Strong Convergence of Subgradient Extragradient Methods for the Variational Inequality Problem in Hilbert Space

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Abstract

We study two projection algorithms for solving the Variational Inequality Problem (VIP) in Hilbert space. One algorithm is a modified subgradient extragradient method in which an additional projection onto the intersection of two half-spaces is employed. Another algorithm is based on the shrinking projection method. We establish strong convergence theorems for both algorithms.

1 Introduction

We are concerned with the *Variational Inequality Problem* (VIP) of finding a point x^* such that

$$x^* \in C \text{ and } \langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C, \tag{1.1}$$

where $\mathcal{H}$ is a real Hilbert space, $f : \mathcal{H} \to \mathcal{H}$ is a given mapping, $C \subseteq \mathcal{H}$ is nonempty, closed and convex and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathcal{H}$. This problem, denoted by VIP(C, f), is a fundamental problem in Optimization Theory. Many algorithms for solving the VIP are projection algorithms that employ projections onto the feasible set C of the VIP, or onto some related set, in order to iteratively reach a solution. In particular, Korpelevich [19] proposed an algorithm for solving the VIP in Euclidean space, known as the Extragradient Method; see also Facchinei and Pang [10, Chapter 12]. In each iteration of her algorithm, in order to get the next iterate x^{k+1}, two orthogonal projections onto C are calculated, according to the following iterative step. Given the current iterate x^k, calculate

$$y^k = P_C(x^k - \tau f(x^k)), \tag{1.2}$$

$$x^{k+1} = P_C(x^k - \tau f(y^k)), \tag{1.3}$$

where τ is some positive number and P_C denotes the Euclidean least distance projection onto C. Figure 1 illustrates the iterative step (1.2) and (1.3). The

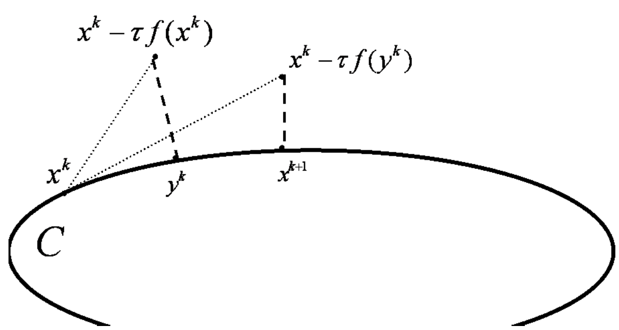

Figure 1: Korpelevich's iterative step.

literature on the VIP is vast and Korpelevich's extragradient method has received considerable attention by many authors who improved it in various ways; see, e.g., [16, 17, 23] and references therein, to name but a few.

Though convergence was proved in [19] under the assumptions of Lipschitz continuity and pseudo-monotonicity, there is still the need to calculate two projections onto C. If the set C is simple enough so that projections onto it can be easily computed, then this method is particularly useful; but if C is a general closed and convex set, a minimal distance problem has to be solved twice in order to obtain the next iterate. This might seriously affect the efficiency of the extragradient method.

As part of our continued efforts to circumvent the need to perform these two projections onto C within the framework of the extragradient method, we developed in [7] the subgradient extragradient algorithm in Euclidean space, in which we replace the second projection (1.3) onto C by a specific subgradient projection. A weak convergence theorem for this algorithm in Hilbert space is presented in [8]. The question of also replacing in this algorithm the first projection onto C with a step that will not involve a projection (onto a nonlinear convex set) and will be easier to compute, remains, to the best of our knowledge, open to this very day. We believe that our work in the present paper will lead to further progress in this direction.

In this paper we study two modifications of the subgradient extragradient method for solving the Variational Inequality Problem (VIP) in Hilbert space. These modifications originate in the work of Haugazeau [14], which was successfully generalized and extended in recent papers by Combettes [9], Solodov and Svaiter [24], Bauschke and Combettes [2, 3], and by Burachik, Lopes and Svaiter [5]. In both modifications we are again able to replace one of two projections onto a closed convex set by a specific subgradient projection. While our work is admittedly of a theoretical nature its potential numerical advantages lie in this fact. Every projection onto a closed convex set that can be replaced by a step that will not involve a projection (onto a nonlinear convex set) and will be easier to compute constitutes a practical saving in the overall algorithmic effort.

Our paper is organized as follows. In Section 3 the first modified subgradient extragradient algorithm is presented. It is analyzed in Section 4. In Section 5 we present another modification of the subgradient extragradient algorithm and then analyze it in Section 6.

2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$, and let D be a nonempty, closed and convex subset of $\mathcal{H}$. We write $x^k \rightharpoonup x$ to indicate that the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ converges weakly to x and $x^k \rightarrow x$ to indicate that the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ converges strongly to x. For each point $x \in \mathcal{H}$, there exists a unique nearest point in D, denoted by $P_D(x)$. That is,

$$\|x - P_D(x)\| \leq \|x - y\| \text{ for all } y \in D. \tag{2.1}$$

The mapping $P_D : \mathcal{H} \rightarrow D$ is called the metric projection of $\mathcal{H}$ onto D. It is well known that P_D is a nonexpansive mapping of $\mathcal{H}$ onto D, i.e.,

$$\|P_D(x) - P_D(y)\| \leq \|x - y\| \text{ for all } x, y \in \mathcal{H}. \tag{2.2}$$

The metric projection P_D is characterized [13, Section 3] by the following two properties:

$$P_D(x) \in D \tag{2.3}$$

and

$$\langle x - P_D(x), P_D(x) - y\rangle \geq 0 \text{ for all } x \in \mathcal{H},\ y \in D, \tag{2.4}$$

and if D is a hyperplane, then (2.4) becomes an equality. It follows that

$$\|x - y\|^2 \geq \|x - P_D(x)\|^2 + \|y - P_D(x)\|^2 \text{ for all } x \in \mathcal{H},\ y \in D. \tag{2.5}$$

We denote by $N_D(v)$ the normal cone of D at $v \in D$, i.e.,

$$N_D(v) := \{d \in \mathcal{H} \mid \langle d, y - v\rangle \leq 0 \text{ for all } y \in D\}. \tag{2.6}$$

We also recall that in a real Hilbert space $\mathcal{H}$,

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2 \tag{2.7}$$

for all $x, y \in \mathcal{H}$ and $\lambda \in [0, 1]$.

Definition 2.1 *Let $B : \mathcal{H} \rightrightarrows 2^{\mathcal{H}}$ be a point-to-set operator defined on a real Hilbert space $\mathcal{H}$. The operator B is called a* `maximal monotone operator` *if B is* `monotone`*, i.e.,*

$$\langle u - v, x - y\rangle \geq 0 \textit{ for all } u \in B(x) \textit{ and } v \in B(y), \tag{2.8}$$

and the graph $G(B)$ of B,

$$G(B) := \{(x,u) \in \mathcal{H} \times \mathcal{H} \mid u \in B(x)\}, \tag{2.9}$$

is not properly contained in the graph of any other monotone operator.

It is clear ([22, Theorem 3]) that a monotone mapping B is maximal if and only if, for any $(x,u) \in \mathcal{H}\times\mathcal{H}$, if $\langle u-v, x-y\rangle \geq 0$ for all $(v,y) \in G(B)$, then it follows that $u \in B(x)$.

The next property is known as the Opial condition [21]. Every Hilbert space has this property.

Condition 2.2 ***(Opial)*** *For any sequence $\left\{x^k\right\}_{k=0}^{\infty}$ in $\mathcal{H}$ that converges weakly to x ($x^k \rightharpoonup x$),*

$$\liminf_{k\to\infty} \|x^k - x\| < \liminf_{k\to\infty} \|x^k - y\| \text{ for all } y \neq x. \tag{2.10}$$

Any Hilbert space $\mathcal{H}$ has the *Kadec-Klee property* [12], that is, if $\left\{x^k\right\}_{k=0}^{\infty}$ is a sequence in $\mathcal{H}$ with $x^k \rightharpoonup x$ and $\|x^k\| \to \|x\|$, then $\|x^k - x\| \to 0$.

Definition 2.3 *A function $g : \mathcal{H} \to (-\infty, +\infty]$ is called (weak) lower semi-continuous if*

$$\liminf_{n\to\infty} g(x^k) \geq g(x) \tag{2.11}$$

for all sequences $\left\{x^k\right\}_{k=0}^{\infty}$ such that ($x^k \rightharpoonup x$) $x^k \to x$.

Notation 2.4 *Any closed and convex set $D \subset \mathcal{H}$ can be represented as*

$$D = \{x \in \mathcal{H} \mid c(x) \leq 0\}, \tag{2.12}$$

where $c : \mathcal{H} \to R$ is an appropriate convex and lower semi-continuous function. Take, for example, $c(x) = \operatorname{dist}(x, D)$, where dist *is the distance function; see, e.g., [15, Chapter B, Subsection 1.3(c)].*

We denote the subdifferential set of c at a point x by

$$\partial c(x) := \{\xi \in \mathcal{H} \mid c(y) \geq c(x) + \langle \xi, y - x\rangle \text{ for all } y \in \mathcal{H}\}. \tag{2.13}$$

For $z \in \mathcal{H}$, take any $\xi \in \partial c(z)$ and define

$$T(z) := \{w \in \mathcal{H} \mid c(z) + \langle \xi, w - z\rangle \leq 0\}. \tag{2.14}$$

This is a half-space the bounding hyperplane of which separates the set D from the point z if $\xi \neq 0$; otherwise $T(z) = \mathcal{H}$; see, e.g., [1, Lemma 7.3].

3 The first modification of the subgradient extragradient algorithm

We assume the following conditions.

Condition 3.1 *The solution set of (1.1), denoted by $SOL(C, f)$, is nonempty.*

Condition 3.2 *The mapping f is* `monotone` *on C, i.e.,*

$$\langle f(x) - f(y), x - y \rangle \geq 0 \text{ for all } x, y \in C. \tag{3.1}$$

Condition 3.3 *The mapping f is* `Lipschitz continuous` *on $\mathcal{H}$ with constant $L > 0$, that is,*

$$\|f(x) - f(y)\| \leq L\|x - y\| \text{ for all } x, y \in \mathcal{H}. \tag{3.2}$$

Next, we present the subgradient extragradient algorithm [7].

Algorithm 3.4 ***The subgradient extragradient algorithm***
Step 0: *Select a starting point $x^0 \in \mathcal{H}$ and $\tau > 0$, and set $k = 0$.*
Step 1: *Given the current iterate x^k, compute*

$$y^k = P_C(x^k - \tau f(x^k)), \tag{3.3}$$

construct the half-space T_k whose bounding hyperplane supports C at y^k,

$$T_k := \{w \in \mathcal{H} \mid \langle (x^k - \tau f(x^k)) - y^k, w - y^k \rangle \leq 0\} \tag{3.4}$$

and calculate the next iterate

$$x^{k+1} = P_{T_k}(x^k - \tau f(y^k)). \tag{3.5}$$

Step 2: *If $x^k = y^k$, then stop. Otherwise, set $k \leftarrow (k + 1)$ and return to* ***Step 1.***

Remark 3.5 *Observe that if c is lower semi-continuous and Gâteaux differentiable at y^k, then $\{(x^k - \tau f(x^k)) - y^k\} = \partial c(y^k) = \{\nabla c(y^k)\}$; otherwise $(x^k - \tau f(x^k)) - y^k \in \partial c(y^k)$. See [1, Facts 7.2] and [11] for more details.*

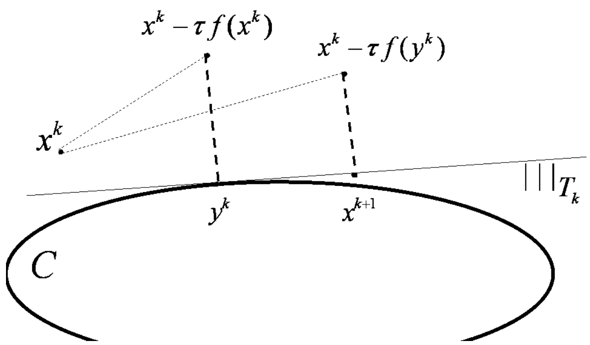

Figure 2: x^{k+1} is a subgradient projection of the point $x^k - \tau f(y^k)$ onto the hyperplane T_k.

Figure 2 illustrates the iterative step of this algorithm.

Inspired by Takahashi and Nadezhkina [20], we now present our first modification of the subgradient extragradient algorithm.

Algorithm 3.6 ***The first modification of the subgradient extragradient algorithm***

Step 0: *Select an arbitrary starting point* $x^0 \in \mathcal{H}$ *and* $\tau > 0$, *and set* $k = 0$.

Step 1: *Given the current iterate* x^k, *compute*

$$\begin{cases} y^k = P_C(x^k - \tau f(x^k)), \\ z^k = \alpha_k x^k + (1 - \alpha_k) P_{T_k}(x^k - \tau f(y^k)), \\ C_k = \left\{ z \in \mathcal{H} \mid \left\| z^k - z \right\| \leq \left\| x^k - z \right\| \right\}, \\ Q_k = \left\{ z \in \mathcal{H} \mid \left\langle x^k - z, x^0 - x^k \right\rangle \geq 0 \right\}, \\ x^{k+1} = P_{C_k \cap Q_k}(x^0), \end{cases} \tag{3.6}$$

where T_k *is as in (3.4) and* $\{\alpha_k\}_{k=0}^{\infty} \subset [0, \alpha]$ *for some* $\alpha \in [0, 1)$.

Step 2: *Set* $k \leftarrow (k + 1)$ *and return to* ***Step 1.***

3.1 Connection with Haugazeau's method

In this subsection we describe the connection between our Algorithm 3.6 and the work of Haugazeau. Haugazeau presented an algorithm for solving the *Best Approximation Problem* (BAP) of finding the projection of a point onto the intersection of m closed convex subsets $\{D_i\}_{i=1}^m \subset \mathcal{H}$. Defining for any pair $x, y \in \mathcal{H}$ the set

$$H(x,y) := \{u \in \mathcal{H} \mid \langle u-y, x-y\rangle \leq 0\}, \tag{3.7}$$

and denoting by $Q(x,y,z)$ the projection of x onto $H(x,y) \cap H(y,z)$, namely, $Q(x,y,z) = P_{H(x,y)\cap H(y,z)}(x)$, he showed, see [14], that for an arbitrary starting point $x^0 \in \mathcal{H}$, any sequence $\{x^k\}_{k=0}^\infty$ generated by the iterative step

$$x^{k+1} = Q(x^0, x^k, P_{k(\mathrm{mod}\, m)+1}(x^k)) \tag{3.8}$$

converges strongly to the projection of x^0 onto $D = \cap_{i=1}^m D_i$. The operator Q requires projecting onto the intersection of two constructible half-spaces; this is not difficult to implement. In [14] Haugazeau introduced the operator Q as an explicit description of the projector onto the intersection of the two half-spaces $H(x,y)$ and $H(y,z)$. So, following, e.g., [4, Definition 3.1], denoting $\pi = \langle x-y, y-z\rangle$, $\mu = \|x-y\|^2$, $\nu = \|y-z\|^2$ and $\rho = \mu\nu - \pi^2$, we have

$$Q(x,y,z) = \begin{cases} z, & \text{if } \rho = 0 \text{ and } \pi \geq 0, \\ x + \left(1 + \frac{\pi}{\nu}\right)(z-y), & \text{if } \rho > 0 \text{ and } \pi\nu \geq \rho, \\ y + \frac{\nu}{\rho}(\pi(x-y) + \mu(z-y)), & \text{if } \rho > 0 \text{ and } \pi\nu < \rho. \end{cases} \tag{3.9}$$

In our Algorithm 3.6 we may write

$$\begin{aligned} C_k &= \left\{z \in \mathcal{H} \mid \left\|z^k - z\right\| \leq \left\|x^k - z\right\|\right\} \\ &= \left\{z \in \mathcal{H} \mid \left\langle x^k - (1/2)(x^k + z^k), z - (1/2)(x^k+z^k)\right\rangle \leq 0\right\} \\ &= H(x^k, (1/2)(x^k + z^k)) \end{aligned} \tag{3.10}$$

and

$$Q_k = \left\{z \in \mathcal{H} \mid \left\langle x^k - z, x^0 - x^k\right\rangle \geq 0\right\} = H(x^0, x^k). \tag{3.11}$$

This leads to the following alternative phrasing of the iterative step of Algorithm 3.6:

$$\begin{cases} y^k = P_C(x^k - \tau f(x^k)), \\ z^k = \alpha_k x^k + (1-\alpha_k) P_{T_k}(x^k - \tau f(y^k)), \\ x^{k+1} = Q(x^0, x^k, (1/2)(x^k + z^k)). \end{cases} \tag{3.12}$$

Observe that using the explicit description (3.9) of the operator Q and the projector onto T_k, the iterative step (3.6) of Algorithm 3.6 can be rewritten even more explicitly as follows.

$$\begin{cases} y^k = P_C(x^k - \tau f(x^k)), \\ \text{denote } a^k := x^k - \tau f(x^k) - y^k \text{ and } w^k := x^k - \tau f(y^k), \\ P_{T_k}(w^k) := t^k = \begin{cases} w^k - \max\left\{0, \frac{\left\langle a^k, w^k - y^k \right\rangle}{\|a^k\|^2}\right\} a^k, & \text{if } a^k \neq 0, \\ w^k, & \text{if } a^k = 0. \end{cases} \\ z^k = \alpha_k x^k + (1-\alpha_k) t^k, \\ \text{denote } \pi_k := \left\langle x^0 - x^k, (1/2)(x^k - z^k) \right\rangle,\ \mu_k := \|x^0 - x^k\|^2, \\ \nu_k := \|(1/2)(x^k - z^k)\|^2 \text{ and } \rho_k := \ \mu_k \nu_k - (\pi_k)^2. \\ \\ \text{Then,} \\ x^{k+1} = \begin{cases} (1/2)(x^k + z^k), & \text{if } \rho_k = 0 \text{ and } \pi_k \geq 0, \\ x^0 + \left(1 + \frac{\pi_k}{\nu_k}\right)(1/2)(z^k - x^k), & \text{if } \rho_k > 0 \text{ and } \pi_k \nu_k \geq \rho_k, \\ y^k + \frac{\nu_k}{\rho_k}(\pi_k \left(x^0 - x^k\right) + \frac{\mu_k}{2}(z^k - x^k)), & \text{if } \rho_k > 0 \text{ and } \pi_k \nu_k < \rho_k. \end{cases} \end{cases} \tag{3.13}$$

4 Convergence of the first modification of the subgradient extragradient algorithm

In this section we establish a strong convergence theorem for Algorithm 3.6. The outline of its proof is similar to that of [20, Theorem 3.1].

Theorem 4.1 *Assume that Conditions 3.1–3.3 hold and $\tau \in (0, 1/L)$. Then any sequences $\left\{x^k\right\}_{k=0}^{\infty}$ and $\left\{y^k\right\}_{k=0}^{\infty}$ generated by Algorithm 3.6 strongly converge to the same point $u^* \in SOL(C, f)$ and furthermore,*

$$u^* = P_{SOL(C,f)}(x^0). \tag{4.1}$$

Proof. First observe that for all $k \geq 0$, Q_k is closed and convex. The set C_k is also closed and convex because

$$\begin{aligned} C_k &= \left\{ z \in \mathcal{H} : \left\| z^k - z \right\|^2 \leq \left\| x^k - z \right\|^2 \right\} \\ &= \left\{ z \in \mathcal{H} : \left\| z^k - x^k \right\|^2 + 2 \left\langle z^k - x^k, x^k - z \right\rangle \leq 0 \right\}. \end{aligned} \tag{4.2}$$

By the definition of Q_k and (2.4), we have

$$x^k = P_{Q_k}(x^0). \tag{4.3}$$

Denote $t^k := P_{T_k}(x^k - \tau f(y^k))$ for all $k \geq 0$. Let $u \in \mathrm{SOL}(C, f)$. Applying (2.5) with $D = T_k$, $x = x^k - \tau f(y^k)$ and $y = u$, we obtain

$$\begin{aligned} \left\| t^k - u \right\|^2 &\leq \left\| x^k - \tau f(y^k) - u \right\|^2 - \left\| x^k - \tau f(y^k) - t^k \right\|^2 \\ &= \| x^k - u \|^2 - \| x^k - t^k \|^2 + 2\tau \left\langle f(y^k), u - t^k \right\rangle \\ &= \| x^k - u \|^2 - \| x^k - t^k \|^2 \\ &+ 2\tau \left(\left\langle f(y^k) - f(u), u - y^k \right\rangle + \left\langle f(u), u - y^k \right\rangle + \left\langle f(y^k), y^k - t^k \right\rangle \right). \end{aligned} \tag{4.4}$$

By Condition 3.2,

$$\left\langle f(y^k) - f(u), u - y^k \right\rangle \leq 0, \tag{4.5}$$

and since $u \in \mathrm{SOL}(C, f)$,

$$\left\langle f(u), u - y^k \right\rangle \leq 0. \tag{4.6}$$

So,

$$\begin{aligned} \left\| t^k - u \right\|^2 &\leq \| x^k - u \|^2 - \| x^k - t^k \|^2 + 2\tau \left\langle f(y^k), y^k - t^k \right\rangle \\ &= \| x^k - u \|^2 - \| x^k - y^k \|^2 - 2 \left\langle x^k - y^k, y^k - t^k \right\rangle \\ &- \| y^k - t^k \|^2 + 2\tau \left\langle f(y^k), y^k - t^k \right\rangle \\ &= \| x^k - u \|^2 - \| x^k - y^k \|^2 - \| y^k - t^k \|^2 \\ &+ 2 \left\langle x^k - \tau f(y^k) - y^k, t^k - y^k \right\rangle. \end{aligned} \tag{4.7}$$

By the definition of T_k,

$$\left\langle \left(x^k - \tau f(x^k) \right) - y^k, t^k - y^k \right\rangle \leq 0, \tag{4.8}$$

so

$$
\begin{aligned}
&\left\langle x^k - \tau f(y^k) - y^k, t^k - y^k \right\rangle \\
&= \left\langle x^k - \tau f(x^k) - y^k, t^k - y^k \right\rangle + \left\langle \tau f(x^k) - \tau f(y^k), t^k - y^k \right\rangle \\
&\leq \left\langle \tau f(x^k) - \tau f(y^k), t^k - y^k \right\rangle \leq \tau \| f(x^k) - f(y^k) \| \| t^k - y^k \| \\
&\leq \tau L \| x^k - y^k \| \| t^k - y^k \|, \qquad (4.9)
\end{aligned}
$$

where the last two inequalities follow from the Cauchy–Schwarz inequality and Condition 3.3. Therefore

$$
\left\| t^k - u \right\|^2 \leq \|x^k - u\|^2 - \|x^k - y^k\|^2 - \|y^k - t^k\|^2 + 2\tau L \|x^k - y^k\| \|t^k - y^k\|. \quad (4.10)
$$

Observe that

$$
\begin{aligned}
0 &\leq \left(\|t^k - y^k\| - \tau L \|x^k - y^k\| \right)^2 \\
&= \|t^k - y^k\|^2 - 2\tau L \|x^k - y^k\| \|t^k - y^k\| + \tau^2 L^2 \|x^k - y^k\|^2, \qquad (4.11)
\end{aligned}
$$

so,

$$
2\tau L \|x^k - y^k\| \|t^k - y^k\| \leq \|t^k - y^k\|^2 + \tau^2 L^2 \|x^k - y^k\|^2. \qquad (4.12)
$$

Thus

$$
\begin{aligned}
\left\| t^k - u \right\|^2 &\leq \|x^k - u\|^2 - \|x^k - y^k\|^2 - \|y^k - t^k\|^2 \\
&+ \|t^k - y^k\|^2 + \tau^2 L^2 \|x^k - y^k\|^2 \\
&= \|x^k - u\|^2 - \|x^k - y^k\|^2 + \tau^2 L^2 \|x^k - y^k\|^2 \\
&= \|x^k - u\|^2 + (\tau^2 L^2 - 1) \|x^k - y^k\|^2 \\
&\leq \|x^k - u\|^2, \qquad (4.13)
\end{aligned}
$$

where the last inequality follows from the fact that $\tau \in (0, 1/L)$. Now by the definition of z^k and (2.7), we get

$$
\begin{aligned}
\|z^k - u\|^2 &= \|\alpha_k x^k + (1 - \alpha_k) t^k - u\|^2 \\
&= \|\alpha_k \left(x^k - u \right) + (1 - \alpha_k) \left(t^k - u \right) \|^2 \\
&= \alpha_k \|x^k - u\|^2 + (1 - \alpha_k) \|t^k - u\|^2 \\
&- \alpha_k (1 - \alpha_k) \| \left(x^k - u \right) - \left(t^k - u \right) \|^2 \\
&\leq \alpha_k \|x^k - u\|^2 + (1 - \alpha_k) \|t^k - u\|^2 \\
&\leq \alpha_k \|x^k - u\|^2 + (1 - \alpha_k) \left(\|x^k - u\|^2 + (\tau^2 L^2 - 1) \|x^k - y^k\|^2 \right) \\
&= \|x^k - u\|^2 + (1 - \alpha_k)(\tau^2 L^2 - 1) \|x^k - y^k\|^2 \\
&\leq \|x^k - u\|^2, \qquad (4.14)
\end{aligned}
$$

so $u \in C_k$ and therefore $\mathrm{SOL}(C,f) \subset C_k$ for all $k \geq 0$. Now we show, by induction, that the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ is well-defined and $\mathrm{SOL}(C,f) \subset C_k \cap Q_k$ for all $k \geq 0$. For $k = 0$ we have $Q_0 = \mathcal{H}$, so it follows that $\mathrm{SOL}(C,f) \subset C_0 \cap Q_0$ and therefore $x^1 = P_{C_0 \cap Q_0}(x^0)$ is well-defined. Now suppose that x^k is given and $\mathrm{SOL}(C,f) \subset C_k \cap Q_k$ for some k. By Condition 3.1, $C_k \cap Q_k$ is nonempty, closed and convex, and therefore $x^{k+1} = P_{C_k \cap Q_k}(x^0)$ is well-defined. By (2.4), we have

$$\left\langle z - x^{k+1}, x^0 - x^{k+1} \right\rangle \leq 0 \text{ for all } z \in C_k \cap Q_k. \tag{4.15}$$

Since $\mathrm{SOL}(C,f) \subset C_k \cap Q_k$,

$$\left\langle u - x^{k+1}, x^0 - x^{k+1} \right\rangle \leq 0 \text{ for all } u \in SOL(C,f), \tag{4.16}$$

which implies that $u \in Q_{k+1}$. Thus $\mathrm{SOL}(C,f) \subset C_{k+1} \cap Q_{k+1}$, as required. Denote $u^* = P_{SOL(C,f)}(x^0)$. It is clear that $u^* \in \mathrm{SOL}(C,f)$. Since $\mathrm{SOL}(C,f) \subset C_k \cap Q_k$, $u^* \in SOL(C,f)$ and $x^{k+1} = P_{C_k \cap Q_k}\left(x^0\right)$, we have

$$\|x^{k+1} - x^0\| \leq \|u^* - x^0\| \text{ for all } k \geq 0. \tag{4.17}$$

This implies, in particular, that $\left\{x^k\right\}_{k=0}^{\infty}$ is bounded, and it follows from (4.13) and (4.14) that so are $\left\{t^k\right\}_{k=0}^{\infty}$ and $\left\{z^k\right\}_{k=0}^{\infty}$. By the definition of x^{k+1}, we have $x^{k+1} \in Q_k$ and by the definition of Q_k, $x^k = P_{Q_k}(x^0)$, so

$$\|x^k - x^0\| \leq \|x^{k+1} - x^0\| \text{ for all } k \geq 0. \tag{4.18}$$

Hence there exists

$$\lim_{k \to \infty} \|x^k - x^0\|. \tag{4.19}$$

Applying (2.5) with $D = Q_k$, $x = x^0$ and $y = x^{k+1}$, we obtain

$$\|x^{k+1} - x^k\|^2 \leq \|x^{k+1} - x^0\|^2 - \|x^k - x^0\|^2 \text{ for all } k \geq 0 \tag{4.20}$$

and so,

$$\lim_{k \to \infty} \|x^{k+1} - x^k\| = 0. \tag{4.21}$$

Since $x^{k+1} \in C_k$, $\|z^k - x^{k+1}\| \leq \|x^k - x^{k+1}\|$, and therefore by the triangle inequality,

$$\|x^k - z^k\| \leq \|x^k - x^{k+1}\| + \|x^{k+1} - z^k\| \leq 2\|x^k - x^{k+1}\|, \tag{4.22}$$

and so,

$$\lim_{k\to\infty}\|x^k - z^k\| = 0. \tag{4.23}$$

By (4.14),

$$\|z^k - u\|^2 \le \|x^k - u\|^2 + (1-\alpha_k)(\tau^2L^2 - 1)\|x^k - y^k\|^2, \tag{4.24}$$

or

$$\begin{aligned}\|x^k - y^k\|^2 &\le \frac{\|x^k-u\|^2 - \|z^k-u\|^2}{(1-\alpha_k)(1-\tau^2L^2)}\\ &= \frac{1}{(1-\alpha_k)(1-\tau^2L^2)}\left(\|x^k-u\| - \|z^k-u\|\right)\left(\|x^k-u\| + \|z^k-u\|\right)\\ &\le \frac{1}{(1-\alpha_k)(1-\tau^2L^2)}\left(\|x^k-u\| + \|z^k-u\|\right)\|x^k - z^k\|. \qquad (4.25)\end{aligned}$$

By (4.23) and the boundedness of $\left\{x^k\right\}_{k=0}^{\infty}$ and $\left\{z^k\right\}_{k=0}^{\infty}$, we obtain

$$\lim_{k\to\infty}\|x^k - y^k\| = 0. \tag{4.26}$$

By Condition 3.3,

$$\lim_{k\to\infty}\|f(x^k) - f(y^k)\| = 0. \tag{4.27}$$

Using a similar argument to the one following (4.10),

$$\begin{aligned}\left\|t^k - u\right\|^2 &\le \|x^k-u\|^2 - \|x^k-y^k\|^2 - \|y^k-t^k\|^2 + 2\tau L\|x^k-y^k\|\|t^k-y^k\|\\ &\le \|x^k-u\|^2 - \|x^k-y^k\|^2 - \|y^k-t^k\|^2 + \|x^k-y^k\|^2 + \tau^2L^2\|y^k-t^k\|^2\\ &\le \|x^k-u\|^2 + (\tau^2L^2-1)\|y^k-t^k\|^2. \qquad (4.28)\end{aligned}$$

Now by (4.14),

$$\|z^k - u\|^2 \le \alpha_k\|x^k-u\|^2 + (1-\alpha_k)\|t^k-u\|^2, \tag{4.29}$$

and by the last inequalities,

$$\begin{aligned}\|z^k-u\|^2 &\le \alpha_k\|x^k-u\|^2 + (1-\alpha_k)\left(\|x^k-u\|^2 + (\tau^2L^2-1)\|y^k-t^k\|^2\right)\\ &= \|x^k-u\|^2 + (1-\alpha_k)(\tau^2L^2-1)\|y^k-t^k\|^2 \le \|x^k-u\|^2. \qquad (4.30)\end{aligned}$$

Thus

$$\begin{aligned}\|y^k - t^k\|^2 &\le \frac{\|x^k-u\|^2 - \|z^k-u\|^2}{(1-\alpha_k)(1-\tau^2L^2)} \\ &= \frac{1}{(1-\alpha_k)(1-\tau^2L^2)}\left(\|x^k-u\| - \|z^k-u\|\right)\left(\|x^k-u\| + \|z^k-u\|\right) \\ &\le \frac{1}{(1-\alpha_k)(1-\tau^2L^2)}\left(\|x^k-u\| + \|z^k-u\|\right)\|x^k-z^k\|. \qquad (4.31)\end{aligned}$$

By (4.23) and the boundedness of $\left\{x^k\right\}_{k=0}^{\infty}$ and $\left\{z^k\right\}_{k=0}^{\infty}$, we obtain

$$\lim_{k\to\infty}\|y^k - t^k\| = 0. \qquad (4.32)$$

By the triangle inequality, we also have

$$\|x^k - t^k\| \le \|x^k - y^k\| + \|y^k - t^k\|, \qquad (4.33)$$

and therefore

$$\lim_{k\to\infty}\|x^k - t^k\| = 0. \qquad (4.34)$$

Since $\left\{x^k\right\}_{k=0}^{\infty}$ is bounded, there exists a subsequence $\left\{x^{k_j}\right\}_{j=0}^{\infty}$ of $\left\{x^k\right\}_{k=0}^{\infty}$ which converges weakly to some $\overline{x} \in \mathcal{H}$. We claim that $\overline{x} \in \mathrm{SOL}(C, f)$. Indeed, let

$$A(v) = \begin{cases} f(v) + N_C(v), & v \in C, \\ \emptyset, & v \notin C, \end{cases} \qquad (4.35)$$

where $N_C(v)$ is the normal cone of C at $v \in C$ (2.6). It is known [22, Theorem 3] that A is a maximal monotone operator and $A^{-1}(0) = \mathrm{SOL}(f, C)$. If $(v, w) \in G(A)$, then $w \in A(v) = f(v) + N_C(v)$, or $w - f(v) \in N_C(v)$. Hence

$$\langle w - f(v), v - y\rangle \ge 0 \text{ for all } y \in C. \qquad (4.36)$$

On the other hand, by the definition of y^k and (2.4),

$$\left\langle x^k - \tau f(x^k) - y^k, y^k - v\right\rangle \ge 0, \qquad (4.37)$$

or

$$\left\langle \left(\frac{y^k - x^k}{\tau}\right) + f(x^k), v - y^k\right\rangle \ge 0 \qquad (4.38)$$

for all $k \geq 0$. Applying (4.36) with $\left\{y^{k_j}\right\}_{j=0}^{\infty}$, we get

$$\left\langle w - f(v), v - y^{k_j}\right\rangle \geq 0. \tag{4.39}$$

Hence,

$$\begin{aligned}
\left\langle w, v - y^{k_j}\right\rangle &\geq \left\langle f(v), v - y^{k_j}\right\rangle \geq \left\langle f(v), v - y^{k_j}\right\rangle \\
&- \left\langle \left(\frac{y^{k_j} - x^{k_j}}{\tau}\right) + f(x^{k_j}), v - y^{k_j}\right\rangle \\
&= \left\langle f(v) - f(y^{k_j}), v - y^{k_j}\right\rangle + \left\langle f(y^{k_j}) - f(x^{k_j}), v - y^{k_j}\right\rangle \\
&- \left\langle \left(\frac{y^{k_j} - x^{k_j}}{\tau}\right), v - y^{k_j}\right\rangle \\
&\geq \left\langle f(y^{k_j}) - f(x^{k_j}), v - y^{k_j}\right\rangle - \left\langle \left(\frac{y^{k_j} - x^{k_j}}{\tau}\right), v - y^{k_j}\right\rangle
\end{aligned} \tag{4.40}$$

and

$$\left\langle w, v - y^{k_j}\right\rangle \geq \left\langle f(y^{k_j}) - f(x^{k_j}), v - y^{k_j}\right\rangle - \left\langle \left(\frac{y^{k_j} - x^{k_j}}{\tau}\right), v - y^{k_j}\right\rangle. \tag{4.41}$$

Since $y^{k_j} \rightharpoonup \bar{x}$ by (4.26), taking the limit as $j \to \infty$ and using (4.26), we obtain

$$\langle w, v - \bar{x}\rangle \geq 0, \tag{4.42}$$

and since A is a maximal monotone operator, it follows that $\bar{x} \in A^{-1}(0) = \mathrm{SOL}(f, C)$, as claimed. From $u^* = P_{SOL(C,f)}(x^0)$, $\bar{x} \in \mathrm{SOL}(f, C)$, (4.17) and the weak lower semi-continuity of the norm it follows that

$$\begin{aligned}
\|u^* - x^0\| \leq \|\bar{x} - x^0\| &\leq \liminf_{j\to\infty} \|x^{k_j} - x^0\| \\
&\leq \limsup_{j\to\infty} \|x^{k_j} - x^0\| \leq \|u^* - x^0\|,
\end{aligned} \tag{4.43}$$

so

$$\lim_{j\to\infty} \|x^{k_j} - x^0\| = \|\bar{x} - x^0\|. \tag{4.44}$$

Hence we have $x^{k_j} - x^0 \rightharpoonup \bar{x} - x^0$ and $\|x^{k_j} - x^0\| \to \|\bar{x} - x^0\|$, and so by the Kadec-Klee property of $\mathcal{H}$ we obtain $\|x^{k_j} - \bar{x}\| \to 0$. Since $x^{k_j} = P_{Q_{k_j}}(x^0)$

and $u^* \in Q_{k_j}$, we see that

$$-\|u^* - x^{k_j}\|^2 = \left\langle u^* - x^{k_j}, x^{k_j} - x^0 \right\rangle + \left\langle u^* - x^{k_j}, x^0 - u^* \right\rangle \geq \left\langle u^* - x^{k_j}, x^0 - u^* \right\rangle. \tag{4.45}$$

Taking the limit as $j \to \infty$, we obtain

$$-\|u^* - \bar{x}\|^2 \geq \left\langle u^* - \bar{x}, x^0 - u^* \right\rangle \geq 0, \tag{4.46}$$

and therefore

$$\lim_{k \to \infty} x^{k_j} = \bar{x} = u^*. \tag{4.47}$$

Since $\left\{x^{k_j}\right\}_{j=0}^{\infty}$ is an arbitrary weakly convergent subsequence of $\left\{x^k\right\}_{k=0}^{\infty}$, we conclude that $\left\{x^k\right\}_{k=0}^{\infty}$ converges strongly to u^*, i.e., $\lim_{k\to\infty} x^k = u^* = P_{SOL(C,f)}(x^0)$, as asserted. ■

5 The second modification of the subgradient extragradient algorithm

Takahashi, Takeuchi and Kubota [26] presented an algorithm for finding a fixed point of a nonexpansive mapping S in Hilbert space. Let $C \subseteq \mathcal{H}$ be a closed and convex subset, and S be a nonexpansive mapping of C into itself such that $\operatorname{Fix}(S) \neq \emptyset$. Their iterative method, known as the shrinking projection method, is presented next.

Algorithm 5.1

Step 0: *Select an arbitrary starting point* $x^0 \in \mathcal{H}$, $C_1 = C$, $x^1 = P_{C_1}(x^0)$, *and set* $k = 1$.

Step 1: *Given the current iterate* x^k, *compute*

$$\begin{cases} y^k = \alpha_k x^k + (1 - \alpha_k) S(x^k), \\ C_{k+1} = \left\{ z \in C_k \mid \left\| y^k - z \right\| \leq \left\| x^k - z \right\| \right\}, \\ x^{k+1} = P_{C_{k+1}}\left(x^0\right), \end{cases} \tag{5.1}$$

where $\{\alpha_k\}_{k=1}^{\infty} \subset [0, \alpha]$ *for some* $\alpha \in [0, 1)$.

Step 2: *Set* $k \leftarrow (k + 1)$ *and return to* ***Step 1.***

They proved that any sequence $\left\{x^k\right\}_{k=0}^{\infty}$ generated by Algorithm 5.1 converges strongly to

$$u^* = P_{\mathrm{Fix}(S)}(x^0). \tag{5.2}$$

A comment on implementability is in order here. Observe that in the Takahashi-Takeuchi-Kubota algorithm, the sets C_{k+1} become increasingly complicated because at every iteration another half-space is used to cut the set. This may render the algorithm unimplementable, unless one uses an inner-loop for calculating an approximation of the projection $x^{k+1} = P_{C_{k+1}}(x^0)$ at each iterative step. Such an inner-loop can indeed be constructed from an iterative projection method that solves the Best Approximation Problem (BAP); see, e.g., [6] and the many references therein. These considerations also apply to our next algorithm.

Inspired by Algorithm 5.1 and a recent development of Sudsukh [25], we now present the following algorithm for solving variational inequalities.

Algorithm 5.2 ***The second modification of the subgradient extragradient algorithm***

Step 0*: Select an arbitrary starting point $x^0 \in \mathcal{H}$, a constant $\tau > 0$, $C_1 = C$, $x^1 = P_{C_1}(x^0)$, and set $k = 1$.*

Step 1*: Given the current iterate x^k, compute*

$$\begin{cases} y^k = P_C(x^k - \tau f(x^k)), \\ z^k = \alpha_k x^k + (1-\alpha_k) P_{T_k}(x^k - \tau f(y^k)), \\ C_{k+1} = \left\{ z \in C_k \mid \left\| z^k - z \right\| \leq \left\| x^k - z \right\| \right\}, \\ x^{k+1} = P_{C_{k+1}}(x^0), \end{cases} \tag{5.3}$$

where $\{\alpha_k\}_{k=1}^{\infty} \subset [0, \alpha]$ for some $\alpha \in (0,1)$ and T_k is as in (3.4).

Step 2: *Set $k \leftarrow (k+1)$ and return to* ***Step 1.***

We now prove a convergence theorem for this algorithm by using arguments which are quite similar to those we employed in the proof of Theorem 4.1.

6 Convergence of the second modification of the subgradient extragradient algorithm

Theorem 6.1 *Assume that Conditions 3.1–3.3 hold and $\tau \in (0, 1/L)$. Then any sequences $\left\{x^k\right\}_{k=0}^{\infty}$ and $\left\{y^k\right\}_{k=0}^{\infty}$ generated by Algorithm 5.2 strongly con-*

verge to the same point $u^* \in SOL(C,f)$ and furthermore,

$$u^* = P_{SOL(C,f)}(x^0). \tag{6.1}$$

Proof. First we show that C_k is closed and convex for all $k \geq 1$. $C_1 = C$ is clearly closed and convex by assumption. Now assume that C_k is closed and convex. Then C_{k+1} is closed and convex as an intersection of C_k and a half-space. Now we prove, using induction, that the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ is well-defined, by showing that $\mathrm{SOL}(C,f) \subset C_k$ for all $k \geq 1$. For $C_1 = C$ this is clear. Now assume that $\mathrm{SOL}(C,f) \subset C_k$. Denote $t^k := P_{T_k}(x^k - \tau f(y^k))$ for all $k \geq 0$. Let $u \in \mathrm{SOL}(C,f)$. Applying (2.5) with $D = T_k$, $x = x^k - \tau f(y^k)$ and $y = u$, we obtain

$$\begin{aligned}
\left\|t^k - u\right\|^2 &\leq \left\|x^k - \tau f(y^k) - u\right\|^2 - \left\|x^k - \tau f(y^k) - t^k\right\|^2 \\
&= \|x^k - u\|^2 - \|x^k - t^k\|^2 + 2\tau \left\langle f(y^k), u - t^k\right\rangle \\
&= \|x^k - u\|^2 - \|x^k - t^k\|^2 \\
&+ 2\tau \left(\left\langle f(y^k) - f(u), u - y^k\right\rangle + \left\langle f(u), u - y^k\right\rangle + \left\langle f(y^k), y^k - t^k\right\rangle\right).
\end{aligned} \tag{6.2}$$

By Condition 3.2,

$$\left\langle f(y^k) - f(u), u - y^k\right\rangle \leq 0, \tag{6.3}$$

and since $u \in \mathrm{SOL}(C,f)$,

$$\left\langle f(u), u - y^k\right\rangle \leq 0. \tag{6.4}$$

So,

$$\begin{aligned}
\left\|t^k - u\right\|^2 &\leq \|x^k - u\|^2 - \|x^k - t^k\|^2 + 2\tau \left\langle f(y^k), y^k - t^k\right\rangle \\
&= \|x^k - u\|^2 - \|x^k - y^k\|^2 - 2\left\langle x^k - y^k, y^k - t^k\right\rangle \\
&- \|y^k - t^k\|^2 + 2\tau \left\langle f(y^k), y^k - t^k\right\rangle \\
&= \|x^k - u\|^2 - \|x^k - y^k\|^2 - \|y^k - t^k\|^2 \\
&+ 2\left\langle x^k - \tau f(y^k) - y^k, t^k - y^k\right\rangle.
\end{aligned} \tag{6.5}$$

By the definition of T_k,

$$\left\langle \left(x^k - \tau f(x^k)\right) - y^k, t^k - y^k\right\rangle \leq 0, \tag{6.6}$$

so

$$
\begin{aligned}
&\left\langle x^k - \tau f(y^k) - y^k, t^k - y^k \right\rangle \\
&= \left\langle x^k - \tau f(x^k) - y^k, t^k - y^k \right\rangle + \left\langle \tau f(x^k) - \tau f(y^k), t^k - y^k \right\rangle \\
&\le \left\langle \tau f(x^k) - \tau f(y^k), t^k - y^k \right\rangle \le \tau \|f(x^k) - f(y^k)\| \|t^k - y^k\| \\
&\le \tau L \|x^k - y^k\| \|t^k - y^k\|, \qquad (6.7)
\end{aligned}
$$

where the last two inequalities follow from the Cauchy–Schwarz inequality and Condition 3.3. Therefore

$$
\left\|t^k - u\right\|^2 \le \|x^k - u\|^2 - \|x^k - y^k\|^2 - \|y^k - t^k\|^2 + 2\tau L \|x^k - y^k\| \|t^k - y^k\|. \quad (6.8)
$$

Observe that

$$
\begin{aligned}
0 &\le \left(\|t^k - y^k\| - \tau L \|x^k - y^k\|\right)^2 \\
&= \|t^k - y^k\|^2 - 2\tau L \|x^k - y^k\| \|t^k - y^k\| + \tau^2 L^2 \|x^k - y^k\|^2, \qquad (6.9)
\end{aligned}
$$

so,

$$
2\tau L \|x^k - y^k\| \|t^k - y^k\| \le \|t^k - y^k\|^2 + \tau^2 L^2 \|x^k - y^k\|^2. \qquad (6.10)
$$

Thus

$$
\begin{aligned}
\left\|t^k - u\right\|^2 &\le \|x^k - u\|^2 - \|x^k - y^k\|^2 - \|y^k - t^k\|^2 \\
&\quad + \|t^k - y^k\|^2 + \tau^2 L^2 \|x^k - y^k\|^2 \\
&= \|x^k - u\|^2 - \|x^k - y^k\|^2 + \tau^2 L^2 \|x^k - y^k\|^2 \\
&= \|x^k - u\|^2 + (\tau^2 L^2 - 1) \|x^k - y^k\|^2 \\
&\le \|x^k - u\|^2, \qquad (6.11)
\end{aligned}
$$

where the last inequality follows from the fact that $\tau \in (0, 1/L)$. Now by the definition of z^k and (2.7), we get

$$
\begin{aligned}
\|z^k - u\|^2 &= \|\alpha_k x^k + (1 - \alpha_k) t^k - u\|^2 \\
&= \|\alpha_k \left(x^k - u\right) + (1 - \alpha_k) \left(t^k - u\right) \|^2 \\
&= \alpha_k \|x^k - u\|^2 + (1 - \alpha_k) \|t^k - u\|^2 \\
&- \alpha_k (1 - \alpha_k) \| \left(x^k - u\right) - \left(t^k - u\right) \|^2 \\
&\le \alpha_k \|x^k - u\|^2 + (1 - \alpha_k) \|t^k - u\|^2 \\
&\le \alpha_k \|x^k - u\|^2 + (1 - \alpha_k) \left(\|x^k - u\|^2 + (\tau^2 L^2 - 1) \|x^k - y^k\|^2\right) \\
&= \|x^k - u\|^2 + (1 - \alpha_k)(\tau^2 L^2 - 1) \|x^k - y^k\|^2 \\
&\le \|x^k - u\|^2, \qquad (6.12)
\end{aligned}
$$

so $u \in C_{k+1}$ and therefore $\mathrm{SOL}(C,f) \subset C_{k+1}$. Denote $u^* = P_{SOL(C,f)}(x^0)$. It is clear that $u^* \in \mathrm{SOL}(C,f)$. Since $\mathrm{SOL}(C,f) \subset C_{k+1}$, $u^* \in SOL(C,f)$ and $x^{k+1} = P_{C_{k+1}}(x^0)$, we have

$$\|x^{k+1} - x^0\| \leq \|u^* - x^0\| \text{ for all } k \geq 0. \tag{6.13}$$

This implies, in particular, that $\left\{x^k\right\}_{k=0}^{\infty}$ is bounded, and it follows from (6.11) and (6.12) that so are $\left\{t^k\right\}_{k=0}^{\infty}$ and $\left\{z^k\right\}_{k=0}^{\infty}$. By the definition of the iterative step, $x^k = P_{C_k}(x^0)$, so by (2.4) we have

$$\left\langle x^k - x^0, z - x^k \right\rangle \geq 0 \text{ for all } z \in C_k. \tag{6.14}$$

Since $\mathrm{SOL}(C,f) \subset C_k$, we have

$$\left\langle x^k - x^0, u - x^k \right\rangle \geq 0 \text{ for all } u \in SOL(C,f). \tag{6.15}$$

So by the Cauchy–Schwarz inequality,

$$\begin{aligned} 0 &\leq \left\langle x^k - x^0, u - x^k \right\rangle = \left\langle x^k - x^0, u - x^0 + x^0 - x^k \right\rangle \\ &= -\|x^k - x^0\|^2 + \left\langle x^k - x^0, u - x^0 \right\rangle \\ &\leq -\|x^k - x^0\|^2 + \|x^k - x^0\|\|u - x^0\| \end{aligned} \tag{6.16}$$

and therefore

$$\|x^k - x^0\| \leq \|u - x^0\| \text{ for all } u \in SOL(C,f). \tag{6.17}$$

Now by the definition of our algorithm, $x^k = P_{C_k}(x^0)$, $x^{k+1} = P_{C_{k+1}}(x^0) \in C_{k+1} \subset C_k$ and (2.4), we have

$$\left\langle x^0 - x^k, x^k - x^{k+1} \right\rangle \geq 0 \text{ for all } k \geq 0. \tag{6.18}$$

Now by (6.18),

$$\begin{aligned} \left\|x^k - x^{k+1}\right\|^2 &= \left\|x^k - x^0 + x^0 - x^{k+1}\right\|^2 \\ &= \left\|x^k - x^0\right\|^2 + 2\left\langle x^k - x^0, x^0 - x^{k+1}\right\rangle + \left\|x^0 - x^{k+1}\right\|^2 \\ &= \left\|x^k - x^0\right\|^2 + 2\left\langle x^k - x^0, x^0 - x^k + x^k - x^{k+1}\right\rangle + \left\|x^0 - x^{k+1}\right\|^2 \\ &= \left\|x^k - x^0\right\|^2 - 2\left\|x^k - x^0\right\|^2 + 2\left\langle x^k - x^0, x^k - x^{k+1}\right\rangle + \left\|x^0 - x^{k+1}\right\|^2 \\ &\leq -\left\|x^k - x^0\right\|^2 + \left\|x^0 - x^{k+1}\right\|^2. \end{aligned} \tag{6.19}$$

Thus,

$$\|x^k - x^0\| \le \|x^0 - x^{k+1}\| \quad \text{for all } k \ge 0, \tag{6.20}$$

and therefore the sequence $\left\{\|x^k - x^0\|\right\}_{k=0}^{\infty}$ is increasing and since it is also bounded, it converges to some $l \in R$, i.e.,

$$\lim_{k\to\infty} \|x^k - x^0\| = l. \tag{6.21}$$

Apply (6.21) to (6.19), to obtain

$$\lim_{k\to\infty} \|x^{k+1} - x^k\| = 0. \tag{6.22}$$

In addition, since $x^{k+1} \in C_{k+1} \subset C_k$,

$$\|z^k - x^{k+1}\|^2 \le \|x^k - x^{k+1}\|^2, \tag{6.23}$$

and by the triangle inequality,

$$\|x^k - z^k\| \le \|x^k - x^{k+1}\| + \|x^{k+1} - z^k\| \le 2\|x^k - x^{k+1}\|, \tag{6.24}$$

so

$$\lim_{k\to\infty} \|x^k - z^k\| = 0. \tag{6.25}$$

By (6.12),

$$\|z^k - u\|^2 \le \|x^k - u\|^2 + (1-\alpha_k)(\tau^2 L^2 - 1)\|x^k - y^k\|^2, \tag{6.26}$$

or

$$\begin{aligned}\|x^k - y^k\|^2 &\le \frac{\|x^k - u\|^2 - \|z^k - u\|^2}{(1-\alpha_k)(1-\tau^2 L^2)} \\ &= \frac{1}{(1-\alpha_k)(1-\tau^2 L^2)} \left(\|x^k - u\| - \|z^k - u\|\right)\left(\|x^k - u\| + \|z^k - u\|\right) \\ &\le \frac{1}{(1-\alpha_k)(1-\tau^2 L^2)} \left(\|x^k - u\| + \|z^k - u\|\right)\|x^k - z^k\|. \quad (6.27)\end{aligned}$$

By (6.25) and the boundedness of $\left\{x^k\right\}_{k=0}^{\infty}$ and $\left\{z^k\right\}_{k=0}^{\infty}$, we get

$$\lim_{k\to\infty} \|x^k - y^k\| = 0. \tag{6.28}$$

By Condition 3.3,

$$\lim_{k\to\infty} \|f(x^k) - f(y^k)\| = 0. \tag{6.29}$$

Since $\{x^k\}_{k=0}^{\infty}$ is bounded, there exists a subsequence $\{x^{k_j}\}_{j=0}^{\infty}$ of $\{x^k\}_{k=0}^{\infty}$ which converges weakly to some $\overline{x} \in \mathcal{H}$. We claim that $\overline{x} \in \mathrm{SOL}(C, f)$. Indeed, let

$$A(v) = \begin{cases} f(v) + N_C(v), & v \in C, \\ \emptyset, & v \notin C, \end{cases} \tag{6.30}$$

where $N_C(v)$ is the normal cone of C at $v \in C$ (2.6). It is known [22, Theorem 3] that A is a maximal monotone operator and $A^{-1}(0) = \mathrm{SOL}(f, C)$. If $(v, w) \in G(A)$, then $w \in A(v) = f(v) + N_C(v)$, or $w - f(v) \in N_C(v)$. Hence

$$\langle w - f(v), v - y \rangle \geq 0 \text{ for all } y \in C. \tag{6.31}$$

On the other hand, by the definition of y^k and (2.4),

$$\left\langle x^k - \tau f(x^k) - y^k, y^k - v \right\rangle \geq 0, \tag{6.32}$$

or

$$\left\langle \left(\frac{y^k - x^k}{\tau}\right) + f(x^k), v - y^k \right\rangle \geq 0 \tag{6.33}$$

for all $k \geq 0$. Applying (6.31) with $\{y^{k_j}\}_{j=0}^{\infty}$, we get

$$\left\langle w - f(v), v - y^{k_j} \right\rangle \geq 0. \tag{6.34}$$

Hence,

$$\begin{aligned} \left\langle w, v - y^{k_j} \right\rangle &\geq \left\langle f(v), v - y^{k_j} \right\rangle \geq \left\langle f(v), v - y^{k_j} \right\rangle \\ &- \left\langle \left(\frac{y^{k_j} - x^{k_j}}{\tau}\right) + f(x^{k_j}), v - y^{k_j} \right\rangle \\ &= \left\langle f(v) - f(y^{k_j}), v - y^{k_j} \right\rangle + \left\langle f(y^{k_j}) - f(x^{k_j}), v - y^{k_j} \right\rangle \\ &- \left\langle \left(\frac{y^{k_j} - x^{k_j}}{\tau}\right), v - y^{k_j} \right\rangle \\ &\geq \left\langle f(y^{k_j}) - f(x^{k_j}), v - y^{k_j} \right\rangle - \left\langle \left(\frac{y^{k_j} - x^{k_j}}{\tau}\right), v - y^{k_j} \right\rangle \end{aligned} \tag{6.35}$$

and

$$\left\langle w, v - y^{k_j} \right\rangle \geq \left\langle f(y^{k_j}) - f(x^{k_j}), v - y^{k_j} \right\rangle - \left\langle \left(\frac{y^{k_j} - x^{k_j}}{\tau} \right), v - y^{k_j} \right\rangle . \tag{6.36}$$

Since $y^{k_j} \rightharpoonup \bar{x}$ by (6.28), taking the limit as $j \to \infty$ and using (6.28), we obtain

$$\langle w, v - \bar{x} \rangle \geq 0, \tag{6.37}$$

and since A is a maximal monotone operator, it follows that $\bar{x} \in A^{-1}(0) = \mathrm{SOL}(f, C)$, as claimed. From $u^* = P_{SOL(C,f)}(x^0)$, $\bar{x} \in \mathrm{SOL}(f, C)$, (6.13) and the weak lower semi-continuity of the norm, it follows that

$$\begin{aligned} \|u^* - x^0\| \leq \|\bar{x} - x^0\| &\leq \liminf_{j\to\infty} \|x^{k_j} - x^0\| \\ &\leq \limsup_{j\to\infty} \|x^{k_j} - x^0\| \leq \|u^* - x^0\|, \end{aligned} \tag{6.38}$$

so

$$\lim_{j\to\infty} \|x^{k_j} - x^0\| = \|\bar{x} - x^0\|. \tag{6.39}$$

Hence we have $x^{k_j} - x^0 \rightharpoonup \bar{x} - x^0$ and $\|x^{k_j} - x^0\| \to \|\bar{x} - x^0\|$, and so by the Kadec-Klee property of $\mathcal{H}$ we obtain $\|x^{k_j} - \bar{x}\| \to 0$. Since $x^{k_j} = P_{C_{k_j+1}}(x^0)$ and $u^* \in C_{k_j+1}$, we see that

$$\begin{aligned} -\|u^* - x^{k_j}\|^2 &= \left\langle u^* - x^{k_j}, x^{k_j} - x^0 \right\rangle + \left\langle u^* - x^{k_j}, x^0 - u^* \right\rangle \\ &\geq \left\langle u^* - x^{k_j}, x^0 - u^* \right\rangle . \end{aligned} \tag{6.40}$$

Taking the limit as $j \to \infty$, we obtain

$$-\|u^* - \bar{x}\|^2 \geq \left\langle u^* - \bar{x}, x^0 - u^* \right\rangle \geq 0, \tag{6.41}$$

and therefore

$$\lim_{k\to\infty} x^{k_j} = \bar{x} = u^*. \tag{6.42}$$

Since $\left\{x^{k_j}\right\}_{j=0}^{\infty}$ is an arbitrary weakly convergent subsequence of $\left\{x^k\right\}_{k=0}^{\infty}$, we conclude that $\left\{x^k\right\}_{k=0}^{\infty}$ converges strongly to u^*, i.e., $\lim_{k\to\infty} x^k = u^* = P_{SOL(C,f)}(x^0)$, as asserted. ■

Remark 6.2 *In Theorems 4.1 and 6.1 we assume that f is Lipschitz continuous on $\mathcal{H}$ with constant $L > 0$ (Condition 3.3). If we assume that f is*

Lipschitz continuous only on C with constant $L > 0$, we can use a Lipschitz extension of f to $\mathcal{H}$ in order to evaluate the function at x^k. Such an extension exists by Kirszbraun's theorem [18], which states that there exists a Lipschitz continuous function $\tilde{f} : \mathcal{H} \to \mathcal{H}$ that extends f and has the same Lipschitz constant L as f. Alternatively, we can take $\tilde{f} = fP_C$. In any case, the extension is not necessarily monotone on $\mathcal{H}$ but preserves monotonicity on C, which is all that we need in the proofs.

Remark 6.3 *Note that in the proofs of Theorems 4.1 and 6.1, once the fact that the weak cluster points are solutions is established, it is possible to refer to [9, Proposition 3.1(vi)] or to [2, Theorem 3.5 (iv)] and deduce the strong convergence to the projection onto the solution set of the initial point.*

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