Opial-Type Theorems and the Common Fixed Point Problem

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Abstract

The well-known Opial theorem says that a sequence of orbits of a nonexpansive and asymptotically regular operator T having a fixed point and defined on a Hilbert space converges weakly to a fixed point of T. In this paper we consider recurrences generated by a sequence of quasi-nonexpansive operators having a common fixed point or by a sequence of extrapolations of an operator satisfying Opial's demiclosedness principle and having a fixed point. We give sufficient conditions for the weak convergence of sequences defined by these recurrences to a fixed point of an operator which is closely related to the sequence of operators. These results generalize in a natural way the classical Opial theorem. We give applications of these generalizations to the common fixed point problem.

1 Introduction

Iterative methods for convex optimization problems in a Hilbert space $\mathcal{H}$ have usually the form of the recurrence $x^{k+1} = U_k x^k$, where $x^0 \in X$, $X \subset \mathcal{H}$

is closed and convex, and $U_k : X \to X$ are operators related to the optimization problem at hand. Some of the methods employ the same operator $U_k = U$ in all iterations. If we suppose that U is a nonexpansive and asymptotically regular operator having a fixed point then it follows from the Opial theorem that the so generated sequence $\{x^k\}_{k=0}^{\infty}$ converges weakly to a fixed point of U (see [Opi67, Theorem 1]). Many iterative methods employ, however, different operators U_k in successive iterations, usually assuming that all operators U_k have a common fixed point. Examples of such methods for solving the common fixed point problem include methods of successive projections (with various control sequences such as the almost cyclic control, the repetitive control, etc.), methods of simultaneous projections (also known as Cimmino-type methods), where the weights depend on the iteration index, surrogate projection methods, etc. Our main aim here is to give, in a unified manner, sufficient conditions for weak convergence of sequences generated by the recurrence $x^{k+1} = U_k x^k$ and to apply the results to the common fixed point problem.

An interesting point related to our current investigation is a *local acceleration* technique of Cimmino's [Cim38] well-known simultaneous projection method for linear equations. This technique is referred to in the literature as the *Dos Santos* (DS) method, see Dos Santos [DS87] and Bauschke and Borwein [BB96, Section 7], although Dos Santos attributes it, in the linear case, to De Pierro's Ph.D. Thesis [DPi81]. The method essentially uses the line through each pair of consecutive Cimmino iterates and chooses the point on this line which is closest to the solution x^* of the linear system $Ax = b$. The nice thing about it is that existence of the solution of the linear system must be assumed, but the method does not need the solution point x^* in order to proceed with the locally accelerated DS iterative process. This approach was also used by Appleby and Smolarski [AS05]. On the other hand, while trying to be as close as possible to the solution point x^* in each iteration, the method is not known to guarantee overall acceleration of the process. Therefore, we call it a *local acceleration* technique. In all the above references the DS method works for *simultaneous projection methods* and one of our questions was whether it can also be extended to handle common fixed point problems. If so, for which classes of operators.

Here we answer this question by focusing on the class of operators $T : \mathcal{H} \to \mathcal{H}$ that have the property that, for any $x \in \mathcal{H}$, the hyperplane through Tx whose normal is $x - Tx$ always "cuts" the space into two half-spaces one of which contains the point x while the other contains the (assumed nonempty) fixed points set of T. This explains the name *cutter operators* or *cutters* that we introduce here. These operators themselves, introduced and investigated by Bauschke and Combettes [BC01, Definition 2.2] and by Combettes [Com01], play an important role in optimization and feasibility theory since many commonly used operators are actually cutters. We define generalized relaxations and extrapolation of cutter operators and construct *extrapolated simultaneous cutter operators.* For these simultaneous extrapolated cutters we present convergence results of successive iteration processes for common fixed point problems which generalize the locally accelerated DS iterative processes, thus, cover some of the earlier results about such methods and present some new ones.

The paper is organized as follows. In Section 2 we give the definition of cutter operators and bring some of their properties that will be used here. Section 3 contains the Opial theorem and its generalization. Opial-type theorems for cutters are presented in Section 4 and applications to the common fixed point problem, including the connection to the DS method (Example 38), are studied in Section 5.

2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with an inner product $\langle \cdot, \cdot \rangle$ and with the norm $\| \cdot \|$. Given $x, y \in \mathcal{H}$ we denote

$$H(x,y) := \{u \in \mathcal{H} \mid \langle u - y, x - y \rangle \leq 0\} . \tag{1}$$

Definition 1 An operator $T : \mathcal{H} \to \mathcal{H}$ is called a *cutter operator* or, in short, a *cutter* iff

$$\operatorname{Fix} T \subseteq H(x, Tx) \ \text{ for all } x \in \mathcal{H}, \tag{2}$$

where $\operatorname{Fix} T$ is the fixed points set of T, equivalently,

$$q \in \operatorname{Fix} T \text{ implies that } \langle Tx - x, Tx - q \rangle \leq 0 \ \text{ for all } x \in \mathcal{H}. \tag{3}$$

The class of cutter operators is denoted by $\mathcal{T}$, i.e.,

$$\mathcal{T} := \{T : \mathcal{H} \to \mathcal{H} \mid \operatorname{Fix} T \subseteq H(x, Tx) \ \text{ for all } x \in \mathcal{H}\} . \tag{4}$$

The class $\mathcal{T}$ of operators was introduced and investigated by Bauschke and Combettes in [BC01, Definition 2.2] and by Combettes in [Com01]. Operators in this class were named *directed operators* by Zaknoon [Zak03] and further employed under this name by Segal [Seg08] and Censor and Segal [CS08, CS08a, CS09]. Cegielski [Ceg08, Def. 2.1] named and studied these operators as *separating operators*. Since both *directed* and *separating* are key words of other, widely-used, mathematical entities we decide to use from now on the term *cutter operators*. This name can be justified by the fact that the bounding hyperplane of $H(x, Tx)$ "cuts" the space into two half-spaces, one which contains the point x while the other contains the set $\operatorname{Fix} T$. We recall definitions and results on cutter operators and their properties as they appear in [BC01, Proposition 2.4] and [Com01], which are also sources for further references.

Bauschke and Combettes [BC01] showed the following:

(i) The set of all fixed points of a cutter operator assumed to be nonempty is closed and convex because $\operatorname{Fix} T = \cap_{x \in \mathcal{H}} H(x, Tx)$.

(ii) Denoting by Id the identity operator,

$$\text{if } T \in \mathcal{T} \text{ then } \operatorname{Id} + \lambda(T - \operatorname{Id}) \in \mathcal{T} \text{ for all } \lambda \in [0, 1]. \tag{5}$$

This class of operators is fundamental because many common types of operators arising in convex optimization belong to the class and because it allows a complete characterization of Fejér-monotonicity [BC01, Proposition 2.7]. The localization of fixed points is discussed by Goebel and Reich in [GR84, pp. 43–44]. In particular, it is shown there that a firmly nonexpansive (FNE) operator, namely, an operator $T : \mathcal{H} \to \mathcal{H}$ that fulfills

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y \rangle \text{ for all } x, y \in \mathcal{H}, \tag{6}$$

which has a fixed point, satisfies (3) and is, therefore, a cutter operator. The class of cutter operators, includes additionally, according to [BC01, Proposition 2.3], among others, the resolvent of a maximal monotone operator, the orthogonal projections and the subgradient projectors. Another family of cutters appeared recently in Censor and Segal [CS08a, Definition 2.7]. Note that every cutter operator belongs to the class of operators $\mathcal{F}^0$, defined by Crombez [Cro05, p. 161],

$$\mathcal{F}^0 := \{T : \mathcal{H} \to \mathcal{H} \mid \|Tx - q\| \leq \|x - q\| \text{ for all } q \in \operatorname{Fix} T \text{ and } x \in \mathcal{H}\}, \tag{7}$$

whose elements are called elsewhere quasi-nonexpansive or paracontracting operators.

Definition 2 Let $T : \mathcal{H} \to \mathcal{H}$ and let $\lambda \in (0, 2)$. We call the operator $T_\lambda := \operatorname{Id} + \lambda(T - \operatorname{Id})$ *a relaxation of* T.

Definition 3 We say that an operator $T : \mathcal{H} \to \mathcal{H}$ with $\operatorname{Fix} T \neq \emptyset$ is *strictly quasi-nonexpansive* if

$$\|Tx - z\| < \|x - z\| \tag{8}$$

for all $x \notin \operatorname{Fix} T$ and for all $z \in \operatorname{Fix} T$. We say that T is *α-strongly quasi-nonexpansive*, where $\alpha > 0$, or, in short, strongly quasi-nonexpansive if

$$\|Tx - z\|^2 \leq \|x - z\|^2 - \alpha\|Tx - x\|^2 \tag{9}$$

for all $x \in \mathcal{H}$ and for all $z \in \operatorname{Fix} T$.

We have the following result from [Com01, Proposition 2.3 (i)-(ii)].

Lemma 4 *Let $X \subset \mathcal{H}$ be a closed and convex set and $U : X \to X$ be an operator having a fixed point.*

(i) *U is a cutter if and only if*

$$\langle z - x, Ux - x\rangle \geq \|Ux - x\|^2 \tag{10}$$

for all $x \in X$ and for all $z \in \operatorname{Fix} U$.

(ii) *Let $\lambda \in (0, 2)$. If U is a cutter then its relaxation U_λ is $\frac{2-\lambda}{\lambda}$-strongly quasi-nonexpansive.*

One can show that the implication converse to (ii) is also true.

Definition 5 We say that an operator $T : \mathcal{H} \to \mathcal{H}$ is *demiclosed* at 0 if for any weakly converging sequence $\{x^k\}_{k=0}^\infty$, $x^k \rightharpoonup y \in \mathcal{H}$ as $k \to \infty$, with $Tx^k \to 0$ as $k \to \infty$, we have $Ty = 0$.

It is well-known that for a nonexpansive operator $T : \mathcal{H} \to \mathcal{H}$, the operator $T - \operatorname{Id}$ is demiclosed at 0, see Opial [Opi67, Lemma 2].

Definition 6 We say that an operator $T : \mathcal{H} \to \mathcal{H}$ is *asymptotically regular* if

$$\|T^{k+1}x - T^k x\| \to 0, \text{ as } k \to \infty, \tag{11}$$

for all $x \in \mathcal{H}$.

3 The Opial theorem and its generalization

Opial proved the following theorem [Opi67, Theorem 1] which is widely applied in processes described by the recurrence

$$x^{k+1} = Ux^k, \tag{12}$$

where $x^0 \in X$ is arbitrary, $U : X \to X$ is a nonexpansive operator and $X \subset \mathcal{H}$ is a closed and convex subset of a Hilbert space $\mathcal{H}$. Many iterative methods for convex optimization problems have the form (12), where the operator U is defined in a natural way by the problem under consideration.

Theorem 7 *Let $X \subset \mathcal{H}$ be a nonempty, closed and convex subset of a Hilbert space $\mathcal{H}$ and let $U : X \to X$ be a nonexpansive and asymptotically regular operator with $\operatorname{Fix} U \neq \emptyset$. Then, for any arbitrary $x \in X$, the sequence $\{U^k x\}_{k=0}^{\infty}$ converges weakly to a fixed point z^* of U.*

An example of a nonexpansive and asymptotically regular operator is a strict relaxation of a firmly nonexpansive operator or, equivalently, an averaged operator. Therefore, the Krasnoselskii–Mann theorem (see, e.g., [Byr04, Theorem 2.1]) follows from the Opial theorem.

Several optimization methods for convex optimization problems have, however, the form

$$x^{k+1} = U_k x^k, \tag{13}$$

where $x^0 \in X$ is arbitrary and $\{U_k\}_{k=0}^{\infty}$, $U_k : X \to X$, is a sequence of operators. The Opial theorem cannot be applied to such methods, even if we suppose that U_k are averaged operators having a common fixed point. Our aim is to give sufficient conditions for the weak convergence of sequences generated by the recurrence (13) to a common fixed point of the operators $\{U_k\}_{k=0}^{\infty}$. Before formulating our main results we extend the definition of an asymptotically regular operator to a sequence of operators.

Definition 8 We say that a sequence of operators $\{U_k\}_{k=0}^{\infty}$, $U_k : X \to X$, is *asymptotically regular*, if for any $x \in X$

$$\lim_{k\to\infty} \|U_k U_{k-1} \dots U_0 x - U_{k-1} \dots U_0 x\| = 0, \tag{14}$$

or, equivalently,

$$\lim_{k\to\infty} \|U_k x^k - x^k\| = 0, \tag{15}$$

where the sequence $\{x^k\}_{k=0}^{\infty}$ is generated by the recurrence (13) with $x^0 = x$.

It is clear that an operator $U : X \to X$ is asymptotically regular, if the constant sequence of operators $U_k = U$ is asymptotically regular. A weaker version of the following theorem was proved in [Ceg07, Theorem 1].

Theorem 9 *Let $X \subset \mathcal{H}$ be nonempty, closed and convex, let $S : X \to \mathcal{H}$ be an operator having a fixed point and such that $S - \mathrm{Id}$ is demiclosed at 0. Let $\{U_k\}_{k=0}^{\infty}$ be an asymptotically regular sequence of quasi-nonexpansive operators $U_k : X \to X$ such that $\bigcap_{k=0}^{\infty} \operatorname{Fix} U_k \supset \operatorname{Fix} S$. Let $\{x^k\}_{k=0}^{\infty}$ be any sequence generated by the recurrence (13). Under these conditions it is true that:*

(i) *if the sequence of operators $\{U_k\}_{k=0}^{\infty}$ has the property*

$$\lim_{k\to\infty} \|U_k x^k - x^k\| = 0 \Longrightarrow \lim_{k\to\infty} \|Sx^k - x^k\| = 0 \tag{16}$$

then $\{x^k\}_{k=0}^{\infty}$ converges weakly to a point $z^* \in \operatorname{Fix} S$.

(ii) *if $\mathcal{H}$ is finite-dimensional and the sequence of operators $\{U_k\}_{k=0}^{\infty}$ has the property*

$$\lim_{k\to\infty} \|U_k x^k - x^k\| = 0 \Longrightarrow \liminf_{k\to\infty} \|Sx^k - x^k\| = 0 \tag{17}$$

then $\{x^k\}_{k=0}^{\infty}$ converges to a point $z^* \in \operatorname{Fix} S$.

Proof. Let $x \in X$, $z \in \operatorname{Fix} S$ and let the sequence $\{x^k\}_{k=0}^{\infty}$ be generated by the recurrence (13). Since U_k is quasi-nonexpansive and $\operatorname{Fix} U_k \supset \operatorname{Fix} S$, we have

$$\|x^{k+1} - z\| = \|U_k x^k - z\| \leq \|x^k - z\|, \text{ for all } k \geq 0. \tag{18}$$

Therefore, $\{x^k\}_{k=0}^{\infty}$ is Fejér-monotone with respect to $\operatorname{Fix} S$, thus bounded.

(i) Suppose that condition (16) is satisfied. By the asymptotic regularity of the sequence $\{U_k\}_{k=0}^{\infty}$ we have $\lim_{k\to\infty} \|U_k x^k - x^k\| = 0$, consequently, $\lim_{k\to\infty} \|Sx^k - x^k\| = 0$. Let $x^* \in X$ be a weak cluster point of $\{x^k\}_{k=0}^{\infty}$ and let $\{x^{n_k}\}_{k=0}^{\infty} \subset \{x^k\}_{k=0}^{\infty}$ be a subsequence converging weakly to x^*. Then $\lim_{k\to\infty} \|Sx^{n_k} - x^{n_k}\| = 0$ and $x^* \in \operatorname{Fix} S$, by the demiclosedness of $S - \mathrm{Id}$ at 0. Since x^* is an arbitrary weak cluster point of $\{x^k\}_{k=0}^{\infty}$ and $\{x^k\}_{k=0}^{\infty}$ is Fejér-monotone with respect to $\operatorname{Fix} S$, the weak convergence of the whole sequence $\{x^k\}_{k=0}^{\infty}$ to x^* follows from [Bro67, Lemma 6] (see also [BB96, Theorem 2.16 (ii)]).

(ii) Let $\mathcal{H}$ be finite-dimensional and suppose that condition (17) is satisfied. By the asymptotic regularity of $\{U_k\}_{k=0}^{\infty}$, we have $\lim_{k\to\infty} \|U_k x^k - x^k\| = 0$, consequently, $\lim_{k\to\infty} \|Sx^{n_k} - x^{n_k}\| = 0$ for a subsequence $\{x^{n_k}\}_{k=0}^{\infty} \subset \{x^k\}_{k=0}^{\infty}$. Since $\{x^{n_k}\}_{k=0}^{\infty}$ is bounded, a subsequence $\{x^{m_{n_k}}\}_{k=0}^{\infty} \subset \{x^{n_k}\}_{k=0}^{\infty}$ which converges to a point $x^* \in X$ exists. Since $S - \mathrm{Id}$ is closed at 0, we have $x^* \in \operatorname{Fix} S$. The convergence of the whole sequence $\{x^k\}_{k=0}^{\infty}$ to x^* follows now from [BB96, Theorem 2.16 (v)]. ■

Note that if $U : X \to X$ is a nonexpansive operator having a fixed point, then U is quasi-nonexpansive and $U - \mathrm{Id}$ is demiclosed at 0 (see [Opi67, Lemma 2]). Therefore, Theorem 9 (i) indeed generalizes the Opial theorem.

Remark 10 It follows from the proof that Theorem 9 remains true if we replace the assumption that $\{U_k\}_{k=0}^{\infty}$ is asymptotically regular and the assumption (16) in case (i) or (17) in case (ii) by a weaker assumption $\lim_{k\to\infty} \|Sx^k - x^k\| = 0$ in case (i) or $\liminf_{k\to\infty} \|Sx^k - x^k\| = 0$ in case (ii), respectively. The formulation presented in Theorem 9 is preferred, because in applications, the operators U_k are often relaxed cutters with relaxation parameters guaranteeing the asymptotic regularity of $\{U_k\}_{k=0}^{\infty}$. Furthermore, various practical algorithms which apply relaxed cutters have properties which yield (16), (17) or some related conditions (see the examples presented in Section 5).

4 Opial-type theorems for cutters

In this section we focus our attention on cutters. We first recall some properties of sequences of real numbers. Let $\alpha_k, \beta_k \geq 0$, for all $k \geq 0$, and let $\sum_{k=0}^{\infty} \alpha_k \beta_k < +\infty$. Then

$$\liminf_{k\to\infty} \alpha_k > 0 \Longrightarrow \sum_{k=0}^{\infty} \beta_k < +\infty \tag{19}$$

or, equivalently,

$$\sum_{k=0}^{\infty} \beta_k = +\infty \Longrightarrow \liminf_{k\to\infty} \alpha_k = 0. \tag{20}$$

If $\lambda_k \in [0,2]$ then the following equivalence holds

$$\liminf_{k\to\infty} \lambda_k(2-\lambda_k) > 0 \Longleftrightarrow \left(\liminf_{k\to\infty} \lambda_k > 0 \text{ and } \limsup_{k\to\infty} \lambda_k < 2\right). \tag{21}$$

Lemma 11 *Let the sequence $\{x^k\}_{k=0}^\infty \subset X$ be generated by the recurrence*

$$x^{k+1} = P_X(x^k + \lambda_k(T_k x^k - x^k)), \tag{22}$$

where $\lambda_k \in [0,2]$ and $\{T_k\}_{k=0}^\infty$ is a sequence of cutters, $T_k : X \to \mathcal{H}$, with $\bigcap_{k=0}^\infty \operatorname{Fix} T_k \neq \emptyset$. Then

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - \lambda_k(2-\lambda_k)\|T_k x^k - x^k\|^2 \tag{23}$$

for all $z \in \bigcap_{k=0}^\infty \operatorname{Fix} T_k$. Consequently,

$$\|x^{k+1} - z\|^2 \leq \|x^0 - z\|^2 - \sum_{l=0}^{k} \lambda_l(2-\lambda_l)\|T_l x^l - x^l\|^2 \tag{24}$$

and

$$\sum_{k=0}^{\infty} \lambda_k(2-\lambda_k)\|T_k x^k - x^k\|^2 \leq d^2(x^0, \bigcap_{k=0}^{\infty} \operatorname{Fix} T_k). \tag{25}$$

Moreover,

(i) *if $\liminf_{k\to\infty} \lambda_k(2-\lambda_k) > 0$ then $\sum_{k=0}^\infty \|T_k x^k - x^k\|^2 < +\infty$,*

(ii) *if $\sum_{k=0}^\infty \lambda_k(2-\lambda_k) = +\infty$ then $\liminf_{k\to\infty} \|T_k x^k - x^k\| = 0$.*

Proof. Let $z \in \bigcap_{k=0}^\infty \operatorname{Fix} T_k$. It is clear that $z \in X$, so that $P_X z = z$. By the nonexpansivity of the metric projection P_X and by Lemma 4 (i), we have

$$\begin{aligned}
\|x^{k+1} - z\|^2 &= \|P_X(x^k + \lambda_k(T_k x^k - x^k)) - z\|^2 \\
&= \|P_X(x^k + \lambda_k(T_k x^k - x^k)) - P_X z\|^2 \\
&\leq \|x^k + \lambda_k(T_k x^k - x^k) - z\|^2 \\
&= \|x^k - z\|^2 + \lambda_k^2\|T_k x^k - x^k\|^2 - 2\lambda_k\langle z - x, T_k x^k - x^k\rangle \\
&\leq \|x^k - z\|^2 + \lambda_k^2\|T_k x^k - x^k\|^2 - 2\lambda_k\|T_k x^k - x^k\|^2,
\end{aligned} \tag{26}$$

which yields (23). Iterating this inequality k times we obtain (24). Since $\|x^{k+1} - z\|^2 \geq 0$, we obtain (25).

(i) Suppose that $\liminf_{k\to\infty} \lambda_k(2-\lambda_k) > 0$. If we set $\alpha_k = \lambda_k(2-\lambda_k)$ and $\beta_k = \|T_k x^k - x^k\|^2$ in (19) we obtain $\sum_{k=0}^\infty \|T_k x^k - x^k\|^2 < +\infty$.

(ii) Suppose that $\sum_{k=0}^\infty \lambda_k(2-\lambda_k) = +\infty$. If we set $\beta_k = \lambda_k(2-\lambda_k)$ and $\alpha_k = \|T_k x^k - x^k\|^2$ in (20) we obtain $\liminf_{k\to\infty} \|T_k x^k - x^k\| = 0$. ■

Proposition 12 *Let $S : X \to \mathcal{H}$ be an operator having a fixed point and such that $S - \mathrm{Id}$ is demiclosed at 0, let $x^0 \in X$ and let the sequence $\{x^k\}_{k=0}^{\infty} \subset X$ be generated by the recurrence (22), where $\lambda_k \in [0, 2]$ for all $k \geq 0$, and $\{T_k\}_{k=0}^{\infty}$, $T_k : X \to \mathcal{H}$, is a sequence of cutters with $\bigcap_{k=0}^{\infty} \operatorname{Fix} T_k \supset \operatorname{Fix} S$.*

(i) *If $\liminf_{k\to\infty} \lambda_k(2 - \lambda_k) > 0$ and*

$$\sum_{k=0}^{\infty} \|T_k x^k - x^k\|^2 < +\infty \Longrightarrow \lim_{k\to\infty} \|Sx^k - x^k\| = 0 \tag{27}$$

then $\{x^k\}_{k=0}^{\infty}$ converges weakly to a fixed point of S.

(ii) *If $\liminf_{k\to\infty} \lambda_k(2 - \lambda_k) > 0$, $\mathcal{H}$ is finite-dimensional and*

$$\sum_{k=0}^{\infty} \|T_k x^k - x^k\|^2 < +\infty \Longrightarrow \liminf_{k\to\infty} \|Sx^k - x^k\| = 0 \tag{28}$$

then $\{x^k\}_{k=0}^{\infty}$ converges to a fixed point of S.

(iii) *If $\sum_{k=0}^{\infty} \lambda_k(2 - \lambda_k) = +\infty$ and*

$$\liminf_{k\to\infty} \|T_k x^k - x^k\| = 0 \Longrightarrow \lim_{k\to\infty} \|Sx^k - x^k\| = 0 \tag{29}$$

then $\{x^k\}_{k=0}^{\infty}$ converges weakly to a fixed point of S.

(iv) *If $\sum_{k=0}^{\infty} \lambda_k(2 - \lambda_k) = +\infty$, $\mathcal{H}$ is finite-dimensional and*

$$\liminf_{k\to\infty} \|T_k x^k - x^k\| = 0 \Longrightarrow \liminf_{k\to\infty} \|Sx^k - x^k\| = 0 \tag{30}$$

then $\{x^k\}_{k=0}^{\infty}$ converges to a fixed point of S.

Proof. Let $C = \bigcap_{k=0}^{\infty} \operatorname{Fix} T_k$ and $z \in C$. Denote $U_k = P_X(\mathrm{Id} + \lambda_k(T_k - \mathrm{Id}))$. By Lemma 11 the sequence $\{x^k\}_{k=0}^{\infty}$ is Fejér-monotone with respect to C, thus bounded. Suppose that $\liminf_{k\to\infty} \lambda_k(2 - \lambda_k) > 0$.

(i) Lemma 11 (i) and (27) yield $\lim_{k\to\infty} \|Sx^k - x^k\| = 0$. Let $x^* \in X$ be a weak cluster point of $\{x^k\}_{k=0}^{\infty}$. By the demiclosedness of $S - \mathrm{Id}$ we have $x^* \in \operatorname{Fix} S$. The weak convergence of $\{x^k\}_{k=0}^{\infty}$ to x^* follows now from [BB96, Theorem 2.16 (ii)].

(ii) Suppose that $\mathcal{H}$ is finite-dimensional. Lemma 11 (i) and (28) yield $\lim_{k\to\infty}\|Sx^{n_k}-x^{n_k}\|=0$ for a subsequence $\{x^{n_k}\}_{k=0}^{\infty}\subset\{x^k\}_{k=0}^{\infty}$. Let $\{x^{m_{n_k}}\}_{k=0}^{\infty}\subset\{x^{n_k}\}_{k=0}^{\infty}$ be a subsequence which converges to a point $x^*\in X$. By the closedness of $S-\mathrm{Id}$ we have $x^*\in\operatorname{Fix}S$. The convergence of $\{x^k\}_{k=0}^{\infty}$ to x^* follows now from [BB96, Theorem 2.16 (v)].

If $\sum_{k=0}^{\infty}\lambda_k(2-\lambda_k)=+\infty$ then (iii) and (iv) can be proved similarly to (i) and (ii) by application of Lemma 11 (ii) and (29), (30), respectively. ■

Special cases of Proposition 12 were proved in [Ceg93, Corollary 3.4.F], where $X=\mathbb{R}^n$ and $S=P_{C_i}$, $i=1,2,...,m$, with $\bigcap_{i=1}^{m}C_i\subset\bigcap_{k=0}^{\infty}\operatorname{Fix}T_k$. Other results which are closely related to Proposition 12 can be found in [Sch91, Section 2], where, instead of assumptions (27)–(30), there appears

$$\liminf_{k\to\infty}\|T_kx^k-x^k\|=0\Longrightarrow\liminf_{k\to\infty}\|x^k-P_Fx^k\|=0, \tag{31}$$

where $F=\bigcap_{k=0}^{\infty}\operatorname{Fix}T_k$. As shown in the next section, the assumptions (27)–(30) are easier to verify than (31).

Remark 13 (a) If $\{a_k\}_{k=0}^{\infty}\subset\mathbb{R}_+$ then $\sum_{k=0}^{\infty}a_k^2<+\infty$ implies $\lim_{k\to\infty}a_k=0$. Therefore, if we replace $\sum_{k=0}^{\infty}\|T_kx^k-x^k\|^2<+\infty$ by $\lim_{k\to\infty}\|T_kx^k-x^k\|=0$ in Proposition 12 (i), we obtain the following weaker result:

(i') If $\liminf_k\lambda_k(2-\lambda_k)>0$ and

$$\lim_{k\to\infty}\|T_kx^k-x^k\|=0\Longrightarrow\lim_{k\to\infty}\|Sx^k-x^k\|=0 \tag{32}$$

then $\{x^k\}_{k=0}^{\infty}$ converges weakly to a fixed point of S.

(b) Since relaxed cutters are quasi-nonexpansive (see, [BC01, equivalence (v)⇔(vi) in Proposition 2.3]), iteration (22) with $X=\mathcal{H}$ is a special case of (13), where $U_k=\mathrm{Id}+\lambda_k(T_k-\mathrm{Id})$, $k\geq 0$. Then inequality (23) for $\lambda_k\in(0,2]$ can be written as

$$\|x^{k+1}-z\|^2\leq\|x^k-z\|^2-\frac{2-\lambda_k}{\lambda_k}\|U_kx^k-x^k\|^2. \tag{33}$$

This shows that result (i') also follows from Theorem 9 (i). Indeed. By (33) $\{U_k\}_{k=0}^{\infty}$ is asymptotically regular. If (32) holds then (16) holds, because of the equivalence $\lim_{k\to\infty}\|T_kx^k-x^k\|=0\Longleftrightarrow\lim_{k\to\infty}\|U_kx^k-x^k\|=0$ which is valid if $\liminf_k \lambda_k > 0$. Now Theorem 9 (i) yields the weak convergence of $\{x^k\}_{k=0}^{\infty}$ to a fixed point of S.

(c) We also see that (28) is weaker than (27), and (30) is weaker than (29), i.e., in the finite-dimensional case convergence holds under weaker assumptions than in the infinite-dimensional one.

Corollary 14 *Let $T : X \to \mathcal{H}$ be a nonexpansive cutter (e.g., a firmly nonexpansive operator) having a fixed point, let $x^0 \in X$ and let a sequence $\{x^k\}_{k=0}^{\infty}$ be generated by the recurrence*

$$x^{k+1} = P_X(x^k + \lambda_k(Tx^k - x^k)), \tag{34}$$

where $\lambda_k \in [0, 2]$.

(i) *If $\liminf_{k\to\infty} \lambda_k(2 - \lambda_k) > 0$ then $\{x^k\}_{k=0}^{\infty}$ converges weakly to a fixed point of T.*

(ii) *If $\mathcal{H}$ is finite-dimensional and $\sum_{k=0}^{\infty} \lambda_k(2 - \lambda_k) = +\infty$ then $\{x^k\}_{k=0}^{\infty}$ converges to a fixed point of T.*

Proof. Denote $T_k = T$, for all $k \geq 0$, and $S = T$. Since S is nonexpansive, $S - \mathrm{Id}$ is demiclosed at 0 (see [Opi67, Lemma 2]). Implications (27) and (30) are obvious. Therefore, (i) follows from Proposition 12 (i), while (ii) follows from Proposition 12 (iv). ■

Remark 15 Since a firmly nonexpansive operator is a cutter and an averaged operator is relaxed firmly nonexpansive, the Krasnoselskii–Mann theorem (see, e.g., [Byr04, Theorem 2.1]) follows from Corollary 14 (i) by setting $X = \mathcal{H}$ and $\lambda_k = \lambda \in (0, 2)$ for $k \geq 0$.

Before formulating our next result, we introduce the notion of a generalized relaxation of an operator (compare [Ceg08, Section 1]).

Definition 16 Let $T : X \to \mathcal{H}$, $\lambda \in [0, 2]$ and let $\sigma : X \to (0, +\infty)$. The operator $T_{\sigma,\lambda} : X \to \mathcal{H}$,

$$T_{\sigma,\lambda}x := x + \lambda\sigma(x)(Tx - x) \tag{35}$$

is called the *generalized relaxation* of T, the value λ is called the *relaxation parameter* and σ is called the *step-size function*. If $\sigma(x) \geq 1$ for all $x \in X$ then the operator $T_{\sigma,\lambda}$ is called an *extrapolation* of T_λ.

Definition 17 We say that an operator $T : X \to \mathcal{H}$ having a fixed point is *oriented* if, for all $x \notin \operatorname{Fix} T$,

$$\delta(x) := \inf \left\{ \frac{\langle z - x, Tx - x \rangle}{\|Tx - x\|^2} \mid z \in \operatorname{Fix} T \right\} > 0. \tag{36}$$

If $\delta(x) > \alpha > 0$ for all $x \notin \operatorname{Fix} T$ then we call the operator T *α-strongly oriented* or *strongly oriented*.

Lemma 4 (i) means that a cutter is 1-strongly oriented. Denoting $T_\sigma = T_{\sigma,1}$ for an operator $T : X \to \mathcal{H}$ and a step-size function $\sigma : X \to (0, +\infty)$, it is clear that $T_{\sigma,\lambda}$ is a λ-relaxation of T_σ, i.e., $T_{\sigma,\lambda} = (T_\sigma)_\lambda$ for any $\lambda \in [0, 2]$.

Lemma 18 *Let $T : X \to \mathcal{H}$ be an oriented operator with $\operatorname{Fix} T \neq \emptyset$. If a step-size function $\sigma : X \to (0, +\infty)$ satisfies the inequality*

$$\sigma(x) \le \frac{\langle z - x, Tx - x \rangle}{\|Tx - x\|^2} \tag{37}$$

for all $x \notin \operatorname{Fix} T$ and for all $z \in \operatorname{Fix} T$, then T_σ is a cutter.

Proof. Let $x \notin \operatorname{Fix} T$ and $z \in \operatorname{Fix} T$. Let $\sigma : X \to (0, +\infty)$ be a step-size function satisfying (37). The existence of σ follows from the assumption that T is oriented. By inequality (37) we have

$$\begin{aligned} \langle z - T_\sigma x, x - T_\sigma x \rangle &= \langle z - x, x - T_\sigma x \rangle + \|x - T_\sigma x\|^2 \\ &= -\langle z - x, \sigma(x)(Tx - x) \rangle + \|x - T_\sigma x\|^2 \\ &\le -\|\sigma(x)(Tx - x)\|^2 + \|x - T_\sigma x\|^2 = 0, \end{aligned} \tag{38}$$

i.e., T_σ is a cutter. ■

Corollary 19 *Let $U : X \to \mathcal{H}$ be a strongly oriented operator having a fixed point and such that $U - \operatorname{Id}$ is demiclosed at 0, and let the sequence $\{x^k\}_{k=0}^\infty \subset X$ be generated by the recurrence*

$$x^{k+1} = P_X U_{\sigma_k, \lambda_k}(x^k), \tag{39}$$

where $x^0 \in X$, $\liminf_{k\to\infty} \lambda_k(2 - \lambda_k) > 0$ and let the step-size functions $\sigma_k : X \to (0, +\infty)$ satisfy the condition

$$\alpha \le \sigma_k(x) \le \frac{\langle z - x, Ux - x \rangle}{\|Ux - x\|^2} \tag{40}$$

for all $x \notin \operatorname{Fix} U$, for all $z \in \operatorname{Fix} U$ and for some $\alpha > 0$. Then $\{x^k\}_{k=0}^\infty$ converges weakly to a fixed point of U.

Proof. Let $z \in \operatorname{Fix} U$. The existence of step-size functions $\sigma_k : X \to (0, +\infty)$ satisfying (40) for all $x \notin \operatorname{Fix} U$ and for some $\alpha > 0$, follows from the assumption that U is strongly oriented. It is clear that the recurrence (39) is a special case of (22) with $T_k = U_{\sigma_k} = U_{\sigma_k,1}$. By Lemma 18 the operator T_k is a cutter. We have

$$\|T_k x^k - x^k\| = \|U_{\sigma_k} x^k - x^k\| = \sigma_k(x^k)\|Ux^k - x^k\| \geq \alpha\|Ux^k - x^k\|. \quad (41)$$

Therefore,

$$\lim_{k\to\infty} \|T_k x^k - x^k\| = 0 \Longrightarrow \lim_{k\to\infty} \|Ux^k - x^k\| \quad (42)$$

which is stronger than condition (27) with $S = U$ (see Remark 13). The weak convergence of $\{x^k\}_{k=0}^{\infty}$ to a fixed point of U follows now from Proposition 12 (i), because $\operatorname{Fix} U_{\sigma_k} = \operatorname{Fix} U$ for all $k \geq 0$. ∎

5 Applications to the common fixed point problem

Let $\mathcal{U} = \{U_i\}_{i\in I}$, where $I := \{1, 2, ..., m\}$, be a finite family of cutters $U_i : \mathcal{H} \to \mathcal{H}$, having a common fixed point. The *common fixed point problem* is to find $x^* \in \bigcap_{i\in I} \operatorname{Fix} U_i$. In this section we study the convergence properties of sequences generated by the recurrence

$$x^{k+1} = x^k + \lambda_k \sigma_k(x^k) \left(\sum_{i\in J_k} w_i^k(x^k) V_i^k x^k - x^k \right), \quad (43)$$

where $\lambda_k \in [0, 2]$, $\sigma_k : \mathcal{H} \to (0, +\infty)$ are step-size functions, $\mathcal{V}^k = \{V_i^k\}_{i\in J_k}$ is a family of cutters $V_i^k : \mathcal{H} \to \mathcal{H}$, $i \in J_k = \{1, 2, ..., m_k\}$ with the property $\bigcap_{i\in J_k} \operatorname{Fix} V_i^k \supset \bigcap_{i\in I} \operatorname{Fix} U_i$ and $w^k : \mathcal{H} \to \Delta_{m_k}$ are weight functions $w^k(x) = (w_1^k(x), w_2^k(x), ..., w_{m_k}^k(x))$ (the subset Δ_m denotes here the standard simplex, i.e., $\Delta_m = \{u \in \mathbb{R}^m : u_i \geq 0,\ i = 1, 2, ..., m, \text{ and } \sum_{i=1}^m u_i = 1\}$). If $\sigma_k(x) = 1$ for all $x \in \mathcal{H}$ and for all $k \geq 0$, then the method defined by the recurrence (43) takes the form

$$x^{k+1} = x^k + \lambda_k \left(\sum_{i\in J_k} w_i^k(x^k) V_i^k x^k - x^k \right), \quad (44)$$

and is called the *simultaneous cutter method.* If $\sigma_k(x) \geq 1$ for all $x \in \mathcal{H}$ and for all $k \geq 0$, then method (43) is called the *extrapolated simultaneous cutter method.* The recurrence (43) can be written in the form

$$x^{k+1} = x^k + \lambda_k \sigma_k(x^k)(V^k x^k - x^k), \tag{45}$$

where $V^k = \sum_{i \in J_k} w_i^k V_i^k$, or in the form

$$x^{k+1} = V_{\sigma_k, \lambda_k} x^k. \tag{46}$$

Remark 20 The sequence of weight functions $\{w^k\}_{k=0}^{\infty}$ induces a *control sequence.* This notion is usually applied in the literature if the values of w^k are extremal points of a standard simplex (see, e.g., [Cen81, Definition 3.2] or [CZ97, Definition 5.1.1]). One can recognize special cases of a sequence of weight functions $\{w^k\}_{k=0}^{\infty}$ as known control sequences. In particular, if the weight functions $\{w^k\}_{k=0}^{\infty}$ are constant, i.e., $w^k(x) = (w_1^k, w_2^k, ..., w_{m_k}^k) \in \Delta_{m_k}$ for all $x \in \mathcal{H}$, $k \geq 0$. A simple example of such a control sequence is the *cyclic control* (see [GPR67, Equality (2)], [Cen81, (3.3)] or [CZ97, Definition 5.1.1]) The sequence $\{w^k\}_{k=0}^{\infty}$ can also be a constant sequence, i.e., $J_k = J$ and $w^k = w : \mathcal{H} \to \Delta_m$ for all $k \geq 0$. A simple example of such a control is the *remotest set control* (see [GPR67, Equality (3')] or [Cen81, (3.5)] or [CZ97, Definition 5.1.1]). Sequences of weights depending on $x \in \mathcal{H}$ enable, however, a more general model and demonstrate the importance of assumptions on the weight functions control.

Definition 21 Let $V_i : \mathcal{H} \to \mathcal{H}$, $i \in J = \{1, 2, ..., l\}$. We say that a weight function $w : \mathcal{H} \to \Delta_l$ is *appropriate with respect to the family* $\mathcal{V} = \{V_i\}_{i \in J}$ or, shortly, *appropriate* if for any $x \notin \bigcap_{i \in J} \operatorname{Fix} V_i$ there exists a $j \in J$ such that

$$w_j(x) \|V_j x - x\| \neq 0. \tag{47}$$

Lemma 22 *Let $V_i : \mathcal{H} \to \mathcal{H}$, $i \in J = \{1, 2, ..., l\}$, be cutters having a common fixed point and let $V = \sum_{i \in J} w_i V_i$, where $w : \mathcal{H} \to \Delta_l$ is appropriate with respect to the family $\mathcal{V} = \{V_i\}_{i \in J}$. Then*

(i) $\operatorname{Fix} V = \bigcap_{i \in J} \operatorname{Fix} V_i$,

(ii) *V is a cutter, consequently, for all $\lambda \in (0, 2)$, the operator V_λ is $\frac{2-\lambda}{\lambda}$-strongly quasi-nonexpansive,*

(iii) *the following inequalities hold*

$$\|V_\lambda x - z\|^2 \leq \|x-z\|^2 - \lambda(2-\lambda)\sum_{i\in J} w_i(x)\|V_i x - x\|^2 \tag{48}$$

$$\leq \|x-z\|^2 - \lambda(2-\lambda)\|Vx - x\|^2 \tag{49}$$

for all $\lambda \in [0,2]$, $x \in \mathcal{H}$ *and* $z \in \operatorname{Fix} V$.

Proof. (i) The inclusion $\bigcap_{i\in J} \operatorname{Fix} V_i \subset \operatorname{Fix} V$ is obvious. We show that $\operatorname{Fix} V \subset \bigcap_{i\in J} \operatorname{Fix} V_i$. If $\bigcap_{i\in J} \operatorname{Fix} V_i = \mathcal{H}$ then the inclusion is clear. Otherwise, suppose that $x \in \operatorname{Fix} V$, $x \notin \bigcap_{i\in J} \operatorname{Fix} V_i$ and that $z \in \bigcap_{i\in J} \operatorname{Fix} V_i$. Since a cutter is strongly quasi-nonexpansive (see Lemma 4 (ii)) we have $\|V_i x - z\| < \|x - z\|$ for any $i \in J$ such that $x \notin \operatorname{Fix} V_i$. The convexity of the norm, the strict quasi-nonexpansivity of V_i and the fact that the weight function w is appropriate yield

$$\|Vx - z\| = \|\sum_{i\in J} w_i(x)(V_i x - z)\| \leq \sum_{i\in J} w_i(x)\|V_i x - z\|$$

$$< \sum_{i\in J} w_i(x)\|x - z\| = \|x - z\|. \tag{50}$$

We get a contradiction, which shows that $\operatorname{Fix} V \subset \bigcap_{i\in J} \operatorname{Fix} V_i$.

(ii) Let $x \in \mathcal{H}$ and $z \in \operatorname{Fix} V$. It follows from (i) that $z \in \bigcap_{i\in J} \operatorname{Fix} V_i$. By Lemma 4 (i) and by the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned}\langle Vx - x, z - x\rangle &= \sum_{i\in J} w_i(x)\langle V_i x - x, z - x\rangle \\ &\geq \sum_{i\in J} w_i(x)\|V_i x - x\|^2 \\ &\geq \|\sum_{i\in J} w_i(x) V_i x - x\|^2 \\ &= \|Vx - x\|^2. \end{aligned} \tag{51}$$

Applying again Lemma 4 (i) we deduce that V is a cutter. By Lemma 4 (ii) the operator V_λ is $\frac{2-\lambda}{\lambda}$-strongly quasi-nonexpansive for any $\lambda \in (0,2)$.

(iii) Let $\lambda \in [0,2]$, $x \in \mathcal{H}$ and $z \in \operatorname{Fix} V$. The convexity of $\|\cdot\|^2$ and Lemma 4 (i) yield

$$\begin{aligned}
&\|V_\lambda x - z\|^2 \\
&= \|x + \lambda \sum_{i\in J} w_i(x)(V_i x - x) - z\|^2 \\
&= \|x - z\|^2 + \lambda^2 \|\sum_{i\in J} w_i(x)(V_i x - x)\|^2 - 2\lambda \sum_{i\in J} w_i(x)\langle z - x, V_i x - x\rangle \\
&\le \|x - z\|^2 + \lambda^2 \sum_{i\in J} w_i(x)\|V_i x - x\|^2 - 2\lambda \sum_{i\in J} w_i(x)\|V_i x - x\|^2 \\
&= \|x - z\|^2 - \lambda(2-\lambda) \sum_{i\in J} w_i(x)\|V_i x - x\|^2, \qquad (52)
\end{aligned}$$

i.e., the inequality (48) holds. Inequality (49) follows from the convexity of the function $\|\cdot\|^2$. ■

Definition 23 Let $\mathcal{V} = \{V_i\}_{i\in J}$ be a finite family of operators $V_i : \mathcal{H} \to \mathcal{H}$, $i \in J$, and let $\beta \in (0,1]$ be a constant. We say that a weight function $w : \mathcal{H} \to \Delta_{|J|}$ is *β-regular with respect to the family of cutters $\mathcal{U} = \{U_i\}_{i\in I}$*, or, shortly, *regular* if for any $x \in \mathcal{H}$ there exists a $j \in J$ such that

$$w_j(x)\|V_j x - x\|^2 \ge \beta \max\left\{\|U_i x - x\|^2 \mid i \in I\right\}. \qquad (53)$$

If $\bigcap_{i\in J} \operatorname{Fix} V_i \supset \bigcap_{i\in I} \operatorname{Fix} U_i$ then a weight function which is regular with respect to the family $\mathcal{U} = \{U_i\}_{i\in I}$ is appropriate with respect to the family $\mathcal{V} = \{V_i\}_{i\in J}$.

Example 24 Let $\mathcal{V} = \mathcal{U}$ and let $I(x) = \{i \in I \mid x \notin \operatorname{Fix} U_i\}$ and let $m(x) = |I(x)|$ be the cardinality of $I(x)$, for $x \in \mathcal{H}$. The following weight functions $w : \mathcal{H} \to \Delta_m$, where $w(x) = (w_1(x), ..., w_m(x))$, are regular:

(a) Positive constant weights, i.e.,

$$w(x) = w \in \operatorname{ri} \Delta_m \qquad (54)$$

for all $x \in \mathcal{H}$, where $\operatorname{ri} \Delta_m = \{w \in \mathbb{R}^m \mid w > 0 \text{ and } \langle e, w\rangle = 1\}$ is the relative interior of Δ_m. A specific example is furnished by equal weights, i.e., $w_i(x) = 1/m$, $i \in I$. To verify that w is regular set $j \in \operatorname{argmax}_{i\in I} \|U_i x - x\|$ and $\beta = \min_{i\in I} w_i$ in Definition 23.

(b) Constant weights for violated constraints, i.e.,

$$w_i(x) := \begin{cases} \dfrac{w_i}{\sum\limits_{j \in I(x)} w_j}, & \text{for } i \in I(x), \\ 0, & \text{for } i \notin I(x), \end{cases} \tag{55}$$

where $w = (w_1, w_2, ..., w_m) \in \operatorname{ri} \Delta_m$. A specific example is

$$w_i(x) := \begin{cases} 1/m(x), & \text{for } i \in I(x), \\ 0, & \text{for } i \notin I(x). \end{cases} \tag{56}$$

To verify that w is regular set $j \in \operatorname{argmax}_{i \in I} \|U_i x - x\|$ and $\beta = \min_{i \in I} w_i$ in Definition 23.

(c) Weights proportional to $\|U_i x - x\|$, i.e.,

$$w_i(x) = \begin{cases} \dfrac{\|U_i x - x\|}{\sum\limits_{j \in I} \|U_j x - x\|}, & \text{for } x \notin \bigcap_{i \in I} \operatorname{Fix} U_i, \\ 0, & \text{for } x \in \bigcap_{i \in I} \operatorname{Fix} U_i. \end{cases} \tag{57}$$

To verify, set $j \in \operatorname{argmax}_{i \in I} \|U_i x - x\|$ and $\beta = 1/m$ in Definition 23.

(d) Weight functions $w : \mathcal{H} \to \Delta_m$ satisfying the condition

$$w_i(x) \geq \delta \text{ for } i \in I(x) \tag{58}$$

for some constant $\delta > 0$. To verify, choose $j(x) \in \operatorname{argmax}_{i \in I} \|U_i x - x\|$ and set $\beta = \delta$ in Definition 23. These weight functions were applied by Combettes in [Com97a, Section III] and in [Com97, Section 1]. Observe that the weight functions defined by (54) and by (55) satisfy (58).

(e) Weight functions $w : \mathcal{H} \to \Delta_m$ for which $w_i(x) = 0$ for all $x \in \mathcal{H}$ and for all $i \notin J_\gamma(x)$, where

$$J_\gamma(x) = \{j \in I \mid \|U_j x - x\| \geq \gamma \max_{i \in I} \|U_i x - x\|\}, \tag{59}$$

for some $\gamma \in (0, 1]$. To verify, set $j = j(x) \in J_\gamma(x)$ with $\omega_j(x) \geq 1/m$ and $\beta = \gamma^2/m$ in Definition 23. The existence of such j follows from the fact that $w_i(x) \geq 0$ for all $i \in J_\gamma(x)$ and $\sum_{i \in J_\gamma(x)} w_i(x) = 1$. Specific examples are obtained as follows:

(i) When $U_i = P_{C_i}$ for a closed convex subset $C_i \subset \mathcal{H}$, $i \in I$, and

$$w_i(x) = \begin{cases} 1, & \text{if } i = \operatorname{argmax}_{j \in I} \|U_j x - x\| \\ 0, & \text{otherwise.} \end{cases} \tag{60}$$

In this case w defines a *remotest set control* (for the definition, see [GPR67, Eq. (3')] or [CZ97, Section 5.1]).

(ii) When $U_i = P_{C_i}$ for a closed convex subset $C_i \subset \mathcal{H}$, $i \in I$, and

$$w_i(x) = \begin{cases} 1, & \text{if } i = j(x) \\ 0, & \text{otherwise,} \end{cases} \tag{61}$$

where $j(x) \in J_\gamma(x)$ for some $\gamma \in (0, 1]$. In this case w is an *approximately remotest set control* (for the definition, see [GPR67, Eq. (3)] or [CZ97, Section 5.1]).

Definition 25 Let $\mathcal{V}^k = \{V_i^k\}_{i \in J_k}$ be a sequence of cutters $V_i^k : \mathcal{H} \to \mathcal{H}$, $i \in J_k = \{1, 2, ..., m_k\}$, $k \geq 0$, and let the sequence $\{x^k\}_{k=0}^{\infty}$ be generated by the recurrence (43). We say that a sequence of appropriate weight functions $w^k : \mathcal{H} \to \Delta_{m_k}$ (applied to the sequence of families $\mathcal{V}^k$) is

- *regular* (with respect to the family $\mathcal{U} = \{U_i\}_{i \in I}$) if there is a constant $\beta \in (0, 1]$ such that w^k are β-regular for all $k \geq 0$,
- *approximately regular* (with respect to the family $\mathcal{U} = \{U_i\}_{i \in I}$) if there exists a sequence $i_k \in J_k$ such that the following implication holds

$$\lim_{k \to \infty} w_{i_k}^k(x^k) \|V_{i_k}^k x^k - x^k\|^2 = 0 \Longrightarrow \lim_{k \to \infty} \|U_i x^k - x^k\| = 0 \text{ for all } i \in I, \tag{62}$$

- *approximately semi-regular* (with respect to the family $\mathcal{U} = \{U_i\}_{i \in I}$) if there exists a sequence $i_k \in J_k$ such that the following implication holds

$$\lim_{k \to \infty} w_{i_k}^k(x^k) \|V_{i_k}^k x^k - x^k\|^2 = 0 \Longrightarrow \liminf_{k \to \infty} \|U_i x^k - x^k\| = 0 \text{ for all } i \in I. \tag{63}$$

Example 26 Here are examples of weight functions which are approximately regular or approximately semi-regular with respect to the family $\mathcal{U} = \{U_i\}_{i \in I}$.

(a) A regular sequence of weight functions is approximately regular.

(b) A sequence containing a regular subsequence of weight functions is approximately semi-regular.

(c) Let $\{x^k\}_{k=0}^{\infty}$ be a sequence generated by the recurrence (44), where $\mathcal{V}^k = \mathcal{U}$ and $w^k = \delta_{i_k}$. We call the sequence $\{i_k\}_{k=0}^{\infty}$ a *control sequence* (see [Cen81, Definition 3.2]). Recurrence (44) can be written as follows

$$x^{k+1} = x^k + \lambda_k(U_{i_k} x^k - x^k). \tag{64}$$

Implication (62) takes the form

$$\lim_{k\to\infty} \|U_{i_k} x^k - x^k\| = 0 \Longrightarrow \lim_{k\to\infty} \|U_i x^k - x^k\| = 0 \text{ for all } i \in I. \tag{65}$$

If (65) is satisfied we say that the control sequence $\{i_k\}_{k=0}^{\infty}$ is *approximately regular*. If we set $U_i = P_{C_i}$ for a closed convex subset $C_i \subset \mathcal{H}$, $i \in I$, then implication (65) can be written in the form

$$\lim_{k\to\infty} \|P_{C_{i_k}} x^k - x^k\| = 0 \Longrightarrow \lim_{k\to\infty} \max_{i\in I} \|P_{C_i} x^k - x^k\| = 0. \tag{66}$$

A sequence $\{i_k\}_{k=0}^{\infty}$ satisfying (66) is called *approximately remotest set control* (see [GPR67, Section 1]).

(d) (Combettes [Com97a, Section II D]). Let I_k be a nonempty subset of I, $k \geq 0$. Suppose that there is a constant $s \geq 1$ such that

$$I = I_k \cup I_{k+1} \cup \ldots \cup I_{k+s-1} \text{ for all } k \geq 0. \tag{67}$$

Let $U_i = P_{C_i}$, where $C_i \subset \mathcal{H}$ is closed and convex. Let $\{x^k\}_{k=0}^{\infty}$ be a sequence generated by the recurrence (44), where $\mathcal{V}^k = \mathcal{U} = \{U_i\}_{i\in I}$, $\lambda_k \in [\varepsilon, 2-\varepsilon]$ for some $\varepsilon \in (0,1)$, and $w^k \in \Delta_m$ is a weight vector such that $\sum_{i\in I_k} w_i^k = 1$ and $w_i^k \geq \delta > 0$ for all $i \in I_k \cap I(x^k)$, $k \geq 0$, and $I(x) = \{i \in I \mid x \notin C_i\}$. Bauschke and Borwein called a sequence of weights satisfying (67) with $I_k = \{i \in I \mid w_i^k > 0\}$ an *intermittent control* (see [BB96, Definition 3.18]). The recurrence (44) can be written in the form $x^{k+1} = T_k x^k$, where $T_k = \mathrm{Id} + \lambda_k(V_k - \mathrm{Id})$ and $V_k = \sum_{i\in I_k} w_i^k P_{C_i}$, or, equivalently, in the form

$$x^{k+1} = x^k + \lambda_k\Big(\sum_{i\in I_k} \omega_i^k P_{C_i} x^k - x^k\Big). \tag{68}$$

One can show that V_k is a cutter. We show that $\{w^k\}_{k=0}^{\infty}$ is approximately regular. Let $i \in I$ be arbitrary and let $r_k \in \{0, 1, \ldots, s-1\}$ be such that $i \in I_{k+r_k}$, $k \geq 0$. By the triangle inequality, we have

$$\begin{aligned} \|x^{k+r_k} - x^k\| &\leq \sum_{i=0}^{r_k-1} \|x^{k+i+1} - x^{k+i}\| = \sum_{i=0}^{r_k-1} \|T_{k+i}x^{k+i} - x^{k+i}\| \\ &\leq \sum_{i=0}^{s-1} \|T_{k+i}x^{k+i} - x^{k+i}\|, \end{aligned} \tag{69}$$

$k \geq 0$. Since T_k are λ_k-relaxed cutters and $\lambda_k \in [\varepsilon, 2-\varepsilon]$, Lemma 22 (iii) yields $\lim_{k\to\infty} \|T_{k+i}x^{k+i} - x^{k+i}\| = 0$, $i = 1, 2, \ldots, s-1$, consequently, $\|x^{k+r_k} - x^k\| \to 0$. Further, by the definition of the metric projection and by the triangle inequality, we have

$$\|P_{C_i}x^k - x^k\| \leq \|P_{C_i}x^{k+r_k} - x^k\| \leq \|P_{C_i}x^{k+r_k} - x^{k+r_k}\| + \|x^{k+r_k} - x^k\|. \tag{70}$$

Let $j_k \in I_k$ be such that $\|P_{C_{j_k}}x^k - x^k\| = \max_{j\in I_k} \|P_{C_j}x^k - x^k\|$, $k \geq 0$. Let $\lim_{k\to\infty} w_{j_k}^k \|P_{C_{j_k}}x^k - x^k\|^2 = 0$. Since $w_{j_k}^k \geq \delta$ for $j_k \in I(x^k)$ we have $\lim_{k\to\infty} \|P_{C_{j_k}}x^k - x^k\| = 0$. Since $i \in I_{k+r_k}$ we have

$$\|P_{C_i}x^{k+r_k} - x^{k+r_k}\| \leq \|P_{C_{j_{k+r_k}}}x^{k+r_k} - x^{k+r_k}\|.$$

consequently, $\lim_{k\to\infty} \|P_{C_i}x^{k+r_k} - x^{k+r_k}\| = 0$. The inequalities (69) and (70) yield now $\lim_{k\to\infty} \|P_{C_i}x^k - x^k\| = 0$, i.e., $\{w^k\}_{k=0}^{\infty}$ is approximately regular.

(e) Let $\mathcal{H} = \mathbb{R}^n$, let $U_i : \mathcal{H} \to \mathcal{H}$, $i \in I$, be cutters having a common fixed point and let $\liminf_{k\to\infty} \lambda_k(2 - \lambda_k) > 0$. Consider a sequence generated by the recurrence (64) with a *repetitive control* $\{i_k\}_{k=0}^{\infty} \subset I$, i.e., a control for which the subset $K_i = \{k \in \mathbb{N} \mid i_k = i\}$ is infinite for any $i \in I$ (see., e.g., [ABC83, Section 3]). It is clear that $\mathbb{N}_0 = K_1 \cup K_2 \cup \ldots \cup K_m$ and that $K_i \cap J_j = \emptyset$ for all $i, j \in I, i \neq j$. The control $\{i_k\}_{k=0}^{\infty}$ is approximately semi-regular. This follows from inequality (25) which guarantees that

$$\sum_{k=0}^{\infty} \lambda_k(2 - \lambda_k)\|U_{i_k}x^k - x^k\|^2 < \infty. \tag{71}$$

Note that the series above is absolutely convergent, thus,

$$\sum_{i=1}^{m}\sum_{k\in K_i}\lambda_k(2-\lambda_k)\|U_i x^k - x^k\|^2 = \sum_{k=0}^{\infty}\lambda_k(2-\lambda_k)\|U_{i_k} x^k - x^k\|^2 < \infty. \tag{72}$$

Therefore,

$$\sum_{k\in K_i}\lambda_k(2-\lambda_k)\|U_i x^k - x^k\|^2 < \infty \text{ for all } i \in I \tag{73}$$

and

$$\lim_{k\to\infty, k\in K_i}\lambda_k(2-\lambda_k)\|U_i x^k - x^k\|^2 = 0 \text{ for all } i \in I. \tag{74}$$

Since $\liminf_{k\to\infty}\lambda_k(2-\lambda_k) > 0$, we have $\lim_{k\to\infty,k\in K_i}\|U_i x^k - x^k\| = 0$ for all $i \in I$, consequently,

$$\liminf_{k\to\infty}\|U_i x^k - x^k\| = 0 \tag{75}$$

for all $i \in I$, and $\{i_k\}_{k=0}^{\infty}$ is approximately semi-regular. One can prove that the approximate semi-regularity also holds for sequences generated by (44), where $J_k = I$ and $\mathcal{V} = \mathcal{U}$ and the sequence of weight functions $\{w^k\}_{k=0}^{\infty}$ has the property $\sum_{i\in I_k}\omega_i^k = 1$ for $I_k \subset I$, $k \geq 0$ and $w_i^k > \delta > 0$ for $i \in I_k$, and $i \in I_k$ for infinitely many k, $i \in I$. Note that a repetitive control is a special case of a sequence $\{w^k\}_{k=0}^{\infty}$ having the above property.

Theorem 27 *Suppose that:*

- $U_i : \mathcal{H} \to \mathcal{H}$, $i \in I$, *are cutters having a common fixed point,*
- $U_i - \mathrm{Id}$ *are demiclosed at* 0, $i \in I$,
- $\mathcal{V}^k = \{V_i^k\}_{i\in J_k}$ *are families of cutters* $V_i^k : \mathcal{H} \to \mathcal{H}$, $i \in J_k$, *with the property* $\bigcap_{i\in J_k}\operatorname{Fix} V_i^k \supset \bigcap_{i\in I}\operatorname{Fix} U_i$, $k \geq 0$,
- $\{w^k\}_{k=0}^{\infty} : \mathcal{H} \to \Delta_{|J_k|}$ *is a sequence of appropriate weight functions,*
- $\liminf_{k\to\infty}\lambda_k(2-\lambda_k) > 0$,

- $\{x^k\}_{k=0}^\infty$ *is generated by the recurrence* (44).

If the sequence of weight functions $\{w^k\}_{k=0}^\infty$ *applied to the sequence of families* $\mathcal{V}^k$*:*

(i) *is approximately regular with respect to the family* $\mathcal{U} = \{U_i\}_{i\in I}$ *then* $\{x^k\}_{k=0}^\infty$ *converges weakly to a common fixed point of* U_i*,* $i \in I$*;*

(ii) *is approximately semi-regular with respect to the family* $\mathcal{U} = \{U_i\}_{i\in I}$ *and* $\mathcal{H}$ *is finite-dimensional, then* $\{x^k\}_{k=0}^\infty$ *converges to a common fixed point of* U_i*,* $i \in I$*.*

Proof. Let $V^k : \mathcal{H} \to \mathcal{H}$ be defined by

$$V^k x = \sum_{i\in J_k} w_i^k(x) V_i^k x \tag{76}$$

and let T_k be the λ_k-relaxation of the operator V^k, i.e.,

$$T_k x = V_{\lambda_k}^k x = x + \lambda_k(V^k x - x). \tag{77}$$

The operators V^k are cutters,

$$\operatorname{Fix} T_k = \operatorname{Fix} V^k = \bigcap_{i\in J_k} \operatorname{Fix} V_i^k \supset \bigcap_{i\in I} \operatorname{Fix} U_i$$

and T_k are strongly quasi-nonexpansive, $k \geq 0$, (see Lemma 22), consequently, $\bigcap_{k=0}^\infty \operatorname{Fix} T_k \supset \bigcap_{i\in I} \operatorname{Fix} U_i$. Let $\varepsilon > 0$ be such that $\liminf_{k\to\infty} \lambda_k \geq \varepsilon$ and $\liminf_{k\to\infty}(2 - \lambda_k) \geq \varepsilon$ and let $z \in \bigcap_{i\in I} \operatorname{Fix} U_i$. For sufficiently large k we have $2 - \lambda_k \geq \varepsilon/2$ and $\dfrac{2-\lambda_k}{\lambda_k} \geq \varepsilon/4$. Now, it follows from Lemma 22 that, for sufficiently large k,

$$\begin{aligned} \|x^{k+1} - z\|^2 &= \|T_k x^k - z\|^2 \\ &\leq \|x^k - z\|^2 - \lambda_k(2-\lambda_k) \sum_{i\in J_k} w_i^k(x^k) \|V_i^k x^k - x^k\|^2 \\ &\leq \|x^k - z\|^2 - \lambda_k(2-\lambda_k) \|V^k x^k - x^k\|^2 \\ &= \|x^k - z\|^2 - \frac{2-\lambda_k}{\lambda_k} \|T_k x^k - x^k\|^2 \\ &\leq \|x^k - z\|^2 - \frac{\varepsilon}{4} \|T_k x^k - x^k\|^2. \end{aligned} \tag{78}$$

Therefore, $\{\|x^k - z\|\}_{k=0}^{\infty}$ decreases and $\sum_{i \in J_k} w_i^k(x^k)\|V_i^k x^k - x^k\|^2 \to 0$. Consequently,

$$w_{i_k}^k(x^k)\|V_{i_k}^k x^k - x^k\|^2 \to 0 \tag{79}$$

for arbitrary $i_k \in J_k$.

(i) Suppose that $\{w^k\}_{k=0}^{\infty}$ is approximately regular with respect to the family $\mathcal{U} = \{U_i\}_{i \in I}$. Let $i_k \in J_k$, $k \geq 0$, be such that the implication (62) holds. Then (79) yields $\lim_{k\to\infty} \|U_i x^k - x^k\| = 0$ for all $i \in I$. Let x^* be a weak cluster point of $\{x^k\}_{k=0}^{\infty}$ and $\{x^{n_k}\}_{k=0}^{\infty}$ be a subsequence of $\{x^k\}_{k=0}^{\infty}$ such that $x^{n_k} \rightharpoonup x^*$ as $k \to \infty$. The demiclosedness of $U_i - \mathrm{Id}$ at 0, $i \in I$, yields that $x^* \in \bigcap_{i \in I} \operatorname{Fix} U_i$. Since x^* is an arbitrary weak cluster point of $\{x^k\}_{k=0}^{\infty}$ and $\{x^k\}_{k=0}^{\infty}$ is Fejér-monotone with respect to $\bigcap_{i \in I} \operatorname{Fix} U_i$, the weak convergence of the whole sequence $\{x^k\}_{k=0}^{\infty}$ to x^* follows from [Bro67, Lemma 6] (see also [BB96, Theorem 2.16 (ii)]).

(ii) Suppose that $\mathcal{H}$ is finite-dimensional and $\{w^k\}_{k=0}^{\infty}$ is approximately semi-regular with respect to the family $\mathcal{U} = \{U_i\}_{i \in I}$. Let $i_k \in J_k$, $k \geq 0$, be such that the implication (63) holds. Then (79) yields $\liminf_{k\to\infty} \|U_i x^k - x^k\| = 0$ for all $i \in I$. Consequently, $\lim_{k\to\infty} \|U_i x^{n_k} - x^{n_k}\| = 0$ for a subsequence $\{x^{n_k}\}_{k=0}^{\infty} \subset \{x^k\}_{k=0}^{\infty}$, $i \in I$. Since $\{x^{n_k}\}_{k=0}^{\infty}$ is bounded, a subsequence $\{x^{m_{n_k}}\}_{k=0}^{\infty} \subset \{x^{n_k}\}_{k=0}^{\infty}$ exists which converges to a point $x^* \in X$. Since $U_i - \mathrm{Id}$ is closed at 0, $i \in I$, we have $x^* \in \bigcap_{i \in I} \operatorname{Fix} U_i$. The convergence of the whole sequence $\{x^k\}_{k=0}^{\infty}$ to x^* follows now from [BB96, Theorem 2.16 (v)]. ∎

Remark 28 Bauschke and Borwein [BB96, Section 3, page 378] consider algorithms which are similar to (44) where $J_k = I$, $\lambda_k = 1$, V_i^k is replaced by a firmly nonexpansive operator U_i^k with $\operatorname{Fix} U_i^k \supset \operatorname{Fix} U_i$, $i \in I$, $k \geq 0$, and $\bigcap_{i \in I} \operatorname{Fix} U_i \neq \emptyset$. They assumed that these algorithms are focusing, strongly focusing or linearly focusing (see [BB96, Definitions 3.7 and 4.8]). These assumptions differ from the assumptions on the regularity, approximate regularity or approximate semi-regularity, but they play a similar role in the proof of convergence of sequences generated by the considered algorithms. The recurrence considered by Bauschke and Borwein has the form

$$x^{k+1} = \sum_{i \in I} v_i^k \left(x^k + \mu_i^k (U_i^k x^k - x^k)\right), \tag{80}$$

where $\{\mu_i^k\}_{k=0}^{\infty} \subset [0, 2]$ are sequences of relaxation parameters, $i \in I$, and $\{v^k\}_{k=0}^{\infty} \subset \Delta_m$ is a sequence of weight vectors (see [BB96, page 378]). Note that (80) can be written in the form

$$x^{k+1} = x^k + \lambda_k(\sum_{i\in I} w_i^k U_i^k x^k - x^k), \tag{81}$$

where $\lambda_k = \sum_{i\in I} \mu_i^k v_i^k$ and $w_i^k = \mu_i^k v_i^k/\lambda_k$. This transformation maintains the assumption $\liminf_{k\to\infty} \mu_i^k(2-\mu_i^k) > 0$, $i \in I$, i.e., if the sequences $\{\mu_i^k\}_{k=0}^\infty$, $i \in I$, satisfies this assumption then $\liminf_{k\to\infty} \lambda_k(2-\lambda_k) > 0$. Furthermore, if the sequence of weight vectors $\{v^k\}_{k=0}^\infty$ applied to the recurrence (80) is regular (approximately regular, approximately semi-regular) and $\liminf_{k\to\infty} \mu_i^k(2-\mu_i^k) > 0$, $i \in I$, then the sequence of weight vectors $\{w^k\}_{k=0}^\infty$ applied to the recurrence (81) is regular (approximately regular, approximately semi-regular). Bauschke and Borwein proved the weak convergence of sequences $\{x^k\}_{k=0}^\infty$ generated by (80) to a point $x \in \bigcap_{i\in I} \operatorname{Fix} U_i$ under the assumptions that (i) the algorithm is focusing and intermittent and (ii) that $\liminf_{k\to\infty, v_i^k>0} v_i^k > 0$ for all $i \in I$ (see [BB96, Theorem 3.20]). Assumption (ii) applied to sequences generated by (80) is equivalent to the following assumption (ii)' $\liminf_{k\to\infty, w_i^k>0} w_i^k > 0$ applied to sequences generated by (81). Note, however, that assumptions (i) as well as (ii)' do not appear in Theorem 27. Assumptions similar to those in [BB96, Theorem 3.20] can be also found in [Com97a, equalities (15)-(17)].

In the following examples we suppose that $C_i \subset \mathcal{H}$, $i \in I$, are closed and convex and that $C = \bigcap_{i\in I} \neq \emptyset$.

Example 29 Consider the recurrence (44), where $J_k = I$ for all $k \geq 0$, $V_i^k = P_{C_i}$, $i \in I$, $\lambda_k = 1$, $k \geq 0$, the sequence of weight functions $\{w^k\}_{k=0}^\infty$ is constant, $w^k = w$, $k \geq 0$, and $w : \mathbb{R}^n \to \Delta_m$ has the form

$$w_i(x) = \begin{cases} \dfrac{v_i}{\sum\limits_{j\in I(x)} v_j}, & \text{for } i \in I(x), \\ 0, & \text{for } i \notin I(x), \end{cases} \tag{82}$$

where $v = (v_1, v_2, ..., v_m) \in \operatorname{ri}\Delta_m$ and $I(x) = \{i \in I \mid x \notin C_i\}$. Since w is regular (see Example 24 (b)), it is approximately regular and it follows from Theorem 27 (i) that $x^k \rightharpoonup x^* \in C$. This convergence was proved by Iusem and De Pierro [IDP86, Corollary 4] for $\mathcal{H} = \mathbb{R}^n$. Note, however, that in finite-dimensional case the convergence holds for any sequence $\{w^k\}_{k=0}^\infty$ containing a subsequence of β-regular weight functions, where $\beta > 0$, e.g., if $w^k = w$ for infinitely many $k \geq 0$.

Example 30 Aharoni and Censor [AC89, Theorem 1] consider the recurrence (44), where $\mathcal{H} = \mathbb{R}^n$, $J_k = I$ for all $k \geq 0$, $V_i^k = P_{C_i}$, $i \in I$, $\lambda_k \in [\varepsilon, 2-\varepsilon]$, where $\varepsilon \in (0,1)$, $w^k \in \Delta_m$ with $\sum_{k=0}^{\infty} w_i^k = +\infty$, $i \in I$. By Lemma 22, for any $z \in C$ we have

$$\|x^{k+1} - z\|^2 \leq \|x^0 - z\|^2 - \sum_{l=0}^{k} \lambda_l(2-\lambda_l) \sum_{i=1}^{m} w_i^l \|P_{C_i}x^l - x^l\|^2. \tag{83}$$

Consequently,

$$\begin{aligned} &\sum_{i=1}^{m}\sum_{k=0}^{\infty} \lambda_k(2-\lambda_k) w_i^k(x^k)\|P_{C_i}x^k - x^k\|^2 \\ &= \sum_{k=0}^{\infty} \lambda_k(2-\lambda_k) \sum_{i=1}^{m} w_i^k \|P_{C_i}x^k - x^k\|^2 < +\infty \end{aligned} \tag{84}$$

and

$$\sum_{k=0}^{\infty} \lambda_k(2-\lambda_k) w_i^k \|P_{C_i}x^k - x^k\|^2 < +\infty \tag{85}$$

for any $i \in I$. The assumption $\liminf_{k\to\infty} \lambda_k(2-\lambda_k) > 0$ yields

$$\sum_{k=0}^{\infty} w_i^k \|P_{C_i}x^k - x^k\|^2 < +\infty, \tag{86}$$

$i \in I$. Since $\sum_{k=0}^{\infty} w_i^k = +\infty$, we have $\liminf_{k\to\infty} \|P_{C_i}x^k - x^k\| = 0$, $i \in I$, i.e., w^k is approximately semi-regular. Theorem 27 (ii) yields now the convergence $x^k \to x^* \in C$.

Example 31 Butnariu and Censor [BC90, Theorem 4.4] consider the recurrence (44), where $\mathcal{H} = \mathbb{R}^n$, $J_k = I$, $V_i = P_{C_i}$, $i \in I$, $\liminf_{k\to\infty} \lambda_k > 0$, $\limsup_{k\to\infty} \lambda_k < 2$, $w^k \in \Delta_m$ has a subsequence converging to a point $w^* \in \operatorname{ri} \Delta_m$. Let $\varepsilon > 0$ be such that $w_i^* > \varepsilon$ for all $i \in I$. Then there exists a subsequence $\{w^{n_k}\}_{k=0}^{\infty} \subset \{w^k\}_{k=0}^{\infty}$ such that $w_i^{n_k} > \varepsilon/2$ for all $i \in I$ and $k \in \mathbb{N}$, consequently, $\{w^{n_k}\}_{k=0}^{\infty}$ is $\frac{\varepsilon}{2}$-regular. Therefore, $\{w^k\}_{k=0}^{\infty}$ is approximately semi-regular. Theorem 27 (ii) yields now $\lim_{k\to\infty} x^k = x^* \in C$. If we suppose that all cluster points of $\{w^k\}_{k=0}^{\infty}$ belong to $\operatorname{ri} \Delta_m$ then $\{w^k\}_{k=0}^{\infty}$ is approximately regular, consequently the weak convergence $x^k \rightharpoonup x^*$ holds in general Hilbert spaces.

Example 32 Consider the recurrence (44), where $J_k = I$ for all $k \geq 0$, $\liminf_{k\to\infty} \lambda_k(2-\lambda_k) > 0$, $U_i = P_{C_i}$ for closed and convex subsets $C_i \subset \mathcal{H}$, $i \in I$, with $C = \bigcap_{i\in I} C_i \neq \emptyset$ and V_i^k are cutters satisfying the inequality

$$\|V_i^k x^k - x^k\| \geq \alpha \|P_{C_i} x^k - x^k\|, \tag{87}$$

$i \in I$, for some $\alpha > 0$ and such that $C \subset \bigcap_{i\in I} \operatorname{Fix} V_i^k$, $k \geq 0$. Furthermore, suppose that the sequence of weight vectors w^k satisfies the following conditions:

(i) $\limsup_{k\to\infty} w_i^k > 0$, $i \in I$,

(ii) $w_i^k \|P_{C_i} x^k - x^k\| \neq 0$ implies $w_i^k > \delta > 0$.

If we set $U_i = P_{C_i}$, $i \in I$, then (87) and (i)-(ii) guarantee that the sequence of weights $\{w^k\}_{k=0}^{\infty}$ is regular and thus all assumptions of Theorem 27 (i) are satisfied. Therefore, $x^k \rightharpoonup x^* \in C$. This convergence was proved by Flåm and Zowe [FZ90, Theorem 1] in case $\mathcal{H} = \mathbb{R}^n$. Actually, they have considered a recurrence which can be reduced to (44). We omit the details.

Results similar to Theorem 27 also hold for sequences generated by extrapolated simultaneous cutters. Before formulating our next theorem, we prove some auxiliary results. The following lemma is an extension of Lemma 22. A part of this lemma can be found in [Com01, Proposition 2.4], where w is a constant weight function with positive coordinates.

Lemma 33 *Let $V_i : \mathcal{H} \to \mathcal{H}$ be cutters having a common fixed point, $i \in J = \{1, 2, ..., l\}$, let $w : \mathcal{H} \to \Delta_l$ be an appropriate weight function and let $\sigma : \mathcal{H} \to (0, +\infty)$ be a step-size function defined by*

$$\sigma(x) = \begin{cases} \dfrac{\sum_{i=1}^{l} w_i(x)\|V_i x - x\|^2}{\|\sum_{i=1}^{l} w_i(x) V_i x - x\|^2}, & \text{if } x \notin \bigcap_{i\in J} \operatorname{Fix} V_i, \\ 1, & \text{otherwise,} \end{cases} \tag{88}$$

and let $V_\sigma := \operatorname{Id} + \sigma(\sum_{i=1}^{l} w_i V_i - \operatorname{Id})$ be a generalized relaxation of the simultaneous cutter $V = \sum_{i=1}^{l} w_i V_i$. Then $\operatorname{Fix} V_\sigma = \bigcap_{i\in J} \operatorname{Fix} V_i$, the operator V_σ is *a cutter and V_σ is an extrapolation of V. Consequently, for all $\lambda \in (0,2)$, the operator $V_{\sigma,\lambda}$ is $\frac{2-\lambda}{\lambda}$-strongly quasi-nonexpansive and*

$$\|V_{\sigma,\lambda}x - z\|^2 \le \|x - z\|^2 - \lambda(2-\lambda)\sigma^2(x)\|Vx - x\|^2 \tag{89}$$

for all $\lambda \in [0,2]$, $x \in \mathcal{H}$ and $z \in \operatorname{Fix} V$.

Proof. Lemma 22 (i) and the positivity of the step-size function σ yield $\operatorname{Fix} V_\sigma = \operatorname{Fix} V = \bigcap_{i\in J} \operatorname{Fix} V_i$. Let $x \in \mathcal{H}$ and $z \in \operatorname{Fix} V_\sigma$. We prove that

$$\langle z - x, V_\sigma x - x\rangle \ge \|V_\sigma x - x\|^2, \tag{90}$$

which is equivalent with V_σ being a cutter; see Lemma 4 (i). The inequality is clear for $x \in \operatorname{Fix} V_\sigma$. For $x \notin \operatorname{Fix} V_\sigma$ we have

$$\begin{aligned}\langle z - x, Vx - x\rangle &= \langle z - x, \sum_{i\in I} w_i(x)(V_i x - x)\rangle \\ &= \sum_{i\in I} w_i(x)\langle z - x, V_i x - x\rangle \\ &\ge \sum_{i\in I} w_i(x)\|V_i x - x\|^2 \\ &= \sigma(x)\|Vx - x\|^2,\end{aligned} \tag{91}$$

thus,

$$\langle z - x, Vx - x\rangle \ge \sigma(x)\|Vx - x\|^2, \tag{92}$$

which is equivalent to (90). By the convexity of the function $\|\cdot\|^2$ we have $\sigma(x) \ge 1$, i.e., V_σ is an extrapolation of V. Lemma 4 (ii) and the fact $V_{\sigma,\lambda} = (V_\sigma)_\lambda$ yield now the $\frac{2-\lambda}{\lambda}$-strong quasi-nonexpansivity of $V_{\sigma,\lambda}$. Inequality (89) follows from the equality $V_{\sigma,\lambda}x - x = \lambda\sigma(x)(Vx - x)$. ■

For a family of cutters $\mathcal{V} = \{V_i\}_{i\in J}$ and for an appropriate weight function $w : \mathcal{H} \to \Delta_{|J|}$ denote

$$\sigma_w(x) = \frac{\sum_{i\in J} w_i(x)\|V_i x - x\|^2}{\|\sum_{i\in J} w_i(x)V_i x - x\|^2}, \tag{93}$$

where $x \notin \bigcap_{i\in J} \operatorname{Fix} V_i$. By Lemma 22, $\bigcap_{i\in J} \operatorname{Fix} V_i = \operatorname{Fix} V$, where $V = \sum_{i\in J} w_i V_i$ and $\sigma_w(x)$ is well-defined.

Definition 34 Let $V_i : \mathcal{H} \to \mathcal{H}$, $i \in J$, be cutters with a common fixed point and let $w : \mathcal{H} \to \Delta_{|J|}$ be a weight function which is appropriate with respect to the family $\mathcal{V} = \{V_i\}_{i\in J}$. We say that the step-size function $\sigma : \mathcal{H} \to (0, +\infty)$ is *α-admissible*, with respect to the family $\mathcal{V}$, where $\alpha \in (0, 1]$, or, shortly, *admissible*, if

$$\alpha\sigma_w(x) \leq \sigma(x) \leq \sigma_w(x) \tag{94}$$

for all $x \notin \bigcap_{i\in J} \operatorname{Fix} V_i$.

Theorem 35 *Suppose that:*

- $U_i : \mathcal{H} \to \mathcal{H}$, $i \in I$, *are cutters having a common fixed point,*
- $U_i - \mathrm{Id}$, $i \in I$, *are demiclosed at* 0,
- $\mathcal{V}^k = \{V_i^k\}_{i\in J_k}$ *are families of cutters* $V_i^k : \mathcal{H} \to \mathcal{H}$, $i \in J_k$, *with the properties* $\bigcap_{i\in J_k} \operatorname{Fix} V_i^k \supset \bigcap_{i\in I} \operatorname{Fix} U_i$, *and* $\max_{i\in J_k} \|V_i^k x - x\| \leq \gamma \max_{i\in I} \|U_i x - x\|$ *for all* $x \in \mathcal{H}$, $k \geq 0$, *and for some constant* $\gamma > 0$,
- $\{w^k\}_{k=0}^{\infty} : \mathcal{H} \to \Delta_{|J_k|}$ *is a sequence of appropriate weight functions,*
- *the step-size* $\sigma_k : \mathcal{H} \to (0, +\infty)$ *is α-admissible with respect to* $\mathcal{V}^k$, $k \geq 0$, *for some* $\alpha \in (0, 1]$,
- $\liminf_{k\to\infty} \lambda_k(2 - \lambda_k) > 0$,
- $\{x^k\}_{k=0}^{\infty}$ *is generated by the recurrence* (43).

If the sequence of weight functions $\{w^k\}_{k=0}^{\infty}$ *applied to the sequence of families* $\mathcal{V}^k$*:*

(i) *is regular with respect to the family* $\mathcal{U} = \{U_i\}_{i\in I}$ *then* $\{x^k\}_{k=0}^{\infty}$ *converges weakly to a common fixed point of* U_i, $i \in I$*;*

(ii) *contains a subsequence which is regular with respect to the family* $\mathcal{U} = \{U_i\}_{i\in I}$ *and* $\mathcal{H}$ *is finite-dimensional, then* $\{x^k\}_{k=0}^{\infty}$ *converges to a common fixed point of* U_i, $i \in I$.

Proof. Let $V^k : \mathcal{H} \to \mathcal{H}$ be defined by

$$V^k x = \sum_{i \in J_k} w_i^k(x) V_i^k x \tag{95}$$

and let T_k be a generalized relaxation of the operator V^k, i.e.,

$$T_k x = V^k_{\sigma_k, \lambda_k} x = x + \lambda_k \sigma_k(x)(V^k x - x). \tag{96}$$

The operators V^k are cutters and $\operatorname{Fix} T_k = \operatorname{Fix} V^k = \bigcap_{i \in J_k} \operatorname{Fix} V_i^k$ (see Lemma 22). Consequently, $\bigcap_{k=0}^{\infty} \operatorname{Fix} T_k \supset \bigcap_{i \in I} \operatorname{Fix} U_i$. Let $\varepsilon > 0$ and $k_0 \in \mathbb{N}$ be such that $\lambda_k \in [\varepsilon, 2 - \varepsilon]$ for $k \geq k_0$. By Lemma 33 the operator $V^k_{\sigma_{w_k}}$ is a cutter. Now, the second inequality in (94) and (5) which remains true also for $\lambda : \mathcal{H} \to [0, 1]$ yield that $V^k_{\sigma_k}$ is a cutter, consequently T_k is a λ_k-relaxed cutter, $k \geq 0$. Lemma 33 also implies that

$$\|x^{k+1} - z\|^2 \leq \|x^k - z\|^2 - \frac{2 - \lambda_k}{\lambda_k} \|T_k x^k - x^k\|^2 \tag{97}$$

for all $z \in \bigcap_{i=1}^{m} \operatorname{Fix} U_i$. Therefore, $\{x^k\}_{k=0}^{\infty}$ is bounded, $\{\|x^k - z\|\}_{k=0}^{\infty}$ is monotone and $\lim_{k \to \infty} \|T_k x^k - x^k\| = 0$.

(i) Let $\beta \in (0, 1]$, $k_1 \geq k_0$ and $j_k \in J_k$ be such that

$$w_{j_k}(x) \|V^k_{j_k} x - x\|^2 \geq \beta \max_{i \in I} \|U_i x - x\|^2 \tag{98}$$

for any $x \in \mathcal{H}$ and for $k \geq k_1$. Since σ_k is α-admissible, the norm is a convex function and $\|V_j^k x^k - x^k\| \leq \gamma \max_i \|U_i x^k - x^k\|$ for all $j \in J_k$, we have

$$\begin{aligned}
\|T_k x^k - x^k\| &= \lambda_k \sigma_k(x^k) \|V^k x^k - x^k\| \\
&\geq \lambda_k \alpha \frac{\sum_{i \in J_k} w_i^k(x^k) \|V_i^k x^k - x^k\|^2}{\|\sum_{i \in J_k} w_i^k(x^k) V_i^k x^k - x^k\|} \\
&\geq \lambda_k \alpha \frac{w_{j_k}^k(x^k) \|V_{j_k}^k x^k - x^k\|^2}{\sum_{i \in J_k} w_i^k(x^k) \|V_i^k x^k - x^k\|} \\
&\geq \frac{\lambda_k \alpha}{\gamma} \frac{\beta \max_{i \in I} \|U_i x^k - x^k\|^2}{(\sum_{i \in J_k} w_i^k(x^k)) \max_{i \in I} \|U_i x^k - x^k\|} \\
&= \frac{\varepsilon \alpha \beta}{\gamma} \max_{i \in I} \|U_i x^k - x^k\|,
\end{aligned} \tag{99}$$

and $\lim_{k\to\infty} \|U_i x^k - x^k\| = 0$ for all $i \in I$. Therefore, condition (16) is satisfied for $U_k = T_k$ and $S = U_i$, $i \in I$. We have proved that all assumptions of Theorem 9 (i) are satisfied for $S = U_i$, $i \in I$. Therefore, $\{x^k\}_{k=0}^\infty$ converges weakly to a common fixed point of U_i, $i \in I$.

(ii) Suppose that $\mathcal{H}$ is finite-dimensional and $\{w^k\}_{k=0}^\infty$ contains an approximately β-regular subsequence $\{w^{n_k}\}_{k=0}^\infty$. Let $\beta \in (0,1]$, $k_1 \geq k_0$ and $j_{n_k} \in I$ be such that

$$v_{j_{n_k}}(x)\|V_{j_{n_k}}^{n_k} x - x\|^2 \geq \beta \max_{i\in I} \|U_i x - x\|^2. \tag{100}$$

Similarly to (i), one can prove that

$$\|T_{n_k} x^{n_k} - x^{n_k}\| \geq \varepsilon\alpha\beta \max_i \|U_i x^{n_k} - x^{n_k}\|. \tag{101}$$

Therefore, $\liminf_{k\to\infty} \|U_i x^k - x^k\| = 0$ for all $i \in I$. If we set $U_k = T_k$ and $S = U_i$, $i \in I$, in Theorem 9 (ii), we obtain the weak convergence of $\{x^k\}_{k=0}^\infty$ to a fixed point of U_i for all $i \in I$. ■

Remark 36 Combettes considers an algorithm which is similar to (43) with $J_k = I$, $w^k = w \in \operatorname{ri}\Delta_m$, $V_i^k = P_{C_i^k}$, where $C_i^k \supset C_i$ are closed and convex, $i \in I$, $k \geq 0$, and with a constant sequence of step-size functions $\sigma_k = \sigma_w$ given by

$$\sigma_w(x) = \frac{\sum_{i\in I} w_i \|P_{C_i} x - x\|^2}{\|\sum_{i\in I} w_i (P_{C_i} x - x)\|^2} \tag{102}$$

for $x \notin C = \bigcap_{i\in I} C_i$ (see [Com97a, equations (33)-(36)]). He proves there weak convergence of sequences generated by this algorithm to a point $x \in C$ under the assumption that the algorithm is focusing (see [Com97a, Theorem 2]). However, the assumption $w \in \operatorname{ri}\Delta_m$ is a special case of a regular sequence of weight functions and the step-size function σ_w, given by (102), is a special case of a sequence of α-admissible step-sizes, which are considered in Theorem 35.

Remark 37 Results closely related to Theorems 27 (ii) and 35 (ii) appear in Kiwiel [Kiw95, Theorem 5.1], for the case $\mathcal{H} = \mathbb{R}^n$. Kiwiel applies some assumptions on weights and on the operators [Kiw95, Assumption 3.10] which differ from the assumptions in Theorems 27 (ii) and 35 on the approximate semi-regularity. Our Theorems 27 and 35 show the importance of the regularity, approximate regularity and the approximate semi-regularity in both the finite- and the infinite-dimensional cases.

Example 38 Dos Santos' [DS87, Section 5] work is related to ours as follows. Let $c_i : \mathcal{H} \to \mathbb{R}$ be continuous and convex, let $C_i = \{x \in \mathcal{H} \mid c_i(x) \leq 0\}$, $i \in I$ and let $C = \bigcap_{i=1}^m C_i \neq \emptyset$. Define $U_i : \mathcal{H} \to \mathcal{H}$ by

$$U_i x = \begin{cases} x - \dfrac{(c_i(x))_+}{\|g_i(x)\|^2} g_i(x), & \text{if } g_i(x) \neq 0, \\ x, & \text{if } g_i(x) = 0, \end{cases} \tag{103}$$

where a_+ denotes a nonnegative part of a real number a, i.e., $a_+ = \max\{0, a\}$, $g_i(x) \in \partial c_i(x) := \{g \in \mathcal{H} \mid \langle g, y - x\rangle \leq c_i(y) - c_i(x), \text{ for all } y \in \mathcal{H}\}$ is a subgradient of the function c_i at the point x, $i \in I$. This operator U_i is called the *subgradient projection* onto C_i, $i \in I$. It follows from the definition of the subgradient that U_i is a cutter. Note that $\operatorname{Fix} U_i = C_i$, and thus $\bigcap_{i=1}^m \operatorname{Fix} U_i \neq \emptyset$. Furthermore, the operator $U_i - \mathrm{Id}$ is demiclosed at 0, $i \in I$. Indeed, let $x^k \rightharpoonup x^*$ and $\lim_{k\to\infty} \|U_i x^k - x^k\| = 0$. Then we have

$$\lim_{k\to\infty} \|U_i x^k - x^k\| = \lim_{k\to\infty} \frac{(c_i(x^k))_+}{\|g_i(x^k)\|} = 0. \tag{104}$$

The sequence $\{x^k\}_{k=0}^\infty$ is bounded due to its weak convergence. Since a continuous convex function is locally Lipschitz-continuous, the subgradients $\{g_i(x^k)\}_{k=0}^\infty$ are bounded. Condition (104) implies now the convergence $\lim_{k\to\infty} c_i(x^k)_+ = 0$. Since c_i is weakly lower semi-continuous, we have $c_i(x^*) = 0$, i.e., $U_i - \mathrm{Id}$ is demiclosed at 0. Consider an extrapolated simultaneous subgradient projection method, i.e., a method which generates sequences $\{x^k\}_{k=0}^\infty$ defined by the recurrence (43) where $V_i^k = U_i$, w^k is a sequence of appropriate weight functions, $\liminf_{k\to\infty} \lambda_k(2 - \lambda_k) > 0$ and $\sigma_k : \mathcal{H} \to (0, +\infty)$ is a sequence of step-size functions defined by

$$\sigma_k(x) = \frac{\sum_{i=1}^m w_i^k(x) (\frac{(c_i(x))_+}{\|g_i(x)\|})^2}{\| \sum_{i=1}^m w_i^k(x) \frac{(c_i(x))_+}{\|g_i(x)\|^2} g_i(x)\|^2}. \tag{105}$$

Note that

$$U_i x - x = -\frac{(c_i(x))_+}{\|g_i(x)\|^2} g_i(x), \tag{106}$$

and so,

$$\sigma_k(x) = \sigma_w(x) = \frac{\sum_{i\in J} w_i^k(x) \|U_i x - x\|^2}{\| \sum_{i\in J} w_i^k(x) U_i x - x\|^2}, \tag{107}$$

and σ_k are 1-admissible. If we suppose that the sequence of weight functions $\{w^k\}_{k=0}^{\infty}$ is regular then, by Theorem 35 (i) the sequence $\{x^k\}_{k=0}^{\infty}$ converges weakly to a point $x^* \in C$. Dos Santos [DS87] considers positive constant weights $w \in \operatorname{ri} \Delta_m$ and proves the convergence in the finite-dimensional case.

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