Common Solutions to Variational Inequalities

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Abstract

We present and study a new variational inequality problem, which we call the Common Solutions to Variational Inequalities Problem (CSVIP). This problem consists of finding common solutions to a system of unrelated variational inequalities corresponding to set-valued mappings in Hilbert space. We present an iterative procedure for solving this problem and establish its strong convergence. Relations with other problems of solving systems of variational inequalities, both old and new, are discussed as well.

Keywords: Hilbert space · Iterative procedure · Maximal monotone mapping · Nonexpansive mapping · Variational inequality

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1 Introduction

In this paper we present a new variational inequality problem which we term the *Common Solutions to Variational Inequalities Problem* (CSVIP). This problem is formulated as follows.

Problem 1.1 *Let $\mathcal{H}$ be a real Hilbert space. Let there be given, for each $i = 1, 2, \ldots, N$, a set-valued mapping $A_i : \mathcal{H} \to 2^{\mathcal{H}}$ and a nonempty, closed and convex subset $K_i \subseteq \mathcal{H}$, with $\bigcap_{i=1}^{N} K_i \neq \emptyset$. The CSVIP is to find a point $x^* \in \bigcap_{i=1}^{N} K_i$ such that, for each $i = 1, 2, \ldots, N$, there exists $u_i^* \in A_i(x^*)$ satisfying*

$$\langle u_i^*, x - x^* \rangle \geq 0 \text{ for all } x \in K_i,\ i = 1, 2, \ldots, N. \tag{1.1}$$

Obviously, if $N = 1$ then the problem is nothing but the well-known *Variational Inequality Problem* (VIP), first introduced (with a single-valued mapping) by Hartman and Stampacchia in 1966 (see [16]). The motivation for defining and studying such CSVIPs with $N > 1$ stems from the simple observation that if we choose all $A_i = 0$, then the problem reduces to that of finding a point $x^* \in \bigcap_{i=1}^{N} K_i$ in the nonempty intersection of a finite family of closed and convex sets, which is the well-known *Convex Feasibility Problem* (CFP). If the sets K_i are the fixed point sets of a family of operators $T_i : \mathcal{H} \to \mathcal{H}$, then the CFP is the *Common Fixed Point Problem* (CFPP). These problems have been intensively studied over the past decades both theoretically (existence, uniqueness, properties, etc. of solutions) and algorithmically (devising iterative procedures which generate sequences that converge, finitely or asymptotically, to a solution).

Since the phrase "system of variational inequalities" has been extensively used in the literature for many different problems, as can be seen from the cases mentioned in Subsection 1.1 below, it seems natural to call our new problem the *Common Solutions to Variational Inequalities Problem*.

The significance of studying the CSVIP lies in the fact that besides its enabling a unified treatment of such well-known problems as the CFP and the CFPP, the CSVIP also opens a path to a variety of new "common point problems" that are created from various special cases of the VIP. For an excellent treatise on variational inequality problems in finite-dimensional spaces, see the two-volume book by Facchinei and Pang [12]. The books by Konnov [21] and Patriksson [27] contain extensive studies of VIPs including applications, algorithms and numerical results. For a wide range of applications of VIPs, see, e.g., the book by Kinderlehrer and Stampacchia [22]. The importance of VIPs stems from the fact that several fundamental problems in Optimization Theory can be formulated as VIPs, as the following few examples show.

Example 1.2 ***Constrained minimization.*** *Let $K \subseteq \mathcal{H}$ be a nonempty, closed and convex subset and let $g : \mathcal{H} \to \mathbb{R}$ be a continuously differentiable function which is convex on K. Then x^* is a minimizer of g over K if and only if x^* solves the VIP*

$$\langle \nabla g(x^*), x - x^* \rangle \geq 0 \textit{ for all } x \in K, \tag{1.2}$$

where ∇g is the gradient of g (see, e.g., [5, Proposition 3.1, p. 210]). When g is not differentiable, we get the VIP

$$\langle u^*, x - x^* \rangle \geq 0 \textit{ for all } x \in K, \tag{1.3}$$

where $u^* \in \partial g(x^*)$ and ∂g is the set-valued subdifferential of g (see, e.g., [15, Chapter 4, Subsection 3.5]).

Example 1.3 *When the Hilbert space $\mathcal{H}$ is $\mathbb{R}^n$ and the set K is $\mathbb{R}^n_+$, then the VIP obtained from (1.1) with $N = 1$ is equivalent to the* ***nonlinear complementarity problem:*** *find a point $x^* \in \mathbb{R}^n_+$ and a point $u^* \in A(x^*)$ such that $u^* \in \mathbb{R}^n_+$ and $\langle u^*, x^* \rangle = 0$.*

Indeed, let $\mathcal{H}$ be $\mathbb{R}^n$ and $K = \mathbb{R}^n_+$. So, if x^* solves (1.1) with $N = 1$ and $A : \mathbb{R}^n \to 2^{\mathbb{R}^n}$, then there exists $x^* \in \mathbb{R}^n_+$ such that $u^* \in A(x^*)$ satisfies

$$\langle u^*, x - x^* \rangle \geq 0 \textit{ for all } x \in \mathbb{R}^n_+. \tag{1.4}$$

So, in particular, if we take $x = 0$ we obtain $\langle u^*, x^* \rangle \leq 0$ and if we take $x = 2x^*$ we obtain $\langle u^*, x^* \rangle \geq 0$. Combining the above two inequalities, we see that $\langle u^*, x^* \rangle = 0$. As a consequence, this yields

$$\langle u^*, x \rangle \geq 0 \textit{ for all } x \in \mathbb{R}^n_+ \tag{1.5}$$

and hence $u^* \in \mathbb{R}^n_+$. Conversely, if x^* solves the nonlinear complementarity problem, then $\langle u^*, x - x^* \rangle = \langle u^*, x \rangle \geq 0$ for all $x \in \mathbb{R}^n_+$ (since $u^* \in \mathbb{R}^n_+$), which means that x^*solves (1.1) with $N = 1$.

Example 1.4 *When the set K is the whole space $\mathcal{H}$, then the VIP obtained from (1.1) with $N = 1$ is equivalent to the problem of finding* ***zeros of a mapping*** *A, i.e., to find an element $x^* \in \mathcal{H}$ such that $0 \in A(x^*)$.*

Example 1.5 *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces, and let K_1 and K_2 be two convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Given a function $g : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathbb{R}$, the* `Saddle-Point Problem` *is to find a point $(u_1^*, u_2^*) \in K_1 \times K_2$ such that*

$$g(u_1^*, u_2) \leq g(u_1^*, u_2^*) \leq g(u_1, u_2^*) \text{ for all } (u_1, u_2) \in K_1 \times K_2. \tag{1.6}$$

This problem can be written as the VIP of finding $(u_1^*, u_2^*) \in K_1 \times K_2$ such that

$$\left\langle \begin{pmatrix} \nabla g_{u_1}(u_1^*, u_2^*) \\ -\nabla g_{u_2}(u_1^*, u_2^*) \end{pmatrix}, \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \begin{pmatrix} u_1^* \\ u_2^* \end{pmatrix} \right\rangle \geq 0 \text{ for all } (u_1, u_2) \in K_1 \times K_2. \tag{1.7}$$

Our main goal in this paper is to present an iterative procedure for solving CSVIPs and prove its strong convergence. Our algorithm, besides generating a sequence which strongly converges to a solution, also solves the, so called, *Best Approximation Problem* (BAP), which consists of finding the nearest point projection of a point onto the (unknown) intersection of N closed and convex subsets (see, e.g., [7] and the references therein). More precisely, our algorithm generates a sequence which converges strongly to the nearest point projection of the starting point onto the solution set of the CSVIP.

A special case of the CSVIP (in the Euclidean space $\mathbb{R}^d$ and with single-valued mappings $A_i : \mathbb{R}^d \to \mathbb{R}^d$, $i = 1, 2, \ldots, N$) was considered in [9, Subsection 7.2]. There we transformed that CSVIP into a Constrained Variational Inequality Problem (CVIP) in an appropriate product space, i.e.,

$$\text{find a point } \boldsymbol{x}^* \in \boldsymbol{K} \cap \boldsymbol{\Delta} \text{ such that } \langle \boldsymbol{A}(\boldsymbol{x}^*), \boldsymbol{x} - \boldsymbol{x}^* \rangle \geq 0 \tag{1.8}$$

$$\text{for all } \boldsymbol{x} = (x^1, x^2, \ldots, x^N) \in \boldsymbol{K}, \tag{1.9}$$

where $\boldsymbol{K} := \Pi_{i=1}^N K_i$, the diagonal set in $\mathbb{R}^{Nd}$

$$\boldsymbol{\Delta} := \{\boldsymbol{x} \in \mathbb{R}^{Nd} \mid \boldsymbol{x}{=}(a, a, \ldots, a),\ a \in \mathbb{R}^d\} \tag{1.10}$$

and $\boldsymbol{A} : \mathbb{R}^{Nd} \to \mathbb{R}^{Nd}$ is defined by

$$\boldsymbol{A}\left((x^1, x^2, \ldots, x^N)\right) = (A_1(x^1), \ldots, A_N(x^N)), \tag{1.11}$$

where $x^i \in \mathbb{R}^d$ for all $i = 1, 2, \ldots, N$. So, problem (1.8)-(1.9) can be solved by [9, Algorithm 4.4]. In the present paper, besides extending this problem to the set-valued case, we propose an algorithm that does not require the transformation into a product space.

The paper is organized as follows. In Subsection 1.1 we describe the connections between our work and some earlier papers and in Section 2 we list several known facts about functions, operators and mappings that we need in the sequel. In Section 3 we present our algorithm for solving the CSVIP and prove its strong convergence through a sequence of Claims. In Section 4 we present five special cases of the CSVIP.

1.1 Relation with previous work

Several variants of systems of variational inequalities appeared during the last decades. We present some of them in detail and show their connection to the CSVIP.

1. Konnov [20] considers the following system of variational inequalities. Let $K \subseteq \mathbb{R}^n$ be a nonempty, closed and convex set and let $A_i : K \to 2^{\mathbb{R}^n}$, $i = 1, 2, \ldots, N$, be N set-valued mappings. The problem is to find a point $x^* \in K$ such that for each $i = 1, 2, \ldots, N$, there exists $u_i^* \in A_i(x^*)$ satisfying

$$\langle u_i^*, x - x^* \rangle \geq 0 \text{ for all } x \in K,\ i = 1, 2, \ldots, N. \tag{1.12}$$

   This means that Konnov solves a CSVIP with $\mathcal{H} = \mathbb{R}^n$ and $K_i = K$ for all $i = 1, 2, \ldots, N$.

2. Ansari and Yao [1] studied the following system of variational inequalities. Let I be an index set and for each $i \in I$, let X_i be a Hausdorff topological vector space with its topological dual X_i^*. Let K_i, $i \in I$, be nonempty, closed and convex subsets of X_i. Let $K = \prod_{i=1}^{N} K_i$ and let $A_i : K \to X_i^*$, $i = 1, 2, \ldots, N$, be single-valued mappings (see also [26] for more details). Ansari and Yao then consider the problem of finding a point $x^* \in K$ such that

$$\langle A_i(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in K_i, i = 1, 2, \ldots, N. \tag{1.13}$$

3. Kassay and Kolumbán [19] solve another system of two variational inequalities. Let X and Y be two reflexive real Banach spaces and let $K_1 \subseteq X$ and $K_2 \subseteq Y$ be nonempty, closed and convex sets. Denote by X^* and Y^* the dual spaces of X and Y, respectively. Consider two set-valued mappings $A_1 : K_1 \times K_2 \to 2^{X^*}$ and $A_2 : K_1 \times K_2 \to 2^{Y^*}$.

Kassay's and Kolumbán's problem is to find a pair $(x_1, x_2) \in K_1 \times K_2$ such that

$$\begin{aligned} \sup_{w \in A_1(x_1,x_2)} \langle w, x - x_1 \rangle &\geq 0 \text{ for all } x \in K_1, \\ \sup_{z \in A_2(x_1,x_2)} \langle z, y - x_2 \rangle &\geq 0 \text{ for all } y \in K_2. \end{aligned} \tag{1.14}$$

4. Recently, Zhao et al. [34] have considered the following system of two variational inequalities in Euclidean spaces. Let K_1 and K_2 be two closed and convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Let $A_1 : K_1 \times K_2 \to \mathbb{R}^n$ and $A_2 : K_1 \times K_2 \to \mathbb{R}^m$ be two single-valued mappings. Then Zhao et al.'s problem is to find a point $(u_1^*, u_2^*) \in K_1 \times K_2$ such that

$$\begin{aligned} \langle A_1(u_1^*, u_2^*), u_1 - u_1^* \rangle &\geq 0 \text{ for all } u_1 \in K_1, \\ \langle A_2(u_1^*, u_2^*), u_2 - u_2^* \rangle &\geq 0 \text{ for all } u_2 \in K_2. \end{aligned} \tag{1.15}$$

The main difference between problems (1.1) and (1.15) is that our system includes any finite number (not only two) of mappings (not only single-valued) defined on different sets. In addition, in our case the problem is formulated in Hilbert space (not only in Euclidean space).

2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. In what follows, a point-to-set function $A : \mathcal{H} \to 2^{\mathcal{H}}$ is called a `set-valued mapping` (or a mapping for short) on $\mathcal{H}$. When each set Ax is either empty or a singleton we call A a `single-valued mapping` (or an operator for short) on $\mathcal{H}$. The `domain` of a mapping A is the set

$$\operatorname{dom} A := \{x \in \mathcal{H} : Ax \neq \emptyset\}. \tag{2.1}$$

The `range` of a mapping A is the set

$$\operatorname{ran} A := \{u \in Ax : x \in \operatorname{dom} A\}. \tag{2.2}$$

The `graph` of a mapping A is the subset of $\mathcal{H} \times \mathcal{H}$ defined by

$$\operatorname{graph} A := \{(x, u) \in \mathcal{H} \times \mathcal{H} : u \in Ax\}. \tag{2.3}$$

We write w-$\lim_{n\to+\infty} x^n = x$ to indicate that the sequence $\{x^n\}_{n\in\mathbb{N}}$ converges weakly to x and $\lim_{n\to+\infty} x^n = x$ to indicate that the sequence $\{x^n\}_{n\in\mathbb{N}}$ converges strongly to x.

The next property is known as the `Opial condition` (see [25]). Every Hilbert space enjoys this property.

Condition 2.1 (Opial condition) *For any sequence $\{x^n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$ that converges weakly to x, we have*

$$\liminf_{n\to+\infty} \|x^n - x\| < \liminf_{n\to+\infty} \|x^n - y\| \tag{2.4}$$

for all $y \neq x$.

Any Hilbert space $\mathcal{H}$ has the `Kadec-Klee property` (see, for instance, [13]), that is, if $\{x^n\}_{n\in\mathbb{N}}$ is a sequence in $\mathcal{H}$ which satisfies w-$\lim_{n\to+\infty} x^n = x$ and $\lim_{n\to+\infty} \|x^n\| = \|x\|$, then $\lim_{n\to+\infty} \|x^n - x\| = 0$.

Definition 2.2 (Weakly lower semicontinuous) *A function $g : \mathcal{H} \to (-\infty, +\infty]$ is called* ***weakly lower semicontinuous*** *if*

$$g(x) \leq \liminf_{n\to+\infty} g(x^n) \tag{2.5}$$

for any sequence $\{x^n\}_{n\in\mathbb{N}}$ which satisfies w-$\lim_{n\to+\infty} x^n = x$.

Definition 2.3 (Monotone mappings) *Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a mapping. We say that*

(i) A is ***monotone*** *if for any $x, y \in \operatorname{dom} A$ we have*

$$\langle u - v, x - y\rangle \geq 0 \text{ for all } u \in A(x) \text{ and } v \in A(y); \tag{2.6}$$

(ii) A is ***maximal monotone*** *if it is monotone and its graph* graph A *is not properly contained in the graph of any other monotone mapping.*

The notion of maximal monotonicity can be equivalently formulated in the following way.

Remark 2.4 (Maximal monotone mappings) *The mapping $A : \mathcal{H} \to 2^{\mathcal{H}}$ is maximal monotone if and only if we have*

$$\left.\begin{array}{l} \forall (y, v) \in \operatorname{graph} A \\ \langle u - v, x - y\rangle \geq 0 \end{array}\right\} \quad \Longrightarrow \quad u \in Ax. \tag{2.7}$$

We now recall two definition. Let $K \subseteq \mathcal{H}$ be a nonempty, closed and convex set. Denote by $\mathrm{CB}(K) \subseteq 2^K$ the family of all nonempty, closed, convex and bounded subsets of K.

Definition 2.5 (Hausdorff metric) *Let $K_1, K_2 \in CB(K)$. The* ***Hausdorff metric*** *on $CB(K)$ is defined by*

$$H(K_1, K_2) := \max \left\{ \sup_{x \in K_2} d(x, K_1), \sup_{y \in K_1} d(y, K_2) \right\}, \tag{2.8}$$

where the distance function is defined by $d(x, K) := \inf \{ \|x - z\| : z \in K \}$.

Definition 2.6 (Nonexpansive mappings) *Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be a mapping such that $A(x) \in CB(\mathcal{H})$ for each $x \in \mathcal{H}$. We say that*

(i) A is ***Lipschitz continuous*** *with constant $L_A > 0$ if*

$$H(A(x), A(y)) \leq L_A \|x - y\| \text{ for all } x, y \in \mathcal{H}. \tag{2.9}$$

So, given $x \in \mathcal{H}$, $u_x \in A(x)$ and $y \in \mathcal{H}$, there exists $v_y \in A(y)$ such that $\|u_x - v_y\| \leq L_A \|x - y\|$.

(ii) A is ***nonexpansive*** *(see, for example, [17]) if it is Lipschitz continuous with $L_A = 1$.*

Let K be a nonempty, closed and convex subset $\mathcal{H}$. For each point $x \in \mathcal{H}$, there exists a unique nearest point in K, denoted by $P_K(x)$. That is,

$$\|x - P_K(x)\| \leq \|x - y\| \text{ for all } y \in K. \tag{2.10}$$

The operator $P_K : \mathcal{H} \to K$ is called the `metric projection` of $\mathcal{H}$ onto K. It is well known that P_K is a nonexpansive operator from $\mathcal{H}$ onto K. The metric projection P_K is characterized (see [14, Section 3]) by the following two properties:

$$P_K(x) \in K \tag{2.11}$$

and

$$\langle x - P_K(x), y - P_K(x) \rangle \leq 0 \text{ for all } x \in \mathcal{H},\ y \in K. \tag{2.12}$$

If K is a hyperplane, then (2.12) becomes an equality. It is easy to check that (2.12) is equivalent to

$$\|x - P_K(x)\|^2 + \|y - P_K(x)\|^2 \leq \|x - y\|^2 \text{ for all } x \in \mathcal{H},\ y \in K. \tag{2.13}$$

We denote by $N_K(v)$ the `normal cone` of K at $v \in K$, i.e.,

$$N_K(v) := \{z \in \mathcal{H} : \langle z, y - v \rangle \leq 0 \text{ for all } y \in K\}. \tag{2.14}$$

We also recall that in a real Hilbert space $\mathcal{H}$,

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2 \tag{2.15}$$

for all $x, y \in \mathcal{H}$ and $\lambda \in [0,1]$.

The following result will be essential in the proof of our main theorem.

Claim 2.7 *Consider the half-space*

$$H(x,y) := \{z \in \mathcal{H} : \langle x - y, z - y \rangle \leq 0\}. \tag{2.16}$$

Given two points x and y in $\mathcal{H}$, set $y_\lambda := \lambda x + (1-\lambda)y$ for any $\lambda \in [0,1]$. Then $H = H(x,y) \subseteq H(x, y_\lambda) =: H_\lambda$.

Proof. Let $z \in H$. In order to show that $z \in H_\lambda$, $\lambda \in [0,1]$, we need to check that $\langle x - y_\lambda, z - y_\lambda \rangle \leq 0$. We have

$$\begin{aligned}
\langle x - y_\lambda, z - y_\lambda \rangle &= \langle x - (\lambda x + (1-\lambda)y), z - (\lambda x + (1-\lambda)y) \rangle \\
&= \langle (1-\lambda)x - (1-\lambda)y, (\lambda z + (1-\lambda)z) - (\lambda x + (1-\lambda)y) \rangle \\
&= (1-\lambda)\langle x - y, (1-\lambda)(z-y) + \lambda(z-x) \rangle \\
&= (1-\lambda)^2 \langle x-y, z-y \rangle + (1-\lambda)\lambda \langle x-y, z-x \rangle \\
&= (1-\lambda)^2 \langle x-y, z-y \rangle + (1-\lambda)\lambda \langle x-y, y-x \rangle \\
&\quad + (1-\lambda)\lambda \langle x-y, z-y \rangle \\
&= (1-\lambda)\langle x-y, z-y \rangle - (1-\lambda)\lambda \|x-y\|^2 \\
&\leq (1-\lambda)\langle x-y, z-y \rangle.
\end{aligned} \tag{2.17}$$

Since $z \in H$, we know that $\langle x-y, z-y \rangle \leq 0$. Hence $z \in H_\lambda$ for any $\lambda \in [0,1]$, as claimed. ■

Definition 2.8 (Fixed point set) *For a mapping $A : \mathcal{H} \to 2^{\mathcal{H}}$, we denote by* Fix A *the **`fixed point set`** of A, i.e.,*

$$\operatorname{Fix} A := \{x \in \mathcal{H} : x \in A(x)\}. \tag{2.18}$$

3 The algorithm

In this section we present a new algorithm for solving the CSVIP. Let $\{K_i\}_{i=1}^N$ be N nonempty, closed and convex subsets of $\mathcal{H}$. Let $\{A_i\}_{i=1}^N$ be a set of N mappings from $\mathcal{H}$ into $2^{\mathcal{H}}$ such that $A_i(x) \in \mathrm{CB}(\mathcal{H})$ for each $x \in \mathcal{H}$ and $i = 1, \ldots, N$. Denote by $\mathrm{SOL}(A_i, K_i)$ the solution set of the Variational Inequality Problem $\mathrm{VIP}(A_i, K_i)$ corresponding to the mapping A_i and the set K_i.

Algorithm 3.1

Initialization: *Select an arbitrary starting point* $x^1 \in \mathcal{H}$.

Iterative step: *Given the current iterate* x^n*, calculate the next iterate as follows:*

$$\begin{cases} y_i^n = P_{K_i}\left(x^n - \lambda_i^n u_i^n\right), \quad u_i^n \in A_i\left(x^n\right), \\ \text{find } v_i^n \in A_i\left(y_i^n\right) \text{ which satisfies Definition 2.6(i) with } u_i^n, \\ z_i^n = P_{K_i}\left(x^n - \lambda_i^n v_i^n\right), \\ C_i^n = \left\{z \in \mathcal{H} : \left\langle x^n - z_i^n, z - x^n - \gamma_i^n\left(z_i^n - x^n\right)\right\rangle \leq 0\right\}, \\ C^n = \bigcap_{i=1}^N C_i^n, \\ W^n = \left\{z \in \mathcal{H} : \left\langle x^1 - x^n, z - x^n\right\rangle \leq 0\right\}, \\ x^{n+1} = P_{C^n \cap W^n}\left(x^1\right). \end{cases} \tag{3.1}$$

This algorithm is quite complex in comparison with more "direct" iterative methods. In order to calculate the next approximation to the solution of the problem, the latter only use a value of one main operator at the current approximation. On the other hand, Algorithm 3.1 generates strongly convergent sequences, as is proved below, and this important property apparently complicates the process. It seems natural to ask by how much and how difficult it is to calculate C_i^n , $C^n = \cap_{i=1}^N C_i^n$, W^n and $C^n \cap W^n$.

Our main interest here is not to develop a practical numerical method and whether our work can help in the design and analysis of more practical algorithms remains to be seen. In Subsection 4.6 we give some simple computational examples to demonstrate the practical difficulties.

In order to prove our convergence theorem we assume that the following conditions hold.

Condition 3.2 *The mappings* $\{A_i\}_{i=1}^N$ *are maximal monotone and Lipschitz continuous with* $L_{A_i} = \alpha_i$.

Condition 3.3 *The common solution set* $F := \bigcap_{i=1}^N SOL(A_i, K_i)$ *is nonempty.*

Condition 3.4 *The sequence* $\{\lambda_i^n\}_{n\in\mathbb{N}} \subset [a, b]$, $i = 1, \ldots, N$, *for some* a *and* b *with* $0 < a < b < 1/\alpha$, *where* $\alpha := \max_{1\le i\le N} \alpha_i$.

Condition 3.5 *The sequence* $\{\gamma_i^n\}_{n\in\mathbb{N}} \subset [\varepsilon, 1/2]$ *for each* $i = 1, \ldots, N$, *where* $\varepsilon \in (0, 1/2]$.

Theorem 3.6 *Assume that Conditions 3.2-3.5 hold. Then any sequences* $\{x^n\}_{n\in\mathbb{N}}$, $\{y_i^n\}_{n\in\mathbb{N}}$ *and* $\{z_i^n\}_{n\in\mathbb{N}}$ *, generated by Algorithm 3.1, converge strongly to* $P_F(x^1)$.

Proof. We divide the proof into four claims.

Claim 3.7 *The projection* $P_F(x^1)$ *and the sequence* $\{x^n\}_{n\in\mathbb{N}}$ *are well-defined.*

Proof. It is known that each $\mathrm{SOL}(A_i, K_i)$, $i = 1, \ldots, N$, is a closed and convex subset of $\mathcal{H}$ (see, e.g., [4, Lemma 2.4(ii)]). Hence F is nonempty (by Condition 3.3), closed and convex, so $P_F(x^1)$ is well defined. Next, it is clear that both C_i^n and W^n are closed half-spaces for all $n \ge 1$. Therefore C^n and $C^n \cap W^n$ are closed and convex for all $n \ge 1$. It remains to be proved that $C^n \cap W^n$ is not empty for all n. When $\gamma_i^n = 1/2$ for all $n \in \mathbb{N}$ and for all $i = 1, \ldots, N$, then the set C_i^n has the following form:

$$\widetilde{C}_i^n := \{z \in \mathcal{H} : \|z_i^n - z\| \le \|x^n - z\|\}. \tag{3.2}$$

From Claim 2.7 it follows that

$$\widetilde{C}_i^n \subset \{z \in \mathcal{H} : \langle x^n - z_i^n, z - x^n - \gamma_i^n (z_i^n - x^n)\rangle \le 0\} = C_i^n. \tag{3.3}$$

Let $\widetilde{C}^n = \bigcap_{i=1}^N \widetilde{C}_i^n$. It is enough to show that $F \subseteq \widetilde{C}^n \cap W^n$ for all $n \in \mathbb{N}$. First we prove that $F \subseteq \widetilde{C}^n$ for all $n \in \mathbb{N}$. To this end, let $s \in F$ and let $w_i \in A_i(s)$ for any $i = 1, \ldots, N$. It now follows from (2.13) that

$$\begin{aligned}
\|z_i^n - s\|^2 &= \|P_{K_i}(x^n - \lambda_i^n v_i^n) - s\|^2 \\
&\le \|(x^n - \lambda_i^n v_i^n) - s\|^2 - \|(x^n - \lambda_i^n v_i^n) - z_i^n\|^2 \\
&= \|x^n - s\|^2 - \|x^n - z_i^n\|^2 + 2\lambda_i^n \langle v_i^n, s - z_i^n\rangle \\
&= \|x^n - s\|^2 - \|x^n - z_i^n\|^2 + 2\lambda_i^n [\langle v_i^n - w_i, s - y_i^n\rangle \\
&+ \langle w_i, s - y_i^n\rangle + \langle v_i^n, y_i^n - z_i^n\rangle]
\end{aligned} \tag{3.4}$$

for any $i = 1, \ldots, N$. Using the monotonicity of A_i and the fact that $s \in \mathrm{SOL}(A_i, K_i)$, we obtain from (3.4) that

$$\begin{aligned}
\|z_i^n - s\|^2 &\le \|x^n - s\|^2 - \|x^n - z_i^n\|^2 + 2\lambda_i^n \langle v_i^n, y_i^n - z_i^n \rangle \\
&= \|x^n - s\|^2 - \|x^n - y_i^n + y_i^n - z_i^n\|^2 + 2\lambda_i^n \langle v_i^n, y_i^n - z_i^n \rangle \\
&= \|x^n - s\|^2 - \|x^n - y_i^n\|^2 - 2\langle x^n - y_i^n, y_i^n - z_i^n \rangle - \\
&\quad \|y_i^n - z_i^n\|^2 + 2\lambda_i^n \langle v_i^n, y_i^n - z_i^n \rangle \\
&= \|x^n - s\|^2 - .\, \|x^n - y_i^n\|^2 - \|y_i^n - z_i^n\|^2 + \\
&\quad 2\langle x^n - \lambda_i^n v_i^n - y_i^n, z_i^n - y_i^n \rangle . \qquad (3.5)
\end{aligned}$$

From (2.12) we have

$$\begin{aligned}
\langle x^n - \lambda_i^n v_i^n - y_i^n, z_i^n - y_i^n \rangle &= \langle x^n - \lambda_i^n u_i^n - y_i^n, z_i^n - y_i^n \rangle \\
&\quad + \lambda_i^n \langle u_i^n - v_i^n, z_i^n - y_i^n \rangle \\
&\le \lambda_i^n \langle u_i^n - v_i^n, z_i^n - y_i^n \rangle \qquad (3.6)
\end{aligned}$$

and from the Cauchy-Schwarz inequality it follows that

$$\langle x^n - \lambda_i^n v_i^n - y_i^n, z_i^n - y_i^n \rangle \le \lambda_i^n \|u_i^n - v_i^n\| \, \|z_i^n - y_i^n\| . \qquad (3.7)$$

Each mapping A_i, $1, \ldots, N$, is Lipschitz continuous with constant α_i. Therefore A_i is obviously Lipschitz continuous with constant α. Using this fact, we obtain

$$\langle x^n - \lambda_i^n v_i^n - y_i^n, z_i^n - y_i^n \rangle \le \lambda_i^n \alpha \|x^n - y_i^n\| \, \|z_i^n - y_i^n\| . \qquad (3.8)$$

Hence

$$\begin{aligned}
\|z_i^n - s\|^2 &\le \|x^n - s\|^2 - \|x^n - y_i^n\|^2 - \|y_i^n - z_i^n\|^2 + \\
&\qquad 2\lambda_i^n \alpha \|x^n - y_i^n\| \, \|z_i^n - y_i^n\| . \qquad (3.9)
\end{aligned}$$

Since

$$\begin{aligned}
0 &\le (\lambda_i^n \alpha \|x^n - y_i^n\| - \|z_i^n - y_i^n\|)^2 \\
&= (\lambda_i^n \alpha)^2 \|x^n - y_i^n\|^2 - 2\lambda_i^n \alpha \|x^n - y_i^n\| \, \|z_i^n - y_i^n\| \\
&\quad + \|z_i^n - y_i^n\|^2 , \qquad (3.10)
\end{aligned}$$

we obtain that

$$2\lambda_i^n\alpha \left\| x^n - y_i^n \right\| \left\| z_i^n - y_i^n \right\| \leq \left(\lambda_i^n\alpha\right)^2 \left\| x^n - y_i^n \right\|^2 + \left\| z_i^n - y_i^n \right\|^2 . \tag{3.11}$$

Thus,

$$\begin{aligned} \left\| z_i^n - s \right\|^2 \leq & \left\| x^n - s \right\|^2 - \left\| x^n - y_i^n \right\|^2 - \left\| y_i^n - z_i^n \right\|^2 + \\ & \left(\lambda_i^n\alpha\right)^2 \left\| x^n - y_i^n \right\|^2 + \left\| z_i^n - y_i^n \right\|^2 . \\ & = \left\| x^n - s \right\|^2 - \left(1 - \left(\lambda_i^n\alpha\right)^2\right) \left\| x^n - y_i^n \right\|^2 . \end{aligned} \tag{3.12}$$

Since $\lambda_i^n < 1/\alpha$ it follows that $\left\| z_i^n - s \right\|^2 \leq \left\| x^n - s \right\|^2$. Therefore $s \in \widetilde{C}^n$. Consequently, $F \subseteq \widetilde{C}^n$ for all $n \geq 1$. Now we prove by induction that the sequence $\{x^n\}_{n\in\mathbb{N}}$ is well defined. Indeed, since $F \subseteq \widetilde{C}^1$ and $F \subseteq W^1 = \mathcal{H}$, it follows that $F \subseteq \widetilde{C}^1 \cap W^1$ and therefore $x^2 = P_{\widetilde{C}^1\cap W^1}(x^1)$ is well defined. Now suppose that $F \subseteq \widetilde{C}^{n-1} \cap W^{n-1}$ for some $n > 2$. Let $x^n = P_{\widetilde{C}^{n-1}\cap W^{n-1}}(x^1)$. Again we have $F \subseteq \widetilde{C}^n$ and for any $s \in F$, it follows from (2.12) that

$$\left\langle x^1 - x^n, s - x^n \right\rangle = \left\langle x^1 - P_{\widetilde{C}^{n-1}\cap W^{n-1}}(x^1), s - P_{\widetilde{C}^{n-1}\cap W^{n-1}}(x^1) \right\rangle \leq 0. \tag{3.13}$$

This implies that $s \in W^n$. Therefore $F \subseteq \widetilde{C}^n \cap W^n$ for any $n \geq 1$, as required. This shows that the sequence $\{x^n\}_{n\in\mathbb{N}}$ is indeed well defined. ■

Claim 3.8 *The sequences* $\{x^n\}_{n\in\mathbb{N}}$, $\{y_i^n\}_{n\in\mathbb{N}}$ *and* $\{z_i^n\}_{n\in\mathbb{N}}$ *are bounded for any* $i = 1, \ldots, N$.

Proof. Since $x^{n+1} = P_{C^n\cap W^n}(x^1)$, we have for any $s \in C^n \cap W^n$,

$$\left\| x^{n+1} - x^1 \right\| \leq \left\| s - x^1 \right\| . \tag{3.14}$$

Therefore $\{x^n\}_{n\in\mathbb{N}}$ is bounded. It follows from the definition of W^n that $x^n = P_{W^n}(x^1)$. Since $x^{n+1} \in W^n$, it follows from (2.13) that

$$\left\| x^{n+1} - x^n \right\|^2 + \left\| x^n - x^1 \right\|^2 \leq \left\| x^{n+1} - x^1 \right\|^2 . \tag{3.15}$$

Thus the sequence $\{\left\| x^n - x^1 \right\|\}_{n\in\mathbb{N}}$ is increasing and bounded, hence convergent. This shows that $\lim_{n\to\infty} \left\| x^n - x^1 \right\|$ exists. In addition, from (3.15) we get that

$$\lim_{n\to\infty} \left\| x^{n+1} - x^n \right\| = 0. \tag{3.16}$$

Since $x^{n+1} \in C_i^n$, $i = 1, \ldots, N$, we have

$$\left\langle x^n - z_i^n, x^{n+1} - x^n - \gamma_i^n \left(z_i^n - x^n\right)\right\rangle \leq 0. \tag{3.17}$$

Thus

$$\gamma_i^n \left\|z_i^n - x^n\right\|^2 \leq \left\langle x^n - z_i^n, x^n - x^{n+1}\right\rangle. \tag{3.18}$$

Hence $\|z_i^n - x^n\| \leq \|x^n - x^{n+1}\|$ and therefore

$$\lim_{n\to\infty} \|z_i^n - x^n\| = 0, \text{ for all } i = 1, \ldots, N. \tag{3.19}$$

Thus $\{z_i^n\}_{n\in\mathbb{N}}$ is a bounded sequence for each $i = 1, \ldots, N$. Using (3.12), we see that

$$\begin{aligned}
\|x^n - y_i^n\|^2 &\leq \left(1 - (\lambda_i^n \alpha)^2\right)^{-1} \left(\|x^n - s\|^2 - \|z_i^n - s\|^2\right) \\
&= \left(1 - (\lambda_i^n \alpha)^2\right)^{-1} \left(\|x^n - s\| - \|z_i^n - s\|\right)\left(\|x^n - s\| + \|z_i^n - s\|\right) \\
&\leq \left(1 - (\lambda_i^n \alpha)^2\right)^{-1} \|x^n - z_i^n\| \left(\|x^n - s\| + \|z_i^n - s\|\right). \qquad (3.20)
\end{aligned}$$

Since both $\{x^n\}_{n\in\mathbb{N}}$ and $\{z_i^n\}_{n\in\mathbb{N}}$ are bounded, Condition 3.4 and (3.19), imply that

$$\lim_{n\to\infty} \|x^n - y_i^n\| = 0 \text{ for all } i = 1, \ldots, N. \tag{3.21}$$

Therefore $\{y_i^n\}_{n\in\mathbb{N}}$ is a bounded sequence for each $i = 1, \ldots, N$, which completes the proof of Claim 3.8. ■

Claim 3.9 *Any weak accumulation point of the sequences* $\{x^n\}_{n\in\mathbb{N}}$,$\{y_i^n\}_{n\in\mathbb{N}}$ *and* $\{z_i^n\}_{n\in\mathbb{N}}$ *belongs to* F.

Proof. Since $\{x^n\}_{n\in\mathbb{N}}$ is bounded (see Claim 3.8), there exists a subsequence $\{x^{n_k}\}_{k\in\mathbb{N}}$ of $\{x^n\}_{n\in\mathbb{N}}$ which converges weakly to x^*. Therefore it follows from (3.21) that there also exists a subsequence $\{y_i^{n_k}\}_{k\in\mathbb{N}}$ of $\{y_i^n\}_{n\in\mathbb{N}}$ which converges to x^* for each $i = 1, \ldots, N$. Define the mapping T_i as follows:

$$T_i(r) = \begin{cases} A_i(r) + N_{K_i}(r), & r \in K_i, \\ \varnothing, & \text{otherwise}, \end{cases} \tag{3.22}$$

where $N_{K_i}(r)$ is the normal cone of K_i at $r \in K_i$. Since A_i is a maximal monotone mapping, it follows from [29, Theorem 5, p. 85] that T_i is a maximal monotone operator and $T_i^{-1}(0) = \mathrm{SOL}(A_i, K_i)$. Let $(r, w) \in \operatorname{graph}(T_i)$ with $r \in K_i$ and let $p_i \in A_i(r)$. Since $w \in T_i(r) = A_i(r) + N_{K_i}(r)$, we get

$w - p_i \in N_{K_i}(r)$. Since $y_i^{n_k} \in K_i$, we obtain $\langle w - p_i, r - y_i^{n_k} \rangle \geq 0$. On the other hand, since $y_i^{n_k} = P_{K_i}(x^{n_k} - \lambda_i^{n_k} u_i^{n_k})$, we also have

$$\langle (x^{n_k} - \lambda_i^{n_k} u_i^{n_k}) - y_i^{n_k}, r - y_i^{n_k} \rangle \leq 0 \tag{3.23}$$

and thus

$$\left\langle \frac{x^{n_k} - y_i^{n_k}}{\lambda_i^{n_k}} - u_i^{n_k}, r - y_i^{n_k} \right\rangle \leq 0. \tag{3.24}$$

Therefore it follows from the monotonicity of the mapping A_i, $i = 1, \ldots, N$, that

$$\begin{aligned} \langle w, r - y_i^{n_k} \rangle &\geq \langle p_i, r - y_i^{n_k} \rangle \\ &\geq \langle p_i, r - y_i^{n_k} \rangle + \left\langle \frac{x^{n_k} - y_i^{n_k}}{\lambda_i^{n_k}} - u_i^{n_k}, r - y_i^{n_k} \right\rangle \\ &= \langle p_i - v_i^{n_k}, r - y_i^{n_k} \rangle + \langle v_i^{n_k} - u_i^{n_k}, r - y_i^{n_k} \rangle \\ &+ \left\langle \frac{x^{n_k} - y_i^{n_k}}{\lambda_i^{n_k}}, r - y_i^{n_k} \right\rangle \\ &\geq \langle v_i^{n_k} - u_i^{n_k}, r - y_i^{n_k} \rangle + \left\langle \frac{x^{n_k} - y_i^{n_k}}{\lambda_i^{n_k}}, r - y_i^{n_k} \right\rangle. \end{aligned} \tag{3.25}$$

From the Cauchy-Schwarz inequality and the Lipschitz continuity with constant α it follows that

$$\begin{aligned} \langle w, r - y_i^{n_k} \rangle &\geq -\alpha \left\| r - y_i^{n_k} \right\| \left\| x^{n_k} - y_i^{n_k} \right\| - \left\| r - y_i^{n_k} \right\| \frac{\left\| x^{n_k} - y_i^{n_k} \right\|}{a} \\ &= -M_i \left(\alpha \left\| x^{n_k} - y_i^{n_k} \right\| + \frac{\left\| x^{n_k} - y_i^{n_k} \right\|}{a} \right), \end{aligned} \tag{3.26}$$

where $M_i = \sup_{k \in \mathbb{N}} \{ \| r - y_i^{n_k} \| \}$. Taking the limit as $k \to \infty$ and using the fact that $\{ \| r - y_i^{n_k} \| \}_{k \in \mathbb{N}}$ is bounded, we see that $\langle w, r - x^* \rangle \geq 0$. The maximality of T_i and Remark 2.4 now imply that $x^* \in T_i^{-1}(0) = \mathrm{SOL}(A_i, K_i)$. Hence $x^* \in F$. ■

Claim 3.10 *The sequences $\{x^n\}_{n \in \mathbb{N}}$,$\{y_i^n\}_{n \in \mathbb{N}}$ and $\{z_i^n\}_{n \in \mathbb{N}}$ converge strongly to $P_F(x^1)$.*

Proof. Since (3.14) holds for all $s \in C^n \cap W^n$ and $F \subseteq C^n \cap W^n$ by the proof of Claim 3.7, we get for $s = P_F(x^1)$ that

$$\left\| x^n - x^1 \right\| \leq \left\| P_F(x^1) - x^1 \right\| \tag{3.27}$$

and furthermore,

$$\lim_{n\to\infty} \left\| x^n - x^1 \right\| \le \left\| P_F(x^1) - x^1 \right\|. \tag{3.28}$$

Now, since the sequence $\{x^n\}_{n\in\mathbb{N}}$ is bounded (see Claim 3.8), there exists a subsequence $\{x^{n_k}\}_{k\in\mathbb{N}}$ of $\{x^n\}_{n\in\mathbb{N}}$ which converges weakly to x^*. From Claim 3.9 it follows that $x^* \in F$. From the weak lower semicontinuity of the norm and (3.28) it follows that

$$\begin{aligned} \|x^* - x^1\| &\le \liminf_{k\to\infty} \|x^{n_k} - x^1\| \\ &= \lim_{n\to\infty} \|x^n - x^1\| = \left\| P_F(x^1) - x^1 \right\|. \end{aligned} \tag{3.29}$$

Since $x^* \in F$, it follows that $x^* = P_F(x^1)$. So, since by Claim 3.9 any weak accumulation point of the sequence $\{x^n\}_{n\in\mathbb{N}}$ belong to F, it follows that $\text{w-lim}_{n\to\infty} x^n = x^* = P_F(x^1)$. Finally,

$$\|x^* - x^1\| \le \liminf_{k\to\infty} \|x^n - x^1\| = \lim_{n\to\infty} \|x^n - x^1\| = \left\| x^* - x^1 \right\|. \tag{3.30}$$

Since $\text{w-lim}_{n\to\infty}(x^n - x^1) = x^* - x^1$ and $\lim_{n\to\infty} \|x^n - x^1\| = \|x^* - x^1\|$, it follows from the Kadec-Klee property of $\mathcal{H}$ that $\lim_{n\to\infty} \|x^n - x^*\| = 0$, as asserted. ■

This completes the proof of Theorem 3.6. ■

Now we present several consequences of our main result.

First, consider the case where Condition 3.2 is replaced with the following condition.

Condition 3.11 *Each one of the mappings $\{A_i\}_{i=1}^N$ is maximal monotone and α_i-**inverse strongly monotone** (α_i-**ism**) with constant $\alpha_i > 0$, that is,*

$$\langle u - v, x - y \rangle \ge \alpha_i \|u - v\|^2 \text{ for all } u \in A_i(x) \text{ and } v \in A_i(y). \tag{3.31}$$

The class of inverse strongly monotone mappings is commonly used in variational inequality problems (see e.g., [18] and references therein). The Cauchy-Schwarz inequality shows that inverse strong monotonicity implies monotonicity and Lipschitz continuity with constant $L_A = 1/\alpha$, where α is the ism constant. Thus it is clear that Theorem 3.6 applies to this case too.

Second, consider the case where we take A_i as single-valued mappings, that is, $A_i : \mathcal{H} \to \mathcal{H}$, and we change Condition 3.2 to the following one: The mappings A_i are monotone and Lipschitz continuous. It is known that, in general, monotonicity and Lipschitz continuity do not imply inverse strong monotonicity. However, our strong convergence theorem is also applicable in this case.

4 Applications

The CSVIP encompasses several previously separately studied problems, as well as some new ones. For example, the following five problems are special cases of the CSVIP.

4.1 The Convex Feasibility Problem

Let $\mathcal{H}$ be a real Hilbert space. Given N nonempty, closed and convex subsets $K_i \subseteq \mathcal{H}$, with $\bigcap_{i=1}^{N} K_i \neq \emptyset$, the *Convex Feasibility Problem* (CFP) is to find a point x^* such that

$$x^* \in \bigcap_{i=1}^{N} K_i. \tag{4.1}$$

This is obviously a special case of the CSVIP with all $A_i = 0$. The literature on the CFP is vast and many algorithms for solving it have been developed (see, e.g., [2, 11]). It plays a fundamental role in many real-world applications. See, e.g., [8].

4.2 The Common Minimizer Problem

A new problem which can be seen as a special case of the CSVIP is the *Common Minimizer Problem* (CMP). Given N nonempty, closed and convex subsets $K_i \subseteq \mathcal{H}$, with $\bigcap_{i=1}^{N} K_i \neq \emptyset$, and functions g_i, $i = 1, 2, \ldots, N$, that are continuously differentiable and convex on K_i, respectively, the CMP is to find a point x^* so that

$$x^* \in \bigcap_{i=1}^{N} K_i \text{ and } x^* = \operatorname{argmin}\{g_i(x) \mid x \in K_i\} \text{ for all } i = 1, 2, \ldots, N. \tag{4.2}$$

The problem of finding a minimizer of a continuously differentiable and convex function over a convex set K is equivalent to solving a certain variational inequality. See Example 1.2. Therefore, this CMP translates to a

CSVIP (with single-valued mappings) by choosing in (1.1) $A_i = \nabla g_i$ for all $i = 1, 2, \ldots, N$.

Replacing ∇g_i by ∂g_i, we see that the CSVIP also includes the case where the g_i are not necessarily differentiable.

4.3 The Common Saddle-Point Problem

The equivalence between certain VIPs and the saddle-point problems of Example 1.5 leads us to present the *Common Saddle-Point Problem* (CSPP). Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let $\{U_i\}_{i=1}^N \subseteq \mathcal{H}_1$ and $\{V_i\}_{i=1}^N \subseteq \mathcal{H}_2$ be nonempty, closed and convex. Set $\mathbf{H} := \mathcal{H}_1 \times \mathcal{H}_2$. Given N functions $\{f_i : \boldsymbol{H} \to \mathbb{R}\}_{i=1}^N$, the CSPP is to find a point $(u_1^*, u_2^*) \in \left(\cap_{i=1}^N U_i\right) \times \left(\cap_{i=1}^N V_i\right)$ such that for all $i = 1 \ldots N$, we have

$$f_i(u_1^*, u_2) \leq f_i(u_1^*, u_2^*) \leq f_i(u_1, u_2^*) \tag{4.3}$$

for all $(u_1, u_2) \in U_i \times V_i$. This problem reduces to the CSVIP when we take in (1.1) $A_i = (\nabla(f_i)_{u_1}, -\nabla(f_i)_{u_2})$ and $K_i = U_i \times V_i$ for all $i = 1, 2, \ldots, N$.

4.4 The Hierarchical Variational Inequality Problem

Next we present another variant of the CSVIP, namely, the *Hierarchical Variational Inequality Problem* (HVIP). Let $\mathcal{H}$ be a real Hilbert space.

1. Let K be a nonempty, closed and convex subset of $\mathcal{H}$ and let $U : K \to K$ and $V : K \to K$ be two nonexpansive single-valued mappings. Consider the operator $B := I - V$. Xu [30] studied the problem of finding a point $x^* \in \operatorname{Fix} U$ such that

$$\langle B(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in \operatorname{Fix} U. \tag{4.4}$$

2. Yao and Liou [33] considered the following *Hierarchical Variational Inequality Problem* (HVIP). Let $\mathcal{H}$ be a real Hilbert space and let $K \subseteq \mathcal{H}$ be a nonempty, closed and convex subset. Given the single-valued mappings $U : K \to \mathcal{H}$ and $V : K \to \mathcal{H}$, set $B := I - V$. Then the HVIP is to find a point $x^* \in \mathrm{SOL}(U, K)$ such that

$$\langle B(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in \mathrm{SOL}(U, K). \tag{4.5}$$

Since it is well known that $x^* \in \mathrm{SOL}(U, K) \Leftrightarrow x^* \in \mathrm{Fix}\ P_K(x^* - \lambda U(x^*))$ for all $\lambda \geq 0$, this problem is essentially a special case of Xu's problem. Both problems can be formulated as a special CSVIP in the following way. Find a point $x^* \in \mathcal{H}$ such that

$$\langle (I - U)(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in \mathcal{H} \tag{4.6}$$

and

$$\langle B(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in \mathrm{Fix}\ U. \tag{4.7}$$

This is a two-set CSVIP with the single-valued mappings $A_1 = I - U$ and $A_2 = B$, and the sets $K_1 = \mathcal{H}$ and $K_2 = \mathrm{Fix}\ U$.

Recently, hierarchical fixed point problems and hierarchical minimization problems have attracted attention because of their connections with some convex programming problems. See, e.g., [23, 24, 32, 30] and the references therein.

4.5 Variational Inequality Problem over the intersection of convex sets

Let H be a real Hilbert space. Given N nonempty, closed and convex subsets $K_i \subseteq H$, $i = 1, 2, \ldots, N$, with $\bigcap_{i=1}^N K_i \neq \emptyset$, we consider the CSVIP (1.1) with $A_i \equiv A$ for each $i = 1, 2, \ldots, N$. We obtain a single variational inequality problem over a nonempty intersection of N nonempty, closed and convex subsets. More precisely, we have to find a point $x^* \in \bigcap_{i=1}^N K_i$ such that there exists $u^* \in A(x^*)$ satisfying

$$\langle u^*, x - x^* \rangle \geq 0 \text{ for all } x \in K_i,\ i = 1, 2, \ldots, N, \tag{4.8}$$

and, in particular,

$$\langle u^*, x - x^* \rangle \geq 0 \text{ for all } x \in \bigcap_{i=1}^N K_i. \tag{4.9}$$

This problem is closely related to the work of Yamada [31] who considered a variational inequality problem with a singled-valued mapping over the intersection of the fixed point sets of nonexpansive mappings, i.e., $K_i = \mathrm{Fix}\ T_i$, for $i = 1, 2, \ldots, N$.

4.6 Implementation

In this subsection we demonstrate, using a simple low-dimensional example, the practical difficulties associated with the implementation of Algorithm 3.1 (see also our comment after the formulation of Algorithm 3.1). We consider a two-disc convex feasibility problem in $\mathbb{R}^2$ and provide an explicit formulation of our Algorithm 3.1, as well as some numerical results. More explicitly, let $K_1 = \{(x,y) \in \mathbb{R}^2 : (x-a_1)^2 + (y-b_1)^2 \le r_1^2\}$ and $K_2 = \{(x,y) \in \mathbb{R}^2 : (x-a_2)^2 + (y-b_2)^2 \le r_2^2\}$ with $K_1 \cap K_2 \neq \emptyset$. Consider the problem of finding a point $(x^*, y^*) \in \mathbb{R}^2$ such that $(x^*, y^*) \in K_1 \cap K_2$. Observe that in this case $A_1 = A_2 = \{0\}$. For simplicity we choose $\gamma_1^n = \gamma_2^n = 1/2$. Given the current iterate $x^n = (u,v)$, the explicit formulation of the iterative step of Algorithm 3.1 becomes:

$$\begin{cases} y_1^n = P_{K_1}\left(x^n\right) = \left(a_1 + \dfrac{r_1\left(u-a_1\right)}{\|(u-a_1, v-b_1)\|}, b_1 + \dfrac{r_1\left(v-b_1\right)}{\|(u-a_1, v-b_1)\|}\right), \\ y_2^n = P_{K_2}\left(x^n\right) = \left(a_2 + \dfrac{r_2\left(u-a_2\right)}{\|(u-a_2, v-b_2)\|}, b_1 + \dfrac{r_2\left(v-b_2\right)}{\|(u-a_2, v-b_2)\|}\right), \\ C_1^n = \left\{z = (s,t) \in \mathbb{R}^2 : \|z - y_1^n\| \le \|z - x^n\|\right\}, \\ C_2^n = \left\{z = (s,t) \in \mathbb{R}^2 : \|z - y_2^n\| \le \|z - x^n\|\right\}, \\ W^n = \left\{z \in \mathbb{R}^2 : \langle x^1 - x^n, z - x^n \rangle \le 0\right\}, \\ x^{n+1} = P_{C_1^n \cap C_2^n \cap W^n}\left(x^1\right). \end{cases} \tag{4.10}$$

In order to calculate x^{n+1}, we solve the following constrained minimization problem:

$$\begin{cases} \min \|x^1 - z\|^2, \\ \text{such that } z \in C_1^n \cap C_2^n \cap W^n. \end{cases} \tag{4.11}$$

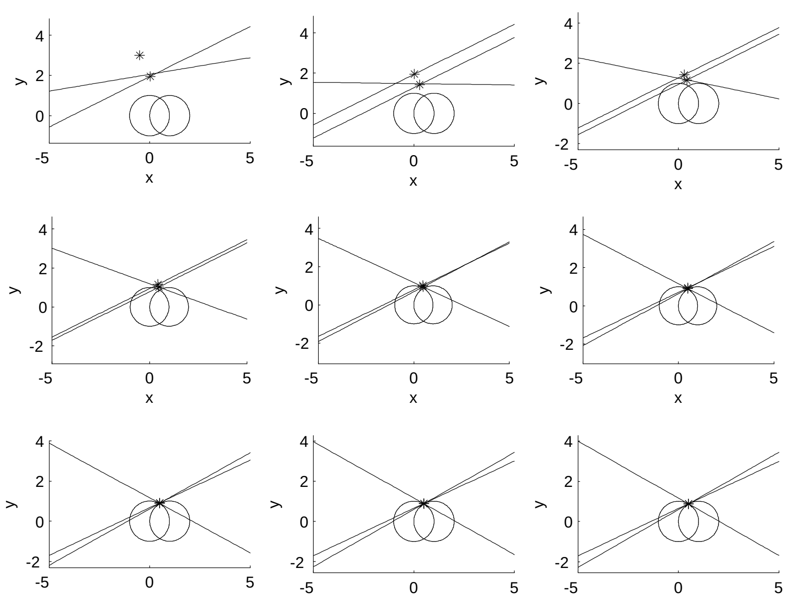
In the case of the metric projection onto two half-spaces, an explicit formula can be found in [3, Definition 3.1] and in [10, Subsection 3.1] Following the same technique, it is possible to obtain the solution to (4.11) even for more than three half-spaces, but there are many subcases in the explicit formula (two to the power of the number of half-spaces).

Now we present some numerical results for the particular case where $K_1 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$ and $K_2 = \{(x,y) \in \mathbb{R}^2 : (x-1)^2 + y^2 \le 1\}$. We choose separately two starting points $(-1/2, 3)$ and $(3,3)$, and for each starting point we present a table with the (x, y) coordinates for the first 10 iterations of Algorithm 3.1. In addition, Figures 4.1 and 4.2 illustrate the geometry in each iterative step, i.e., the discs and the three half-spaces C_1^n, C_2^n and W^n.

Case 4.1 *Starting point* $x^1 = (-1/2, 3)$ *with the first* 10 *iterations of the algorithm.*

Iteration Number	x-value	y-value
1	-0.500000000	3.0000000000
2	0.0263507717	1.9471923798
3	0.2898391508	1.4209450920
4	0.4211545167	1.1576070220
5	0.4687763141	1.0169184232
6	0.4862238741	0.9429308114
7	0.4935428246	0.9048859275
8	0.4968764116	0.8855650270
9	0.4984644573	0.8758239778
10	0.4992386397	0.8709324060

Table 1: 10 iterations with the starting point $x^1 = (-1/2, 3)$

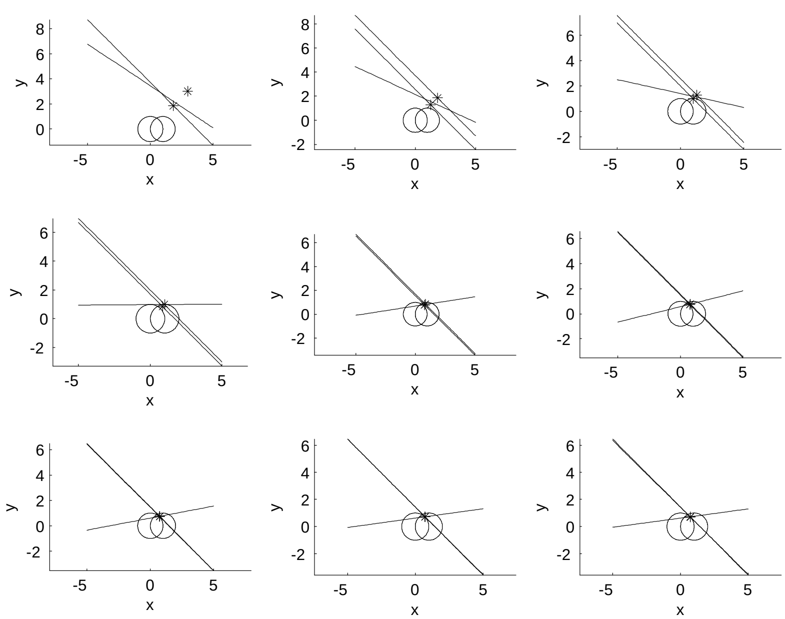


Geometric illustration of Algorithm 3.1 in each iterative step, i.e., the discs and the three half-spaces C_1^n, C_2^n and W^n, with the starting point $x^1 = (-1/2, 3)$

Case 4.2 *Starting point is* $x^1 = (3,3)$ *with the first* 10 *iterations of the algorithm.*

Iteration Number	x-value	y-value
1	3.0000000000	3.0000000000
2	1.8536075595	1.8534992168
3	1.2802790276	1.2803811470
4	0.9937807510	0.9936561265
5	0.8503033752	0.8505218683
6	0.7789970157	0.7785224690
7	0.7423971596	0.7434698006
8	0.7264747366	0.7235683325
9	0.7115677773	0.7205826742
10	0.7260458319	0.6973591138

Table 2: 10 iterations with the starting point $x^1 = (3,3)$



Geometric illustration of Algorithm 3.1 in each iterative step, i.e., the discs and the three half-spaces C_1^n, C_2^n and W^n, with the starting point $x^1 = (3,3)$

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