Covariant Representations of Subproduct Systems

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Technion

April 8, 2010

Definition

Let $\mathscr{M}$ be a C^*-algebra. A *Hilbert C^*-module* over $\mathscr{M}$ is a linear space, which is a right $\mathscr{M}$-module E with a function $\langle \cdot, \cdot \rangle : E \times E \to \mathscr{M}$ (called a rigging), satisfying

1. $\langle \zeta, \zeta \rangle \geq 0$, and equality holds iff $\zeta = 0$
2. $\langle \zeta, \cdot \rangle$ is linear and $\langle \zeta, \eta a \rangle = \langle \zeta, \eta \rangle \, a$
3. $\langle \zeta, \eta \rangle^* = \langle \eta, \zeta \rangle$

that is complete with respect to the norm $\|\zeta\| := \left\| \langle \zeta, \zeta \rangle^{1/2} \right\|_{\mathscr{M}}$.

Examples

1. $\mathscr{M} := \mathbb{C}$ and $E := \mathcal{H}$ is a Hilbert space.
2. $E := \mathscr{M}$, with the rigging $\langle a, b \rangle := a^* b$. Denoted by $\mathscr{M}_{\mathscr{M}}$.
3. X is a locally compact Hausdorff space and $\mathcal{H}$ is a Hilbert space. Take $\mathscr{M} := C_0(X)$ and $E := C_0(X, \mathcal{H})$.

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Direct sums

If $(E_i)_I$ is a family of Hilbert C^*-modules over $\mathscr{M}$, let $\bigoplus_I E_i$ be the Hilbert C^*-module defined to be the set of all $(\zeta_i)_I \in \prod_I E_i$ such that

$$\sum_I \langle \zeta_i, \zeta_i \rangle \text{ converges in } \mathscr{M}.$$

The rigging is defined "as usual":

$$\langle (\zeta_i)_I, (\eta_i)_I \rangle := \sum_I \langle \zeta_i, \eta_i \rangle .$$

Definition (Adjointable operators)

Let E, F be Hilbert C^*-modules over $\mathscr{M}$. We denote by $\mathcal{L}(E, F)$ the Banach space of all *adjointable* operators from E to F; that is, all functions $T : E \to F$ admitting a function $T^* : F \to E$ satisfying

$$(\forall \zeta \in E, \eta \in F) \qquad \langle T\zeta, \eta \rangle_F = \langle \zeta, T^*\eta \rangle_E .$$

Such a function is necessarily a linear operator, an $\mathscr{M}$-module map ($T(\zeta a) = (T\zeta)a$) and bounded with respect to the norms on E, F. The space $\mathcal{L}(E) := \mathcal{L}(E, E)$ is a C^*-algebra.

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Not all bounded module maps are adjointable

Take $\mathscr{M} := C([0,1])$, $\mathscr{J} := \{f \in \mathscr{M} : f(0) = 0\} \trianglelefteq \mathscr{M}$ and $E := \mathscr{M} \oplus \mathscr{J}$. Then $T : E \to E$ defined by $T(f, g) := (g, 0)$ is a bounded module map, but it is not adjointable.

Definition

A Hilbert C^*-module E over $\mathscr{M}$ is a *C^*-correspondence* if it is also a *left* $\mathscr{M}$-module, with multiplication on the left given by adjointable operators.

That is: there exists a *-homomorphism $\varphi : \mathscr{M} \to \mathcal{L}(E)$ such that $a \cdot \zeta$ is defined to be $\varphi(a)\zeta$ for $a \in \mathscr{M}$ and $\zeta \in E$.

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Examples

1. $\mathscr{M} = \mathbb{C}$, $E = \mathcal{H}$ and $\varphi(\alpha)\zeta = \alpha\zeta$.
2. $E = \mathscr{M}$ and φ is an endomorphism of $\mathscr{M}$.

Definition (Interior tensor product)

Suppose that:

1. E, F are Hilbert C^*-modules over $\mathscr{M}, \mathscr{N}$ respectively.
2. $\sigma : \mathscr{M} \to \mathcal{L}(F)$ is a *-homomorphism.

Denote by $E \otimes_{\text{alg}} F$ the algebraic tensor product of E and F balanced by σ, that is: $(\zeta a) \otimes \eta = \zeta \otimes \sigma(a)\eta$. This is an $\mathscr{N}$-module. Give it the rigging

$$\langle \zeta_1 \otimes \eta_1, \zeta_2 \otimes \eta_2 \rangle := \langle \eta_1, \sigma\left(\langle \zeta_1, \zeta_2 \rangle\right) \eta_2 \rangle_F .$$

The interior tensor product of E and F, denoted by $E \otimes_\sigma F$, is the completion of this module. It is a Hilbert C^*-module over $\mathscr{N}$.

Two important examples

1. E, F are both C^*-correspondences over $\mathscr{M}$. Take $\sigma = \varphi_F$ (the implementation of left multiplication in F). Then $E \otimes_{\varphi_F} F$ is a C^*-correspondence over $\mathscr{M}$.
2. E is a Hilbert C^*-module over $\mathscr{M}$, $\mathcal{H}$ is a Hilbert space, and σ is a (perhaps degenerate) C^*-representation of $\mathscr{M}$ on $\mathcal{H}$. Then $E \otimes_\sigma \mathcal{H}$ is a Hilbert space.

Fix a C^*-correspondence E over $\mathscr{M}$.

Definition

A pair (T, σ) is called a *covariant representation* of E on $\mathcal{H}$ if:

1. σ is a nondegenerate C^*-representation of $\mathscr{M}$ on $\mathcal{H}$.
2. $T : E \to B(\mathcal{H})$ is a linear mapping.
3. T is a bimodule map with respect to σ, that is: $T(a\zeta) = \sigma(a)T(\zeta)$, $T(\zeta a) = T(\zeta)\sigma(a)$ for all $\zeta \in E$ and $a \in \mathscr{M}$.

(T, σ) is called *completely contractive* in case T is completely contractive with respect to the structure of the "linking algebra" of $\mathscr{M}$ and E.

(T, σ) is called *isometric* if the following condition holds for all $\zeta, \eta \in E$:

$$T(\zeta)^* T(\eta) = \sigma(\langle \zeta, \eta \rangle)$$

Examples

1. Take $E = \mathscr{M} = \mathbb{C}$. There is a bijection between completely contractive, covariant representations of E on $\mathcal{H}$ and *contractions* in $B(\mathcal{H})$ given by $(T, \sigma) \mapsto T(1)$.
   (T, σ) is isometric $\iff T(1)$ is an isometry.

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   (T, σ) is isometric $\iff$ $T(1)$ is an isometry.
2. Take $\mathscr{M} = \mathbb{C}$ and $E = \mathbb{C}^d$. There is a bijection between completely contractive, covariant representations of E on $\mathcal{H}$ and *row contractions* of length d in $B(\mathcal{H})$ given by $(T, \sigma) \mapsto (T(e_1), \ldots, T(e_d))$.
   (T, σ) is isometric $\iff$ $T(e_1), \ldots, T(e_d)$ are all isometries.

Definition (The Fock space)

$$\mathcal{F}(E) := \bigoplus_{n \in \mathbb{Z}_+} E^{\otimes n} = \mathscr{M} \oplus E \oplus E^{\otimes 2} \oplus \dots$$

Definition

Given $a \in \mathscr{M}$, define the operator $\varphi_\infty(a) \in \mathcal{L}(\mathcal{F}(E))$ of left multiplication by a as follows:

$$\varphi_\infty(a)(\zeta_0 \oplus \zeta_1 \oplus \zeta_2 \oplus \dots) := a\zeta_0 \oplus a\zeta_1 \oplus a\zeta_2 \oplus \dots$$

Given $\zeta \in E$, define the creation (shift) operator $S(\zeta) \in \mathcal{L}(\mathcal{F}(E))$ by "left tensoring" with ζ. That is, for all $n \in \mathbb{Z}_+$ and $\eta \in E^{\otimes n}$,

$$S(\zeta)\eta := \zeta \otimes \eta \in E^{\otimes(n+1)}.$$

The pair (S, φ_∞) is an *isometric* covariant representation of E on $\mathcal{F}(E)$:

1. $\varphi_\infty : \mathscr{M} \to \mathcal{L}(\mathcal{F}(E))$ is a *-homomorphism.
2. $S : E \to \mathcal{L}(\mathcal{F}(E))$ is linear and $S(a\zeta) = \varphi_\infty(a)S(\zeta)$, $S(\zeta a) = S(\zeta)\varphi_\infty(a)$.
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Remark

This is actually not accurate, as $\mathcal{F}(E)$ is not necessarily a Hilbert space. To overcome this “obstacle”, let π denote a *faithful* C^*-representation of $\mathcal{L}(\mathcal{F}(E))$ on some Hilbert space $\mathcal{H}$. Now consider the pair $(\pi \circ S, \pi \circ \varphi_\infty)$ instead of (S, φ_∞).

Definitions

1. The *Toeplitz algebra*, $\mathcal{T}(E)$, is the C^*-subalgebra of $\mathcal{L}(\mathcal{F}(E))$ generated by $\{\varphi_\infty(a) : a \in \mathscr{M}\}$ and $\{S(\zeta) : \zeta \in E\}$.
2. The *tensor algebra*, $\mathcal{T}_+(E)$, is the non-selfadjoint subalgebra of $\mathcal{L}(\mathcal{F}(E))$ generated by the same operators.

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- $\mathcal{F}(E) \cong \ell_2(\mathbb{Z}_+) \cong H^2(\mathbb{T})$.
- $\mathcal{T}_+(E)$ is the non-selfadjoint algebra generated by the unilateral shift taking e_n to e_{n+1}. Therefore $\mathcal{T}_+(E) \cong A(\mathbb{D})$, the disc algebra (consisting of all functions in $C(\overline{\mathbb{D}})$ that are analytic on $\mathbb{D}$).

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- $\mathcal{T}(E)$ is the C^*-algebra generated by the unilateral shift. It equals the subalgebra $\{T_f : f \in C(\mathbb{T})\} + \mathbb{K}$ of $B(H^2(\mathbb{T}))$.

Example

Take $\mathscr{M} = \mathbb{C}$ and $E = \mathbb{C}^d$. Then:

1. $\mathcal{F}(E) = \mathbb{C} \oplus \mathbb{C}^d \oplus (\mathbb{C}^d)^{\otimes 2} \oplus \dots$.
2. $\mathcal{T}_+(E)$ is Popescu's non-commutative, multidimensional disc algebra $\mathscr{A}_d$.
3. $\mathcal{T}(E)$ is the Toeplitz extension of the Cuntz algebra $\mathcal{O}_d$.

Example

Take $E = \mathscr{M}$ and let φ be an automorphism of $\mathscr{M}$.

1. $\mathcal{T}(E)$ is the Toeplitz extension of $\mathscr{M} \rtimes_\varphi \mathbb{Z}$.
2. $\mathcal{T}_+(E)$ is the "analytic crossed product" of $\mathscr{M}$ by $\mathbb{Z}$ determined by φ.

Theorem (Pimsner, 1997)

If (T, σ) is an <u>*isometric*</u> *covariant representation of E on $\mathcal{H}$, then there exists a C^*-representation π of $\mathcal{T}(E)$ on $\mathcal{H}$, such that $\pi(S(\zeta)) = T(\zeta)$ and $\pi(\varphi_\infty(a)) = \sigma(a)$.*

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In other words: there is a bijection between isometric covariant representations of E and C^*-representations of $\mathcal{T}(E)$.

In other words (2): the Toeplitz algebra is the <u>*universal*</u> *C^*-algebra generated by an isometric covariant representation of E.*

Theorem (Muhly and Solel, 1998)

If (T, σ) is a completely contractive, covariant representation of E on $\mathcal{H}$, then there exists a (completely contractive) representation π of $\mathcal{T}_+(E)$ on $\mathcal{H}$, such that $\pi(S(\zeta)) = T(\zeta)$ and $\pi(\varphi_\infty(a)) = \sigma(a)$.

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In other words: there is a bijection between completely contractive, covariant representations of E and completely contractive representations of $\mathcal{T}_+(E)$.

In other words (2): the tensor algebra is the universal non-selfadjoint algebra generated by a completely contractive, covariant representation of E.

Theorem (Wold decomposition for isometries)

Every isometry V may be written as the direct sum $V = (S \otimes I_{\mathcal{D}}) \oplus U$ where:

1. *S is the unilateral shift.*
2. *$\mathcal{D}$ is some Hilbert space.*
3. *U is unitary.*

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Theorem (Muhly and Solel, 1999)

Every isometric covariant representation V of E on $\mathcal{H}$ may be written as the direct sum $V(\zeta) = (S(\zeta) \otimes I_{\mathcal{D}}) \oplus V^{\mathrm{f}}(\zeta)$, where[a]:

1. *$\mathcal{D}$ is a subspace of $\mathcal{H}$.*
2. *V^{f} is a fully coisometric, isometric, covariant representation of E.*

[a] $S(\cdot) \otimes I_{\mathcal{D}}$ is the *induced representation* of $S(\cdot)$ on $\mathcal{F}(E) \otimes_{\sigma} \mathcal{D}$

Definition

A subproduct system[a] is a family $X = (X(n))_{n\in\mathbb{Z}_+}$ of C^*-correspondences over the C^*-algebra $\mathscr{M} := X(0)$, such that

$$X(n+m) \subseteq X(n) \otimes X(m),$$

and moreover, $X(n+m)$ is *orthogonally complementable* in $X(n) \otimes X(m)$, for all $n, m \in \mathbb{Z}_+$.

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[a] in the "standard" form

Setting $E := X(1)$, we have $X(n) \subseteq E^{\otimes n}$. Denote by $p_n \in \mathcal{L}(E^{\otimes n})$ the orthogonal projection of $E^{\otimes n}$ on $X(n)$.

Definition (The X-Fock space)

$$\mathcal{F}_X := \bigoplus_{n \in \mathbb{Z}_+} X(n) = \mathscr{M} \oplus E \oplus X(2) \oplus X(3) \oplus \ldots \subseteq \mathcal{F}(E)$$

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Definition (The creation operators (X-shifts))

Given $n \in \mathbb{Z}_+$ and $\zeta \in X(n)$, define an operator $S_n^X(\zeta) \in \mathcal{L}(\mathcal{F}_X)$ by

$$S_n^X(\zeta)\eta := p_{n+m}(\zeta \otimes \eta)$$

for $m \in \mathbb{Z}_+$ and $\eta \in X(m)$.
That is, upon writing $P := \bigoplus_{n\in\mathbb{Z}_+} p_n \in \mathcal{L}(\mathcal{F}(E))$, we have

$$S_n^X(\zeta) = PS_n(\zeta)_{|\mathcal{F}_X}.$$

Definitions

1. The C^*-subalgebra of $\mathcal{L}(\mathcal{F}_X)$ generated by $\{S_n^X(\zeta) : n \in \mathbb{Z}_+, \zeta \in X(n)\}$ is called the *Toeplitz algebra* of X. It is denoted by $\mathcal{T}(X)$.
2. The non-selfadjoint subalgebra of $\mathcal{L}(\mathcal{F}_X)$ generated by the same operators is called the *tensor algebra* of X. It is denoted by $\mathcal{T}_+(X)$.

Example

Fix a C^*-algebra $\mathscr{M}$, and take $X(n) := E^{\otimes n}$, $n \in \mathbb{Z}_+$. This subproduct system is called a *product* system. We have:

- $\mathcal{F}_X = \mathcal{F}(E)$.
- $S_0^X(a) = \varphi_\infty(a)$ and $S_1^X(\zeta) = S(\zeta)$.
- $\mathcal{T}(X) = \mathcal{T}(E)$ and $\mathcal{T}_+(X) = \mathcal{T}_+(E)$.

Let $X = (X(n))_{n\in\mathbb{Z}_+}$ be a fixed subproduct system.

Definition

A family $T = (T_n)_{n\in\mathbb{Z}_+}$ is called a *covariant representation* of X if the following conditions hold with $\sigma := T_0$:

1. For every $n \in \mathbb{Z}_+$, (T_n, σ) is a covariant representation of the C^*-correspondence $X(n)$.
2. For every $n, m \in \mathbb{Z}_+$, $\zeta \in X(n)$ and $\eta \in X(m)$,

$$T_{n+m}(p_{n+m}(\zeta \otimes \eta)) = T_n(\zeta)T_m(\eta).$$

The covariant representation is called *completely contractive* if T_n is completely contractive for all n.

If X is a *product* system, there is a bijection between completely contractive, covariant representations of X on $\mathcal{H}$

and

completely contractive, covariant representations of E on $\mathcal{H}$, given by

$$T \mapsto (T_1, T_0)$$

Definition

A completely contractive, covariant representation T of a subproduct system X *on* $\mathcal{H}$ *extends to a* C^**-representation* if there exists a C^*-representation π of $\mathcal{T}(X)$ on $\mathcal{H}$ such that

$$\pi(S_n^X(\zeta)) = T_n(\zeta).$$

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As we have seen, if X is a *product* system, then

$$T \text{ extends to a } C^*\text{-representation} \iff T \text{ is isometric.}$$

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As we have seen, if X is a *product* system, then

$$T \text{ extends to a } C^*\text{-representation} \iff T \text{ is isometric.}$$

This is not true for general subproduct systems. In fact, even in the simplest examples, there is not "convenient" relation describing compositions such as $S_n^X(\zeta)^* S_m^X(\eta)$ and $S_n^X(\zeta) S_m^X(\eta)^*$.

Questions

- Are the algebras $\mathcal{T}(X)$, $\mathcal{T}_+(X)$ *universal* in some sense?
- When does a (completely contractive) covariant representation extend to a C^*-representation?

Theorem (V., 2009)

If T is a completely contractive, covariant representation of X on $\mathcal{H}$, then there exists a (completely contractive) representation π of $\mathcal{T}_+(X)$ on $\mathcal{H}$, such that $\pi(S_n^X(\zeta)) = T_n(\zeta)$ for all $n \in \mathbb{Z}_+$, $\zeta \in X(n)$.

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In other words: there is a bijection between completely contractive, covariant representations of X and completely contractive representations of $\mathcal{T}_+(X)$.

In other words (2): the tensor algebra is the universal non-selfadjoint algebra generated by a completely contractive, covariant representation of X.

Definition

Fix $d \in \mathbb{N}$. The symmetric tensor product $\left(\mathbb{C}^d\right)^{\circledS n}$ is defined to be the subspace of $\left(\mathbb{C}^d\right)^{\otimes n}$ spanned by $\left\{z \otimes \cdots \otimes z : z \in \mathbb{C}^d\right\}$.

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The projection p_n of $(\mathbb{C}^d)^{\otimes n}$ on $(\mathbb{C}^d)^{\circledS n}$ is defined by

$$p_n(z_1 \otimes \cdots \otimes z_n) = \frac{1}{n!} \sum_{\pi} z_{\pi(1)} \otimes \cdots \otimes z_{\pi(n)},$$

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Example

Take $d = 2$. Then $\left(\mathbb{C}^2\right)^{\circledS 2}$ is spanned by $e_1 \otimes e_1$, $e_2 \otimes e_2$ and $e_1 \otimes e_2 + e_2 \otimes e_1$. In particular, $e_1 \otimes e_2 - e_2 \otimes e_1$ does *not* belong to $\left(\mathbb{C}^2\right)^{\circledS 2}$.

Definition

The subproduct system defined by $\mathrm{SSP}_d := \left((\mathbb{C}^d)^{\circledS n}\right)_{n\in\mathbb{Z}_+}$ is called the *symmetric subproduct system*.
Particularly, $\mathscr{M} = \mathbb{C}$ and $E = \mathbb{C}^d$.

There is a bijection between the completely contractive, covariant representations of SSP_d on $\mathcal{H}$

and

commuting row contractions of length d on $\mathcal{H}$, given by
$T \mapsto (T_1(e_1), \ldots, T_1(e_d))$.

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A d-tuple of operators $(T(1),\dots,T(d))$ over $\mathcal{H}$ is *spherical* if $T(1),\dots,T(d)$ are commuting normal operators satisfying $T(1)T(1)^*+\ldots+T(d)T(d)^*=I_{\mathcal{H}}$.
This is a d-dimensional "counterpart" of unitary operators.

A completely contractive, covariant representation Z of SSP_d if called *spherical* if $(Z_1(e_1),\dots,Z_1(e_d))$ is spherical.

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Example

If B_d denotes the unit ball of $\mathbb{C}^d$, then the tuple $(M_{z_1}, \ldots, M_{z_d})$ is spherical in $L^2(\partial B_d)$.

Theorem (Arveson, 1998)

Let T be a completely contractive, covariant representations of SSP_d. Then T extends to a C^*-representation $\iff$ there exist a Hilbert space $\mathcal{D}$ and a spherical covariant representation Z of SSP_d such that

$$T_1(e_k) \cong (S_1^{\mathrm{SSP}_d}(e_k) \otimes I_{\mathcal{D}}) \oplus Z_1(e_k)$$

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The proof:

Relies on the fine structure of $\mathcal{T}(\mathrm{SSP}_d)$:

$$\mathcal{T}(\mathrm{SSP}_d)/\mathbb{K} = C(\partial B_d).$$

It is thus not reproducible in the general case.

Let E be a C^*-correspondence over a C^*-algebra $\mathscr{M}$.

Definition

Given a covariant representation (T, σ) of E on $\mathcal{H}$, define an operator $\widetilde{T} : E \otimes_\sigma \mathcal{H} \to \mathcal{H}$ by

$$\widetilde{T}(\zeta \otimes h) := T(\zeta)h.$$

$\widetilde{T}$ is convenient to use since it is an operator between two Hilbert *spaces*.

Proposition

1. T is completely contractive $\iff$ $\widetilde{T}$ is a well-defined contraction.
2. T is isometric $\iff$ $\widetilde{T}$ is an isometry.

Corollary

The following are equivalent:

1. T extends to a C^*-representation.
2. T is an isometric covariant representation.
3. $\widetilde{T}$ is an isometry.

Suppose that $X = (X(n))_{n\in\mathbb{Z}_+}$ is a subproduct system and $T = (T_n)_{n\in\mathbb{Z}_+}$ is a completely contractive, covariant representation of X on $\mathcal{H}$.

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1. For $n \in \mathbb{Z}_+$, $T_n : X(n) \to B(\mathcal{H})$ is a completely contractive, covariant representation of $X(n)$ on $\mathcal{H}$. Hence $\widetilde{T}_n : X(n) \otimes_\sigma \mathcal{H} \to \mathcal{H}$ is a well-defined contraction.

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2. The sequence $\widetilde{T}_n \widetilde{T}_n^*{}_{n\in\mathbb{Z}_+}$ is a decreasing sequence of positive contractions in $B(\mathcal{H})$. It thus possesses a strong limit, Q. T is called *pure* if $Q = 0$.

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3. T is said to be *fully coisometric* in case $\widetilde{T}_n \widetilde{T}_n^* = I_{\mathcal{H}}$ for all $n \in \mathbb{Z}_+$.
   (It is enough to check for $n = 1$: i.e., that $\widetilde{T}_1 \widetilde{T}_1^* = I_{\mathcal{H}}$.)

Examples

1. If $X = (X(n))_{n\in\mathbb{Z}_+}$ is a subproduct system and $\mathcal{D}$ is a Hilbert space, then the *induced* covariant representation $\left(S_n^X(\cdot) \otimes I_{\mathcal{D}}\right)_{n\in\mathbb{Z}_+}$ is *pure*.
2. If $(T(1), \ldots, T(d))$ is a *spherical* tuple (of commuting operators), then the matching covariant representation of SSP_d is *fully coisometric*.

The C*-representability question

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Motivated by a Wold decomposition-like dilation theorem, we divide first the problem to two cases: the *pure* and the *fully coisometric*.

Definition

A completely contractive, covariant representation T of a subproduct system X on $\mathcal{H}$ is called relatively isometric if:

1. The maps $\widetilde{T}_n$, $n \in \mathbb{Z}_+$, are all *partial isometries*. Denote by Δ_* the projection $I_{\mathcal{H}} - \widetilde{T}_1 \widetilde{T}_1^*$.
2. For all $n \in \mathbb{Z}_+$ and $\zeta \in X(n)$,

$$\Delta_* T_n(\zeta)^* T_n(\zeta) \Delta_* = \sigma\left(\langle \zeta, \zeta \rangle\right) \Delta_* .$$

Theorem (V., 2010)

The following are equivalent:

1. *T is relatively isometric.*
2. *There exist Hilbert spaces $\mathcal{U}, \mathcal{D}$ and a fully coisometric, covariant representation Z of X on $\mathcal{U}$ such that*

$$T_n(\zeta) = (S_n^X(\zeta) \otimes I_{\mathcal{D}}) \oplus Z_n(\zeta).$$

Corollary

If T is relatively isometric and pure, then $T_n(\zeta) = S_n^X(\zeta) \otimes I_{\mathcal{D}}$, i.e., T is an induced representation. It therefore extends to a C^*-representation ($\pi : \mathcal{T}(X) \to \mathcal{L}(\mathcal{F}_X \otimes_\sigma \mathcal{D})$ is defined by $\pi(A) = A \otimes I_{\mathcal{D}}$).

The corollary gives only sufficiency. What about necessity?

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Proposition

If X is a *product system* and T is a *pure* completely contractive, covariant representation of X, then the following are equivalent:

1. T is isometric ($\Longleftrightarrow$ T extends to a C^*-representation).
2. T is relatively isometric.

The corollary gives only sufficiency. What about necessity?

Proposition

If X is a *product system* and T is a *pure* completely contractive, covariant representation of X, then the following are equivalent:

1. T is isometric ($\iff$ T extends to a C^*-representation).
2. T is relatively isometric.

Proposition

If X is a subproduct system such that $E = X(1)$ is a *finite dimensional Hilbert space* and T is a *pure* completely contractive, covariant representation of X, then the following are equivalent:

1. T extends to a C^*-representation.
2. T is relatively isometric.

Example: $X = \mathrm{SSP}_d$.

Example

$X = (X(n))_{n\in\mathbb{Z}_+}$ is a subproduct system satisfying $X(n) = \{0\}$ for all $n \geq n_0$.

Fix a completely contractive, covariant representation T of X on $\mathcal{H}$. Recall that $\widetilde{T}_n : X(n) \otimes_\sigma \mathcal{H} \to \mathcal{H}$ is defined by $\widetilde{T}_n(\zeta \otimes h) := T_n(\zeta)h$. Hence $\widetilde{T}_n = 0$ for all $n \geq n_0$, and thus $Q = \text{s-lim}_{n\to\infty} \widetilde{T}_n \widetilde{T}_n^* = 0$, that is, T is *automatically pure*. Thus T extends to a C^*-representation if it is *relatively isometric*.

Theorem (V., 2010)

Let T be a fully coisometric, covariant representation of the subproduct system X on $\mathcal{H}$ that satisfies

$$\lim_{\ell\to\infty} \left\|(p_\ell \otimes I_{\mathcal{H}})(\eta \otimes \widetilde{T}^*_{\ell-m} h)\right\|_{X(\ell)\otimes_\sigma \mathcal{H}} = \|T_m(\eta)h\|_{\mathcal{H}} \tag{1}$$

for all $m \in \mathbb{N}$, $\eta \in X(m)$ and $h \in \mathcal{H}$. Then T extends to a C^*-representation.

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for all $m \in \mathbb{N}$, $\eta \in X(m)$ and $h \in \mathcal{H}$. Then T extends to a C^*-representation.

Remark

The sequence $\left\{\left\|(p_\ell \otimes I_\mathcal{H})(\eta \otimes \widetilde{T}^*_{\ell-m}h)\right\|_{X(\ell)\otimes_\sigma \mathcal{H}}\right\}_{\ell\geq m}$ is decreasing, so that the its limit always exists, and it is greater than or equal to $\|T_m(\eta)h\|_\mathcal{H}$.

What about necessity?

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Proposition

If X is a *product system* and T is a *fully coisometric*, covariant representation of X, then the following are equivalent:

1. T is isometric ($\iff T$ extends to a C^*-representation).
2. Condition (1) holds.

And if $E = X(1)$ is a finite dimensional Hilbert space?

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Theorem

Let T be a fully coisometric, covariant representation of SSP_d. Then the following are equivalent:

1. *T is spherical ($\iff$ T extends to a C^*-representation)*
2. *T satisfies Condition (1).*

Being more specific, we prove that the limit in (1) equals $\|T_m(\eta)^* h\|$. Therefore (1) holds if and only if $\|T_m(\eta)^* h\| = \|T_m(\eta)h\|$, that is, $T_m(\eta)$ is normal for all m, η, as desired.

Recall the definition $Q := \text{s-lim}_{n\to\infty} \widetilde{T}_n \widetilde{T}_n^*$.

Theorem

If T is a completely contractive, covariant representation of X on $\mathcal{H}$, such that

1. *T is relatively isometric, and*
2. $\lim_{\ell\to\infty} \left\| (p_\ell \otimes Q)(\eta \otimes \widetilde{T}_{\ell-m}^* h) \right\|_{X(\ell)\otimes_\sigma \mathcal{H}} = \| T_m(\eta) Q h \|_{\mathcal{H}}$ *for all suitable m, η, h.*

Then there exist Hilbert spaces $\mathcal{U}, \mathcal{D}$ and a fully coisometric, covariant representation Z of X on $\mathcal{U}$, which extends to a C^*-representation, such that

$$T_n(\zeta) = (S_n^X(\zeta) \otimes I_{\mathcal{D}}) \oplus Z_n(\zeta).$$

In particular, T extends to a C^*-representation.

- Dilations of completely contractive, covariant representations.
- Von Neumann inequalities
- The W^*-setting:
  - $\mathscr{M}$ is a von Neumann algebra
  - Hilbert W^*-modules
  - W^*-correspondences
  - covariant representations
  - Fock space
  - ...

Questions?