# "An Improved Method of Total Variation Superiorization Applied to Reconstruction in Proton Computed Tomography" by B. Schultze, Y. Censor, P. Karbasi, K.E. Schubert, and R.W. Schulte. Manuscript no. TMI-2018-1299
# SUPPLEMENTAL MATERIALS

## I. OTVS ALGORITHM

The pseudocode definition of the OTVS algorithm is written as follows:

set $k = 0$
set $\ell = 0$
set $\beta = 1$
set $x^k = \bar{x}$
**while** $k < K$ **do**
    set $v^k = \phi'(x^k)$
    set $loop = true$
    **while** $loop$ **do**
        set $z = x^k + \beta v^k$
        **if** $\phi(z) \leq \phi(x^k)$ **then**
            set $x^k = z$
            set $loop = false$
        **end if**
        set $\ell = \ell + 1$
        set $\beta = (\frac{1}{2})^\ell$     (originally $\beta \leftarrow \beta/2$)
    **end while**
    set $x^{k+1} = P_T(x^k)$
    set $k = k + 1$
**end while**

## II. SIMULATED CTP404 DATA SET

### A. Number of TVS steps ($N$)

The number of TV perturbations per feasibility-seeking iteration, $N$, was varied between 1 and 12 in increments of 1. Figure 1 shows the dependence of TV as a function of $N$ for each of the first four feasibility-seeking iterations with the TV reduction requirement excluded. As will be shown later, a similar pattern was observed with the TV reduction requirement included. The general effect of increasing $N$ was a reduction in TV that leveled off after $N \geq 5$ steps, as best seen in the $k = 1$ iteration plot (top left of Figure 1). An irregular oscillation in TV as a function of increasing $N$ appeared for $k \geq 2$ and increased in magnitude as the number of feasibility-seeking iterations, $k$, increased.

To determine whether the observed fluctuations were random, an analysis of 8 separate reconstructions with $N = 5$, $\alpha = 0.5$, and the TV reduction requirement excluded were

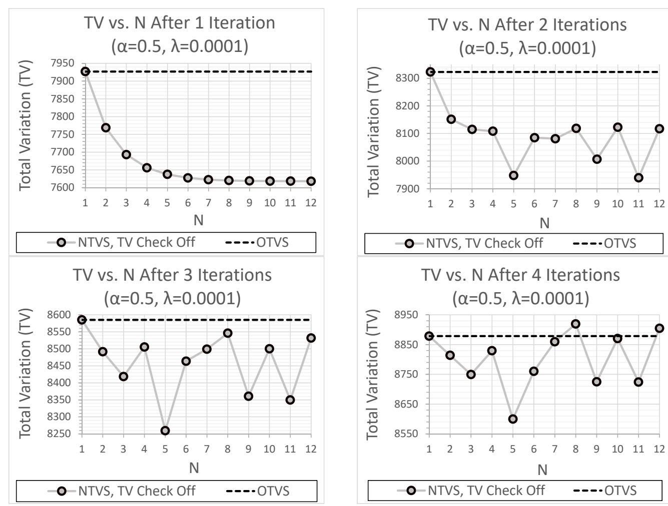
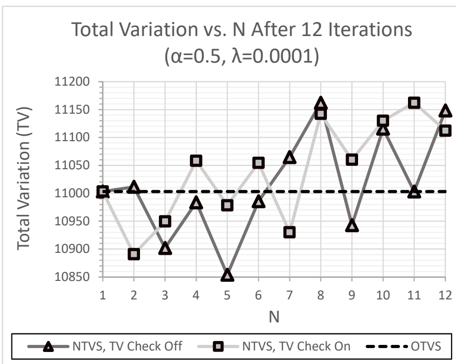



Fig. 1: TV as a function of $N$ after each of the first 4 feasibility-seeking iterations for the simulated CTP404 data set using OTVS and NTVS (TV reduction requirement excluded) with $\lambda = 0.0001$ and $\alpha = 0.5$.

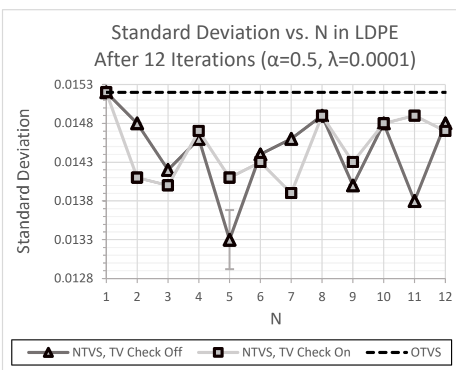
performed for $k = 12$ feasibility-seeking iterations. The random increase in $\ell$ between feasibility-seeking iterations was governed by a random number generator that was seeded with a value based on the Julian time at execution, resulting in an effectively random seeding of the random number generator. The standard deviation within the LDPE insert varied between reconstructions with a standard deviation of $\sigma_{LDPE} = 0.00038$ (shown as an error bar on the point at $N = 5$ in Figure 2(b)); similar variations were also seen in the ROI of the other materials. Note that the standard deviation obtained within the LDPE insert at $N = 5$ with the TV reduction requirement excluded was nearly $2\sigma_{LDPE}$ less than that obtained with the requirement included and just under $4\sigma_{LDPE}$ less than that obtained with OTVS. In addition, the standard deviation obtained with $N = 5$ was at least $1.5\sigma_{LDPE}$ less than that obtained with any other value of $N$. These differences are large enough to conclude that the observed fluctuation in standard deviation as a function of $N$ was not random.

For $3 \leq N \leq 6$, there was a benefit from NTVS compared to OTVS, which persisted throughout all twelve feasibility-seeking iterations (see Figure 2(a) and (b)). However, for $N \geq 7$ the benefits of NTVS were increasingly lost as $N$ and $k$ increased. This can be explained by the decreasing magnitude of TV reducing perturbations with increasing $N$ and the overall increase in TV from each feasibility-seeking iteration. Although not shown here, a similar dependence on $N$ and $k$ was seen for regional standard deviations. However, the benefit of NTVS in terms of standard deviation was consistently seen, including for $N \geq 7$, after twelve feasibility-seeking iterations (see, e.g., Figure 2(b)).

### *B. Inclusion/Exclusion of TV Reduction Requirement*




(a)



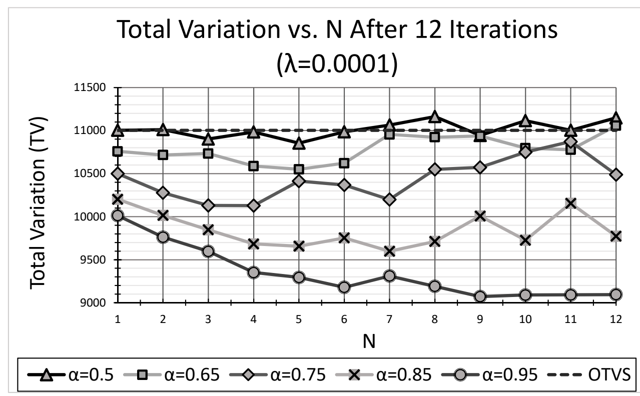

(b)

Fig. 2: (a) TV and (b) standard deviation (LDPE) as a function of $N$ after 12 feasibility-seeking iterations for the simulated CTP404 data set using OTVS and NTVS including and excluding the TV reduction requirement with $\lambda = 0.0001$ and $\alpha = 0.5$. The error bar at $N = 5$ denotes the variation in standard deviation ($\sigma = 0.00038$) between 8 repetitions of reconstruction with $N = 5$.

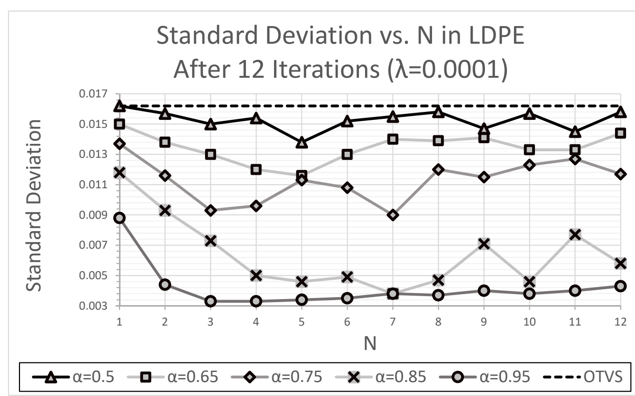
To determine if the exclusion of the TV reduction requirement in the definition of the NTVS algorithm (Appendix B) is an appropriate decision, reconstructions were also performed with a variation of the NTVS algorithm that included the TV reduction requirement; the definition of the algorithm used for these investigations is provided for reference at the end of Appendix B. Figures 2(a) and 2(b) show the comparison of TV and standard deviation, respectively, for OTVS and NTVS with relaxation parameter $\lambda = 0.0001$, median filter radius $r = 2$ applied to the initial iterate [1], and 12 feasibility-seeking iterations. In each plot, the results for NTVS with and without inclusion of the TV reduction requirement are shown as a function of $N$. The horizontal line corresponds to the result of OTVS ($N = 1$, $\alpha = 0.5$).

In the range of $3 \leq N \leq 6$, including the TV reduction requirement had practically no benefit, whereas its removal yields up to a 5.7% reduction in the standard deviation in RSP within the LDPE material insert and up to a 1.2% reduction in overall TV. Similar results were obtained for other values of $\alpha$, $\lambda$, and, in the case of standard deviation, for different materials. One can conclude that imposing the TV reduction requirement does not provide a consistent benefit in terms of TV and standard deviation. Therefore, for the remainder of the parameter space exploration, the TV reduction requirement was excluded.

### *C. Perturbation Kernel ($\alpha$)*




(a)

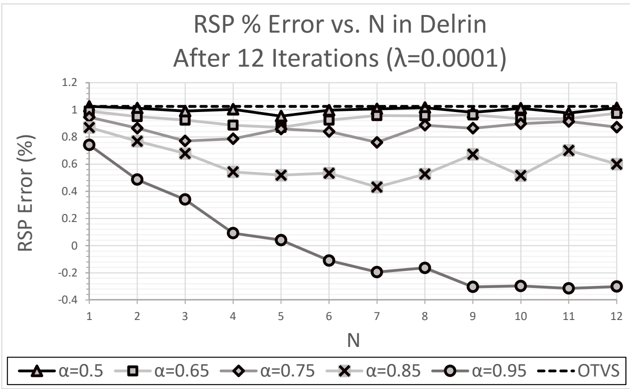



(b)

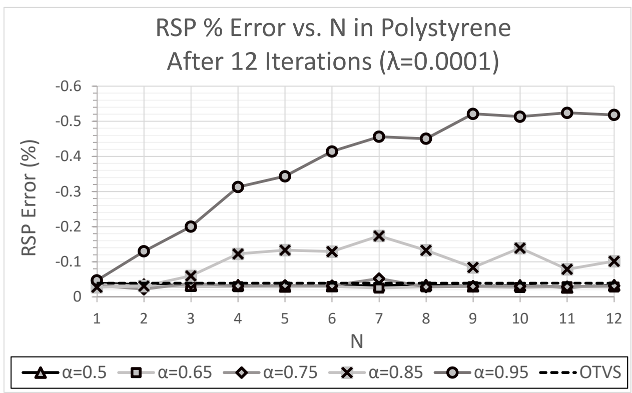
Fig. 3: (a) TV and (b) standard deviation (LDPE) as a function of $N$ after 12 feasibility-seeking iterations for the simulated CTP404 data set using OTVS and NTVS (TV reduction requirement excluded) with $\lambda = 0.0001$ and $\alpha = 0.5$.

Further investigations were performed to determine the effect of the perturbation kernel $\alpha$ (see step (10) of the NTVS algorithm in Appendix B) on TV and standard deviation for $0.5 \leq \alpha \leq 0.95$ and $1 \leq N \leq 12$. Increasing $\alpha$ produces larger perturbations and results in the perturbation magnitude

$\beta_k$ converging to zero more slowly. Thus, one can expect a larger reduction of TV and standard deviation for larger values of $\alpha$. Figures 3(a) and 3(b) demonstrate this effect.




(a)



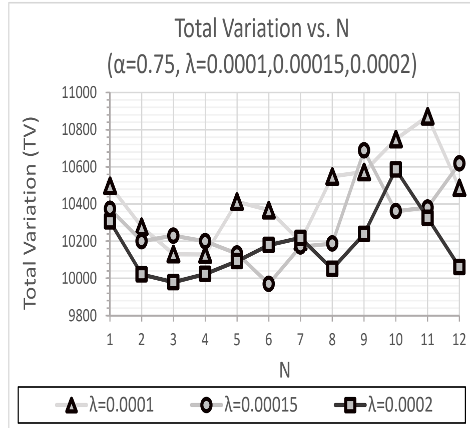

(b)

Fig. 4: RSP error in the (a) Delrin and (b) polystyrene ROIs as a function of $N$ after 12 feasibility-seeking iterations for the simulated CTP404 data set using OTVS and NTVS (TV reduction requirement excluded) with $\lambda = 0.0001$ and varying $\alpha$.

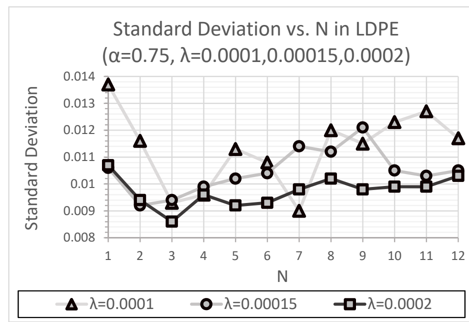
Figures 4(a) and 4(b) show the effect of $\alpha$ on the accuracy of reconstructed RSP values in the Delrin and polystyrene inserts, respectively. These two materials were chosen because they were most affected by the value of $\alpha$. From these plots, one can see that for $\alpha > 0.75$, perturbations have a growing effect on RSP accuracy as $\alpha$ and $N$ increase. This leads to changes in error greater than $1\%$ for Delrin and greater than $0.5\%$ for polystyrene. Although increasing $\alpha$ to decrease TV and standard deviation is a worthwhile goal, one cannot do so without considering its effect on RSP error. On the other hand, increasing $\alpha$ from $\alpha = 0.5$ to $\alpha = 0.75$ yielded up to a $39.3\%$ reduction in the standard deviation in RSP within the LDPE material insert and up to an $8.2\%$ reduction in overall TV without negatively impacting RSP error.

### *D. Relaxation Parameter ($\lambda$)*

Increasing the relaxation parameter accelerates the rate of convergence of the feasibility-seeking algorithm. To investigate the impact of NTVS independent of convergence rate, the number of iterations was adjusted for $\lambda = 0.00015$ and $\lambda = 0.0002$ to obtain the same RSP accuracy as for $\lambda = 0.0001$ and 12 iterations. For this comparison, $\alpha = 0.75$ was chosen.




(a)




(b)

Fig. 5: (a) TV and (b) standard deviation (soft tissue) as a function of $N$ for $\lambda = 0.0001$, $k = 12$; $\lambda = 0.00015$, $k = 8$; and $\lambda = 0.0002$, $k = 6$ iterations, respectively, and $\alpha = 0.75$ for the simulated CTP404 data set.

Figures 5(a) and 5(b) show TV and standard deviation within the LDPE ROI, respectively, for three combinations of $\lambda$ and $k$. For most values of $N$, the relative improvements in TV and standard deviation increased as $\lambda$ increased. Note that the trend for standard deviation was not as pronounced for other materials, but increasing $\lambda$ consistently produced comparable or larger reductions in TV and standard deviation in each material region.

## References

[1] B. E. Schultze, P. Karbasi, V. Giacometti, T. E. Plautz, K. E. Schubert, and R. W. Schulte, "Reconstructing highly accurate relative stopping powers in proton computed tomography," in *Proceedings of the IEEE Nuclear Science Symposium & Medical Imaging Conference (NSS/MIC) 2015*, Oct 2015, pp. 1–3.