Contents

Chapter 1

Binary Steering of Non-Binary Iterative Algorithms

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ABSTRACT
Existing algorithms for binary image reconstruction that can handle two-dimensional problems are mainly of a combinatorial nature. This has, so far, hindered their direct application to fully three-dimensional binary problems. This chapter proposes a steering scheme by which non-binary iterative reconstruction algorithms can be steered towards a binary solution of a binary problem. Experimental studies show the viability of this approach.

1.1 Introduction: Problem definition, approach and motivation

Let $Ax = b$ be a system of linear equations representing the fully discretized model of a two-dimensional image reconstruction from projections problem. The vector $x = (x_j)_{j=1}^n \in \mathbf{R}^n$, in the n-dimensional Euclidean space, is the *image vector* whose j-th component x_j has the value of the uniform grayness at the j-th pixel. The vector $b = (b_i)_{i=1}^m \in \mathbf{R}^m$ is the *measurements vector* whose i-th component b_i is the value of the i-th line integral through the unknown image. The $m \times n$ *projection matrix* A is a 0-1 matrix having its i-th row and j-th column element a_j^i equal to zero if the i-th ray does not intersect the j-th pixel, and equal to one if it does. The *Binary Reconstruction Problem* is to find a 0-1 vector x^* that is an acceptable approximation to a solution of the system $Ax = b$.

There is a large body of literature on this problem and renewed current interest, see, e.g., Herman and Kuba [1]. Due to their mainly combinatorial nature, existing algorithms for this problem, such as Chang [2], do not lend themselves to extension to three-dimensional problems of binary

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reconstruction.

The adoption of *non-binary* iterative image reconstruction algorithms such as ART, MART, EM, and others (see, e.g., Bauschke and Borwein [3], Censor and Zenios [4], Byrne [5, 6] or Herman [7]) to binary reconstruction is problematic because such algorithms do not preserve the binary nature of the iterates even if initialized at a binary vector x^0. This difficulty exists in Vardi and Lee [8, Section I(A), point 6], and in Fishburn *et al.* [9] where simple thresholding is used.

On the other hand, the temptation to apply non-binary algorithms exists because there is an abundance of such algorithms which have proven their usefulness in non-binary image reconstruction problems. Moreover, they could, technically speaking, be extended without hindrance to three-dimensional problems, which are harder then two-dimensional problems from the point of view of computational complexity and thus cause combinatorial algorithms to be much more involved, see, e.g., Gritzmann *et al.* [10]. Therefore, if the route that we propose here will be successful for two-dimensional binary reconstruction problems then it will also be immediately applicable to three-dimensional problems.

Our approach to rely on non-binary iterative image reconstruction algorithms (for fully discretized image reconstruction problems) has its roots in, and is inspired by, the work of Herman [11]. The *binary steering mechanism* we propose here extends and replaces the ad hoc steps devised there by Herman.

Non-binary iterative reconstruction algorithms of various kinds (asymptotically) solve (depending on the relevant solution concept), or find good approximate solutions of, the linear system of equations of the form $Ax = b$. Several of these algorithms have been shown to perform very efficiently, handle linear inequalities, treat nonnegativity constraints, generate acceptable approximations even in the inconsistent case (i.e., when there exists no nonnegative solution of the system of equations), lend themselves to parallel computations, or have other favorable features.

These algorithms can solve fully discretized real three-dimensional image reconstruction problems because such problems can be modeled into, admittedly much bigger, systems of linear equations.

We present a *mathematical mechanism* that, when used in conjunction with any non-binary iterative reconstruction algorithm, will steer to acceptable approximate solutions of the binary reconstruction problem.

1.2 The steering mechanism

The iterative non-binary reconstruction algorithms that are considered here are of the general form described in Fig. 1.1.

In a typical iterative step the algorithm first calculates a quantity called

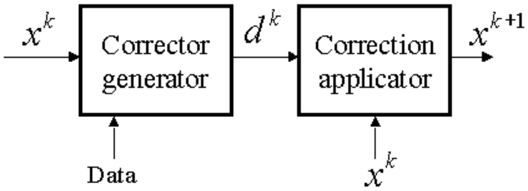

FIGURE 1.1. General structure of an iterative step of a non-binary iterative image reconstruction algorithm.

corrector which is then applied to the current iterate. In sequential reconstruction algorithms like ART, ART2, ARM, ART3, MART, etc. (see, e.g., Herman [12], Censor and Zenios [4]), the corrector d^k is a vector whose components are applied to the current iterate x^k in one of several possible ways (addition in ART, componentwise multiplication in MART, etc.) to obtain the next iterate x^{k+1}. The proposed *binary steering mechanism* consists of two additional operations which we describe next.

Given a real number x and two real parameters α and β such that $0 \leq \alpha < \beta \leq 1$ we define $\tilde{x}$ by

$$\tilde{x} = \begin{cases} 0, & \text{if } x \leq \alpha, \\ 1, & \text{if } x \geq \beta, \\ x, & \text{otherwise}, \end{cases} \tag{1.1}$$

and we say that $\tilde{x}$ is the *(partial) binarization of x with respect to the pair* (α, β). Applying this notion to a sequence of vectors leads to the next definition.

Definition 1.1. *Let $\{x^k\}_{k\geq 0}$ be a sequence of vectors $x^k = (x_j^k)_{j=1}^n \in \mathbf{R}^n$ and let $\alpha = \{\alpha_k\}_{k\geq 0}$, $\beta = \{\beta_k\}_{k\geq 0}$ and $\{t_k\}_{k\geq 0}$ be three real sequences such that $0 \leq \alpha_k < t_k$, $\alpha_k < \alpha_{k+1}$, $t_k < \beta_k \leq 1$, and $\beta_{k+1} < \beta_k$, for $k \geq 0$, where t_k is a threshold at the given iteration k. The sequence $\{\tilde{x}^k\}_{k\geq 0}$ defined, for $k \geq 0$ and $j = 1, 2, \ldots, n$, by*

$$\tilde{x}_j^k = \begin{cases} 0, & \text{if } x_j^k \leq \alpha_k, \\ 1, & \text{if } x_j^k \geq \beta_k, \\ x_j^k, & \text{otherwise}, \end{cases} \tag{1.2}$$

is called the (partial) sequential binarization of $\{x^k\}_{k\geq 0}$ with respect to the pair of sequences (α, β) and the threshold sequence $\{t_k\}_{k\geq 0}$.

As will be seen below, we binarize each iterate x^k prior to feeding it to the non-binary iterative algorithm, of the form of Fig. 1.1, at hand. After the iteration has been performed, a conflict might arise between x^k and the output y^k of the non-binary iterative algorithm (see Fig. 1.2). The meaning of the term *conflict* here and the manner in which this conflict is dealt with become clear from the next definition.

Definition 1.2. *Given two real numbers x and y, two real parameters α and β such that $0 \leq \alpha < t$ and $t < \beta \leq 1$, where t is a given threshold, and a fixed ϵ, $0 < \epsilon < 0.1$, we define z by*

$$z = \begin{cases} t - \epsilon, & \text{if } x \leq \alpha \text{ and } y \geq t, \\ t + \epsilon, & \text{if } x \geq \beta \text{ and } y \leq t, \\ y, & \text{otherwise}, \end{cases} \tag{1.3}$$

and we say that z settles the conflict between x and y with respect to the pair (α, β), the threshold t, and ϵ.

Actually we use this notion for sequences via the following definition.

Definition 1.3. *Let there be given two vector sequences $\{x^k\}_{k \geq 0}$ and $\{y^k\}_{k \geq 0}$, and two real sequences $\alpha = \{\alpha_k\}_{k \geq 0}$ and $\beta = \{\beta_k\}_{k \geq 0}$, a sequence $\{t_k\}_{k \geq 0}$ of threshold values, having the same properties as in Definition 1.1, and a fixed ϵ with $0 < \epsilon < 0.1$. The sequence $\{z^k\}_{k \geq 0}$, defined for $k \geq 0$ and $j = 1, 2, \ldots, n$, by*

$$z_j^k = \begin{cases} t_k - \epsilon, & \text{if } x_j^k \leq \alpha_k \text{ and } y_j^k \geq t_k, \\ t_k + \epsilon, & \text{if } x_j^k \geq \beta_k \text{ and } y_j^k \leq t_k, \\ y, & \text{otherwise}, \end{cases} \tag{1.4}$$

is said to settle sequentially the conflict between $\{x^k\}_{k \geq 0}$ and $\{y^k\}_{k \geq 0}$ with respect to the pair (α, β), the threshold sequence $\{t_k\}_{k \geq 0}$, and ϵ.

With the above definitions we describe our proposed steering mechanism which is depicted in Fig. 1.2, here the two middle boxes with the corrector generator and the correction applicator are the same as those in Fig. 1.1. The steering mechanism consist of adding to any such non-binary algorithm the *Binarizer* and the *Conflict settler* as explained next.

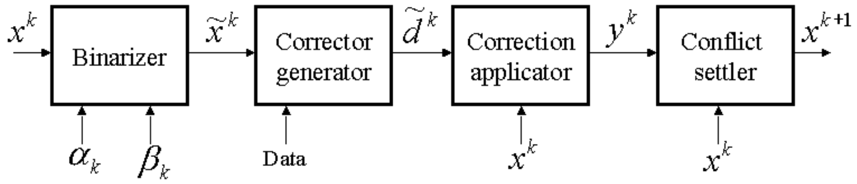

FIGURE 1.2. Binary steering of the non-binary algorithm is accomplished by additional operations of binarization and conflict settlement.

Each iteration of the overall process begins with a *(partial) binarization* of the current iterate x^k to form $\tilde{x}^k$. As iterations proceed the values α_k keep increasing and the values of β_k decrease so that more and more components fit into the desired binary 0-1 nature of the vector. The corrector $\tilde{d}^k$ is based on $\tilde{x}^k$, not on the original x^k, but it is applied by the *correction applicator* to x^k itself (not to $\tilde{x}^k$). If the resulting y^k has a component y_j^k which is larger or equal to the current threshold value t_k while its previous value x_j^k was below α_k then we say that there is a conflict and we prefer not to make a binary decision about this component but rather "settle the conflict" by allowing x_j^k to be only as much as $t_k - \epsilon$. A similar argument explains the rest of (1.4).

When a predetermined iteration index K has been reached or the iterations are stopped at K due to some other stopping criterion then a final thresholding is used to define the final (approximate) solution $x^* = (x_j^*)_{j=1}^n$ by

$$x_j^* = \begin{cases} 0, & \text{if } x_j^K \leq 0.5, \\ 1, & \text{if } x_j^K > 0.5. \end{cases} \tag{1.5}$$

1.3 Experimental study

In the experimental results presented here we use a fixed threshold value $t_k = 0.5$, for all $k \geq 0$, and the value $\epsilon = 0.05$. We constructed the sequence α_k by the formula $\alpha_k = (k/K)t_k$, where k is the iteration index, K is the predetermined number of iterations at which the reconstruction is stopped, and we define $\beta_k = 1 - (k/K)(1 - t_k)$, for $k \geq 0$. The non-binary iterative reconstruction algorithm that we use is the fully simultaneous Cimmino algorithm; see, e.g., Gastinel [13] or Censor and Zenios [4]. Starting from an arbitrary $x^0 \in \mathbf{R}^n$, given a current iterate x^k, Cimmino's algorithm calculates the next iterate x^{k+1} by

$$x^{k+1} = x^k + \lambda_k \left(\sum_{i=1}^{m} w_i^k \, \frac{b_i - \langle a^i, x^k \rangle}{\|a^i\|^2} \, a^i \right). \tag{1.6}$$

In this formula $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ are the Euclidean norm and the inner product in $\mathbf{R}^n$, respectively, m is the number of views, a^i is the i-th column of A^T (the transpose of A), λ_k are the relaxation parameters, and w_i^k are positive iteration dependent weights which must sum up (over i) to one, for every $k \geq 0$.

We employed this algorithm with unity relaxation, i.e., $\lambda_k = 1$, for all $k \geq 0$, with equal and constant weights, meaning that $w_i^k = 1/m$ for all i and all k, and using a uniform starting image x^0, for which for any given view (projection angle) v

$$\sum_{j=0}^{n} x_j^0 \;=\; \sum_{i \in I_v} \langle a^i, x^0 \rangle \;, \tag{1.7}$$

where I_v is a set of all projection lines from the view v.

For testing our binary steering method with Cimmino's reconstruction algorithm, under the above described circumstances, we used the three test phantoms (see Figs. 1.3, 1.4 and 1.5, top left images) of Fishburn *et al.* [9]. These same phantoms were also used by Vardi and Lee [8], by Salzberg, Rivera-Vega and Rodriguez [14], by Gritzmann *et al.* [10], and by Patch [15].

We demonstrate the performance of the binary steering method by showing along with each of the three phantoms its reconstructions using 2, 3 and 4 views. Two views are the two orthogonal views along the Cartesian axes in the plane, the three views include additionally a diagonal view at 45° (view of the direction $(1, 1)$ in [9]) and the four views include also another diagonally oriented view at 135°. The three view run is employing the same view directions as used by Fishburn *et al.* [9].

The only reconstruction parameter which is being changed in the tests presented here is the number of iterations, where an iteration is counted as a complete sweep through all equations. The number of iterations was chosen so as to minimize the data error e_p, i.e., to minimize the sum of the absolute values of the differences between the line sums in the phantom and the reconstructed image. Following are the observations based on our tests.

In the tests using two views the data error dropped quickly down to a certain value already after a small number of iterations. Top right images in Figs. 1.3, 1.4 and 1.5 show, by way of example, the reconstructions using 25, 25 and 200 iterations, respectively, having data errors 16, 28 and 34, respectively. The number of image errors e_i, i.e., the number of locations at which the reconstructed image disagrees with the phantom, is 34, 50 and 96, respectively. Further decrease of the data error was (for the given values of our reconstruction parameters: ϵ, and sequences λ_k, t_k, α_k, β_k) only very slow when increasing the number of iterations. For example, it took as much as $90,000$ iterations to decrease the data errors to 8, 4, and 12, respectively. The image errors at the same time slightly increased to the values 48, 70 and 110, respectively.

In the tests using three views we needed 220 and 70 iterations to find a solution for the phantoms 1 and 2, respectively, (see bottom right images in Figs. 1.3 and 1.4; note that the solution we obtained for phantom 1 differs from the phantom by the switching chain of length six). Since phantom 3 is substantially more difficult to reconstruct, it needed more iterations. The data error for it stabilized at about $3,000$ iterations (see Fig. 1.5, bottom right; $e_p = 6$, $e_i = 36$). To find a solution free of data errors it took as much as $500,000$ iterations for our choice of parameters. For other reconstruction parameters the exact solution might be obtained earlier. This solution differes from the phantom at 58 locations, representing a set of switching chains. It is as good as any other solution based on the data alone. To be able to reconstruct the particular solution of the original phantom we would need to utilize some additional prior information on the reconstructed images (see, e.g., chapter by Matej, Vardi, Herman and Vardi in this volume).

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FIGURE 1.3. Phantom 1 (top left) and its reconstructions using 2 views (top right; $K = 25$, $e_p = 16$ and $e_i = 34$, where K is the number of iterations, e_p is the data error and e_i represents the number of places at which the reconstructed image disagrees with the phantom), 3 views (bottom right; $K = 220$, $e_p = 0$ and $e_i = 6$) and 4 views (bottom left; $K = 22$, $e_p = 0$ and $e_i = 0$); "." and "1" represent the values zero and one, respectively, in the phantom and at the correct locations in the reconstructions; "-" and "*" represent incorrect values of zero and one, respectively, in the reconstructions; the "o"s in the phantom (top left) show directions of the line sums in the three views case.

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FIGURE 1.4. Phantom 2 (top left) and its reconstructions using 2 views (top right; $K = 25$, $e_p = 28$ and $e_i = 50$), 3 views (bottom right; $K = 70$, $e_p = 0$ and $e_i = 0$) and 4 views (bottom left; $K = 22$, $e_p = 0$ and $e_i = 0$); "." and "1" represent values zero and one, respectively, in the phantom and at the correct locations in the reconstructions; "-" and "*" represent incorrect values of zero and one, respectively, in the reconstructions.

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18     ..1111111111111111111111111...1111111111..    ..11111111111111111111111-1*..1111111111..
19     ...11111111111111111111111....1111111111..    ..*1111111111111111111-111....1111111111..
20     ...11111111111111111111111....1111111111..    ...11111111111111111111111....1111111111..
21     ...1111111111111111111111......111111111..    ..*11111111111111111111-1......111111111..
22     ...111111111111111111111.......111111111..    ...111111111111111111111.......111111111..
23     ..*111111111111111111-1.........11111111..    ...11111111111111111111.........11111111..
24     ...1111111111111....11..........11111111..    ...11111-1111111.***1-..........11111111..
25     ....-....1111111...*.............1111111..    ....-...*1111111.................1111111..
26     .........1111111...................1111...    .........1111111...................1111...
27     .........111111......................1....    .........111111......................1....
28     .........111111...........................    ......*..111111...........................
29     .........111111...........................    ....*....1111-1...........................
30     .........111111...........................    .........111111...........................
31     ..........1..11...........................    ..........1..1-...........................
32     ..........1...............................    ..........1...............................
33     ..........1...............................    ..........1...............................
34     ..........1...............................    ..........1...............................
35     ..........................................    ..........................................
36     ..........................................    ..........................................
```

FIGURE 1.5. Phantom 3 (top left) and its reconstructions using 2 views (top right; $K = 200$, $e_p = 34$ and $e_i = 96$), 3 views (bottom right; $K = 3000$, $e_p = 6$ and $e_i = 36$) and 4 views (bottom left; $K = 775$, $e_p = 0$ and $e_i = 8$); "." and "1" represent values zero and one, respectively, in the phantom and at the correct locations in the reconstructions; "-" and "*" represent incorrect values of zero and one, respectively, in the reconstructions.

Finally, regarding the four views situation, we needed only 22, 22 and 775 iterations to find solutions (see bottom left images in Figs. 1.3, 1.4 and 1.5) for the given three phantoms, respectively. Note, that the solution for the third phantom differs from the phantom by a switching chain of length eight.

While adding more directions to the first two makes the discrete reconstruction problem "harder" from the computational complexity point of view, we experience in the computations better initial results. This is, of course, so because adding directions supplies the iterative algorithm with more information, thus enabling it to work "better".

Although the results of our preliminary computational experiments, presented in this section, are in no way exhausting, they clearly suggest that the underlying ideas of Herman [11], as refined and extended here, are a viable tool for binary steering of non-binary iterative reconstruction algorithms.

1.4 Conclusions

The method proposed in this paper is to reconstruct binary images by using non-binary iterative reconstruction algorithms in conjunction with an additional mechanism to steer the non-binary iterates towards an acceptable binary solution. The steering mechanism is independent of the particular choice of the non-binary reconstruction algorithm and the overall process can be applied to three-dimensional binary image reconstruction once posed as a system of linear equations. The numerical results obtained by our preliminary computational experimentations encourage us to continue this line of research.

Efforts need to be invested in studying the effects of different sequences α_k, β_k, t_k and other parameters involved in the binarization and conflict settlement operations. Different non-binary reconstruction algorithms need to be tried out within this methodology and full three-dimensional binary reconstruction problems must be solved with it to asses its efficiency. It should be possible to study the proposed scheme also from a mathematical point of view to determine bounds on the errors as functions of the parameters involved.

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