Cuntz-Pimsner algebras for subproduct systems

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C^*-correspondences

Let $\mathscr{M}$ denote a C^*-algebra throughout.

Definition

A (right) Hilbert C^*-module E over $\mathscr{M}$ is a *C^*-correspondence* if it is also a *left* $\mathscr{M}$-module, with multiplication on the left given by adjointable operators.

That is: there exists a $*$-homomorphism $\varphi : \mathscr{M} \to \mathcal{L}(E)$ such that $a \cdot \zeta$ is defined to be $\varphi(a)\zeta$ for $a \in \mathscr{M}$ and $\zeta \in E$.

Examples

1. $\mathscr{M} = \mathbb{C}$, $E = \mathcal{H}$ and $\varphi(\alpha)\zeta := \alpha\zeta$.
2. $E = {}_{\alpha}\mathscr{M}$ where α is an endomorphism of $\mathscr{M}$ ($E = \mathscr{M}$ as sets and $\varphi(a)\zeta := \alpha(a)\zeta$).

The Toeplitz algebra

Definition

Let E denote a C^*-correspondence over $\mathscr{M}$.

- The Fock space is the correspondence

$$\mathcal{F}_E := \bigoplus_{n \in \mathbb{Z}_+} E^{\otimes n} = \mathscr{M} \oplus E \oplus E^{\otimes 2} \oplus \dots .$$

- For $a \in \mathscr{M}$ and $\zeta \in E$, let $\varphi_\infty(a), S(\zeta) \in \mathcal{L}(\mathcal{F}_E)$ be given by

$$\varphi_\infty(a) : \eta \mapsto a \cdot \eta \qquad S(\zeta) : \eta \mapsto \zeta \otimes \eta$$

  $(\eta \in X(m),\, m \in \mathbb{Z}_+)$.
  It is a simple calculation that $S(\xi)^* S(\zeta) = \varphi_\infty(\langle \xi, \zeta \rangle)$.
- The *Toeplitz algebra* $\mathcal{T}(E)$ is the C^*-subalgebra of $\mathcal{L}(\mathcal{F}_E)$ generated by the operators $\varphi_\infty(\cdot)$, $S(\cdot)$.

The Cuntz-Pimsner algebra

Definition (Pimsner, 1995)

Let E denote a *faithful* C^*-correspondence over $\mathscr{M}$.

- The ideal $\mathcal{J} \trianglelefteq \mathscr{M}$ is defined by

$$\mathcal{J} := \varphi^{-1}\left(\mathcal{K}(E)\right).$$

- We have $\mathcal{K}(\mathcal{F}_E\mathcal{J}) \trianglelefteq \mathcal{T}(E)$. More precisely,

$$\mathcal{T}(E) \cap \mathcal{K}(\mathcal{F}_E) = \mathcal{K}(\mathcal{F}_E\mathcal{J}).$$

- The *Cuntz-Pimsner* algebra is

$$\mathcal{O}(E) := \mathcal{T}(E)/\mathcal{K}(\mathcal{F}_E\mathcal{J}).$$

Universal property (1) + examples

Embed $\mathcal{K}(E) \hookrightarrow \mathcal{T}(E)$ by $\Psi : \zeta \otimes \eta^* \mapsto S(\zeta)S(\eta)^*$.

Theorem (Pimsner, 1995)

A C^*-representation π of $\mathcal{T}(E)$ factors through $\mathcal{O}(E)$ $\Leftrightarrow$ for all $a \in \mathcal{J}$ we have $\pi\big(\Psi(\varphi(a))\big) = \pi\big(\varphi_\infty(a)\big)$.

Examples

- $\mathscr{M} = E = \mathbb{C} \rightsquigarrow \mathcal{T}(E)$ is the (classical) Toeplitz algebra, $\mathcal{O}(E) = C(\mathbb{T})$.
- G is a finite graph of d vertices, E is the graph correspondence of G (with $\mathscr{M} = \mathbb{C}^d$) $\rightsquigarrow \mathcal{O}(E)$ is the Cuntz-Krieger algebra of G.
- $\mathscr{M}$ is a unital C^*-algebra, $\alpha \in \operatorname{Aut} \mathscr{M}$, $E := {}_\alpha\mathscr{M} \rightsquigarrow \mathcal{O}(E) \cong \mathscr{M} \rtimes_\alpha \mathbb{Z}$.
- This could be generalized further to crossed products of Hilbert bimodules.

Universal property (2) – gauge invariance

The Toeplitz algebra $\mathcal{T}(E)$ has a gauge action: for $\lambda \in \mathbb{T}$ there is $\alpha_\lambda \in \mathrm{Aut}(\mathcal{T}(E))$ with

$$\varphi_\infty(a) \mapsto \varphi_\infty(a) \qquad S(\zeta) \mapsto \lambda S(\zeta).$$

An ideal $\mathcal{I} \trianglelefteq \mathcal{T}(E)$ is called *gauge invariant* if $\alpha_\lambda(\mathcal{I}) = \mathcal{I}$ for all λ.

Recall that $\mathcal{O}(E) = \mathcal{T}(E)/\mathcal{K}(\mathcal{F}_E\mathcal{J})$.

The gauge-invariant uniqueness theorem (Katsura, 2007)

The ideal $\mathcal{K}(\mathcal{F}_E\mathcal{J})$ is the largest among ideals $\mathcal{I}$ of $\mathcal{T}(E)$ s.t.:

1. $\varphi_\infty(\mathscr{M}) \cap \mathcal{I} = \{0\}$.
2. $\mathcal{I}$ is gauge invariant.

Subproduct systems

Definition

A subproduct system is a family $X = (X(n))_{n\in\mathbb{Z}_+}$ of C^*-correspondences over the C^*-algebra $\mathscr{M} := X(0)$, such that

$$X(n+m) \subseteq X(n) \otimes X(m),$$

and moreover, $X(n+m)$ is *orthogonally complementable* in $X(n) \otimes X(m)$, for all $n, m \in \mathbb{Z}_+$.

Product systems – the "trivial" example

E is an (essential) C^*-correspondence over $\mathscr{M}$ and $X(n) = E^{\otimes n}$ for all $n \in \mathbb{Z}_+$.

Examples

SSP_d (the symmetric subproduct system), $d \in \mathbb{N}$

$X(n) = \left(\mathbb{C}^d\right)^{Ⓢn}$ (the n-fold *symmetric* tensor product of $\mathbb{C}^d$) for all n.

SSP_∞ (the infinite-dimensional symmetric subproduct system)

$X(n) = \left(\ell_2\right)^{Ⓢn}$. Here $\dim X(n)$ is infinite for all $n \in \mathbb{N}$.

$P \in M_d$, $P_{ij} \geq 0$ for all i, j

$X(n)$ is the "support" *quiver* of the matrix P^n.

***cp*-semigroups**

A subproduct system can be associated with any *cp*-semigroup over $\mathbb{Z}_+$.

The Toeplitz algebra for subproduct systems

Let $X = (X(n))_{n\in\mathbb{Z}_+}$ be a subproduct system.

Definition (The X-Fock space and creation operators (X-shifts))

- Setting $E := X(1)$, we have $X(n) \subseteq E^{\otimes n}$.
- Denote by $p_n \in \mathcal{L}(E^{\otimes n})$ the orthogonal projection of $E^{\otimes n}$ on $X(n)$.
- Define $\mathcal{F}_X := \bigoplus_{n\in\mathbb{Z}_+} X(n) = \mathscr{M} \oplus X(1) \oplus X(2) \oplus X(3) \oplus \ldots$.
- For $n \in \mathbb{Z}_+$ and $\zeta \in X(n)$, define $S_n^X(\zeta) \in \mathcal{L}(\mathcal{F}_X)$ by

$$(\forall m \in \mathbb{Z}_+, \eta \in X(m)) \qquad S_n^X(\zeta)\eta := p_{n+m}(\zeta \otimes \eta) \in X(n+m)$$

Definition

The *Toeplitz algebra $\mathcal{T}(X)$* of X is the C^*-subalgebra of $\mathcal{L}(\mathcal{F}_X)$ generated by $\left\{S_n^X(\zeta) : n \in \mathbb{Z}_+, \zeta \in X(n)\right\}$.

What about the Cuntz-Pimsner algebra?

The definition of the Toeplitz algebra for subproduct systems is natural.

But how should one define the Cuntz-Pimsner algebra for subproduct systems?

We will present a possible "candidate", and try to "justify" it by demonstrating some of its virtues.

The Cuntz-Pimsner algebra for subproduct systems

Assume henceforth that $X(n)$ is *faithful* for all n.
Write $Q_n \in \mathcal{L}(\mathcal{F}_X)$ for the projection on the direct summand $X(n)$.

Proposition

The set

$$\mathcal{I} := \left\{ S \in \mathcal{T}(X) : \lim_{n\to\infty} \|SQ_n\| = 0 \right\}$$

is a gauge-invariant *ideal* of $\mathcal{T}(X)$ (and in fact, $\mathcal{I} = \langle \mathcal{I} \cap \mathcal{T}_0(X) \rangle$, where $\mathcal{T}_0(X)$ is the 0th spectral subset of $\mathcal{T}(X)$ w.r.t. the gauge action).

Clearly $\mathcal{K}(\mathcal{F}_X\mathcal{J}) \subseteq \mathcal{T}(X) \cap \mathcal{K}(\mathcal{F}_X) \subseteq \mathcal{I}$.

Definition (V.)

The *Cuntz-Pimsner algebra* of X is defined as $\mathcal{O}(X) := \mathcal{T}(X)/\mathcal{I}$.

Proposition

If X is a product system, then $\mathcal{I} = \mathcal{K}(\mathcal{F}_X\mathcal{J})$. Thus, $\mathcal{O}(X) = \mathcal{O}(E)$.

Examples

1. $X = \mathrm{SSP}_d \rightsquigarrow \mathcal{I} = \mathbb{K}$ and $\mathcal{O}(X) = C(\partial B_d)$ (Arveson's construction)
2. More generally: $\mathscr{M} = \mathbb{C}$ and E is a finite dimensional Hilbert space $\rightsquigarrow \mathcal{I} = \mathbb{K}$
3. $X = \mathrm{SSP}_\infty \rightsquigarrow \mathcal{O}(X) = C(B)$, where B is the closed unit ball of ℓ_2 with the Tychonoff topology (by the way: $\mathcal{O}_\infty = ?$)
   Question: is $\mathcal{I}$ *simple* in this case? (our guess: *no*)
4. If $Q_n \in \mathcal{T}(X)$ for all n then $\mathcal{I} = \langle Q_n : n \in \mathbb{Z}_+ \rangle$
   Example: the subproduct system of $P \in M_d$ with $P_{ij} \geq 0$ for all i, j

Gauge-invariant uniqueness theorem? No!

Example

The Toeplitz algebra of SSP_2 does not admit a largest ideal which does not contain the unit I, and which is gauge invariant.

Sketch of proof.

Suppose that such ideal $\mathcal{P} \trianglelefteq \mathcal{T}(\mathrm{SSP}_2)$ exists.

1. $\mathcal{P}$ is largest $\rightsquigarrow \mathbb{K} \subseteq \mathcal{P}$
2. $0 \to \mathbb{K} \to \mathcal{T}(\mathrm{SSP}_2) \to C(\partial B_2)$
3. $\mathcal{P}/\mathbb{K}$ has a clear structure as an ideal of $C(\partial B_2)$

Now it is easy to find a larger ideal with the desired properties. □

An equivalent definition

- Consider the subspaces $\mathcal{L}(\oplus_{k=0}^{n} X(k))$ of $\mathcal{L}(\mathcal{F}_X)$
- Let $\mathcal{B}$ be the $*$-algebra $\bigcup_{n=0}^{\infty} \mathcal{L}(\oplus_{k=0}^{n} X(k))$
- $\mathcal{T}(X)$ is contained in the *multiplier algebra* $M(\overline{\mathcal{B}})$
- Let $q : M(\overline{\mathcal{B}}) \to M(\overline{\mathcal{B}})/\overline{\mathcal{B}}$ be the quotient map.

Easy proposition

$\mathcal{O}(X) \cong q(\mathcal{T}(X))$. That is, $\ker q|_{\mathcal{T}(X)} = \mathcal{I}$.

The proposition generalizes a result of Pimsner (1995). In fact, this was the original definition of the Cuntz-Pimsner algebra.

Essential representations

Definition

A C^*-representation π of $\mathcal{T}(X)$ on $\mathcal{H}$ is *essential* if for every n,

$$\overline{\text{span}} \bigcup_{\zeta \in X(n)} \operatorname{Im} \pi(S_n(\zeta)) = \mathcal{H}.$$

Remark

If X is a *product* system, then π is essential $\Leftrightarrow$ it is fully coisometric.

Theorem (Hirshberg (2005), Skeide (2009))

Let E be a faithful and essential C^*-correspondence. Then

$$\bigcap_{\substack{\pi \text{ is an essential} \\ \text{representation of } \mathcal{T}(E)}} \ker \pi = \mathcal{K}(\mathcal{F}_E \mathcal{J}).$$

Open question

Is it true that under (mild) hypotheses we have

$$\bigcap_{\substack{\pi \text{ is an essential} \\ \text{representation of } \mathcal{T}(X)}} \ker \pi = \mathcal{I} \quad ?$$

We conjecture that it is.
What if "essential" is replaced by "fully coisometric"?

Proposition

$$\bigcap_{\substack{\pi \text{ is an essential} \\ \text{representation of } \mathcal{T}(X)}} \ker \pi \supseteq \mathcal{I}.$$

Essential representations (cont.)

The conjecture (in its strict version) is true in many interesting cases. For instance:

1. "Finite dimensional" subproduct systems:
   - $X(1)$ is a finite-dimensional Hilbert space (e.g.: $X = \mathrm{SSP}_d$, $d \in \mathbb{N}$)
   - The subproduct system of $P \in M_d$ with $P_{ij} \geq 0$ for all i,j
2. But also SSP_∞.

Morita equivalence

Definition (Muhly and Solel (2000))

Let E, F be C^*-correspondences over $\mathscr{A}, \mathscr{B}$. E is *strongly Morita equivalent* to F if $\mathscr{A}$ is ME to $\mathscr{B}$ via an equivalence bimodule M, and there exists an isomorphism $W : \mathrm{M} \otimes F \to E \otimes \mathrm{M}$. Notation: $E \overset{\text{SME}}{\sim}_{\mathrm{M}} F$.

If $E \overset{\text{SME}}{\sim}_{\mathrm{M}} F$, define isomorphisms $W_n : \mathrm{M} \otimes F^{\otimes n} \to E^{\otimes n} \otimes \mathrm{M}$ by $W_1 := W$ and $W_n := (I_E \otimes W_{n-1})(W \otimes I_{F^{\otimes(n-1)}})$.

Definition (V.)

Subproduct systems X, Y are *strongly Morita equivalent* if $X(1) \overset{\text{SME}}{\sim}_{\mathrm{M}} Y(1)$ and

$$W_n(\mathrm{M} \otimes Y(n)) = X(n) \otimes \mathrm{M}$$

for all n.

Morita equivalence (cont.)

For a subproduct system X, the *tensor algebra* $\mathcal{T}_+(X)$ is the operator subalgebra of $\mathcal{T}(X)$ generated by all X-shifts.

The following generalizes a theorem of Muhly and Solel (2000) for *product systems*.

Theorem (V.)

If X is strongly Morita equivalent to Y, then:

1. *$\mathcal{T}_+(X)$ is strongly Morita equivalent[a] to $\mathcal{T}_+(Y)$*
2. *$\mathcal{T}(X)$ is Morita equivalent to $\mathcal{T}(Y)$*
3. *The Rieffel correspondence of $\mathcal{T}(X) \sim \mathcal{T}(Y)$ carries $\mathcal{I}(X)$ to $\mathcal{I}(Y)$. Therefore $\mathcal{O}(X)$ is Morita equivalent to $\mathcal{O}(Y)$.*

[a] as operator algebras, *a la* Blecher, Muhly and Paulsen (2000)

This is another evidence that our definition of the Cuntz-Pimsner algebra for subproduct systems is “natural”.

More open questions

1. Is there a “strong” universality characterization of $\mathcal{O}(X)$?
2. What is the ideal structure of $\mathcal{O}(X)$?
   (The general case seems hopeless; what about specific families?)
3. In the spirit of Cuntz (1977), Pimsner used an “extension of scalars” method to find a C^*-algebra that is naturally isomorphic to $\mathcal{O}(E)$, and for which there is a *semi-split* exact sequence with the Toeplitz algebra[1].
   Could this be done in our context?
4. Is there a relation between $\mathcal{O}(X)$ and $C^*_{\mathrm{env}}(\mathcal{T}_+(X))$?
   Different cases have very different answers:

$$C^*_{\mathrm{env}}(\mathcal{T}_+(E)) = \mathcal{O}(E),$$

   but

$$C^*_{\mathrm{env}}(\mathcal{T}_+(\mathrm{SSP}_d)) = \mathcal{T}(\mathrm{SSP}_d) \qquad (d \in \mathbb{N}).$$

   We do not know what $C^*_{\mathrm{env}}(\mathcal{T}_+(\mathrm{SSP}_\infty))$ is.

[1] Pimsner used this to obtain a *KK*-theoretical six-term exact sequence.

Thank you for listening!