots, m. \tag{61}$$

Denote $C_i := \{x \in \mathcal{H} \mid c_i(x) \le 0\}$, $i = 1, 2, \ldots, m$, and assume that $C = \bigcap_{i=1}^m C_i \ne \emptyset$. Define the operators $U_i : \mathcal{H} \to \mathcal{H}$ by (42), $i = 1, 2, \ldots, m$. Letting $S_0 := I$, $S_i := U_i U_{i-1} \cdots U_1$, $u^i = S_i x$ and $y^i = u^i - u^{i-1}$ for all $i = 1, 2, \ldots, m$, we have, by Remark 10,

$$
\begin{aligned}
\sum_{i=1}^m \langle S_m x - S_{i-1}x, S_i x - S_{i-1}x \rangle &= \sum_{i=1}^m \langle S_i x - x, S_i x - S_{i-1}x \rangle \\
&= \sum_{i=1}^m \langle U_i u^{i-1} - x, U_i u^{i-1} - u^{i-1} \rangle \\
&= -\sum_{i=1}^m \frac{(c_i(u^{i-1}))_+}{\|g_i(u^{i-1})\|^2} \langle U_i u^{i-1} - x, g_i(u^{i-1}) \rangle,
\end{aligned}
\tag{62}
$$

and the step size given by (23) can be written in the form

$$
\sigma_{\max}(x) = -\frac{\displaystyle\sum_{i=1}^m \frac{(c_i(u^{i-1}))_+}{\|g_i(u^{i-1})\|^2} \langle U_i u^{i-1} - x, g_i(u^{i-1}) \rangle}{\|Ux - x\|^2}, \tag{63}
$$

where $x \notin \operatorname{Fix} U$.

Let $U_{\sigma,\lambda}$ be defined by (14), where $\lambda \in (0,2)$ and $\sigma(x)$ is given by (63). We obtain the following corollary.

Corollary 16 *Let* $U := U_m U_{m-1} \cdots U_1$, *where* U_i *is given by* (42), $i = 1, 2, \ldots, m$. *Let the sequence* $\{x^k\}_{k=0}^\infty \subset \mathcal{H}$ *be defined by the recurrence*

$$x^{k+1} = U_{\sigma,\lambda_k} x^k = x^k + \lambda_k \sigma(x^k) \frac{Ux^k - x^k}{\|Ux^k - x^k\|^2}, \tag{64}$$

for all $x^k \notin \operatorname{Fix} U$, where $x^0 \in \mathcal{H}$ is arbitrary, $\sigma := \sigma_{\max}$ is given by (63), and $\lambda_k \in [\varepsilon, 2 - \varepsilon]$ for an arbitrary constant $\varepsilon \in (0, 1)$. Then $\{x^k\}_{k=0}^{\infty}$ converges weakly to an element of C.

As mentioned in Remark 12, Corollary 16 remains true, is we set $\sigma := \max((m + 1)/(2m), \sigma_{\max})$.

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