Covariant Representations of Subproduct Systems

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C^*-correspondences

Let $\mathscr{M}$ denote a C^*-algebra throughout.

Definition

A (right) Hilbert C^*-module E over $\mathscr{M}$ is a *C^*-correspondence* if it is also a *left* $\mathscr{M}$-module, with multiplication on the left given by adjointable operators.

That is: there exists a $*$-homomorphism $\varphi : \mathscr{M} \to \mathcal{L}(E)$ such that $a \cdot \zeta$ is defined to be $\varphi(a)\zeta$ for $a \in \mathscr{M}$ and $\zeta \in E$.

Examples

1. $\mathscr{M} = \mathbb{C}$, $E = \mathcal{H}$ and $\varphi(\alpha)\zeta := \alpha\zeta$.
2. $E = \mathscr{M}$, α is an endomorphism of $\mathscr{M}$ and $\varphi(a)\zeta := \alpha(a)\zeta$.

Subproduct systems

Definition

A subproduct system is a family $X = (X(n))_{n \in \mathbb{Z}_+}$ of C^*-correspondences over the C^*-algebra $\mathscr{M} := X(0)$, such that

$$X(n+m) \subseteq X(n) \otimes X(m),$$

and moreover, $X(n+m)$ is *orthogonally complementable* in $X(n) \otimes X(m)$, for all $n, m \in \mathbb{Z}_+$.

Setting $E := X(1)$, we have $X(n) \subseteq E^{\otimes n}$. Denote by $p_n \in \mathcal{L}(E^{\otimes n})$ the orthogonal projection of $E^{\otimes n}$ on $X(n)$.

Examples

Example (Product systems)

E is a C^*-correspondence over $\mathscr{M}$ and $X(n) = E^{\otimes n}$ for all $n \in \mathbb{Z}_+$.

Example (The symmetric subproduct system)

$X(n) = \left(\mathbb{C}^d\right)^{\circledS n}$ (the n-fold *symmetric* tensor product of $\mathbb{C}^d$) for all n.
Denoted by SSP_d.

Creation operators over the X-Fock space

Definition (The X-Fock space)

$$\mathcal{F}_X := \bigoplus_{n \in \mathbb{Z}_+} X(n) = \mathscr{M} \oplus E \oplus X(2) \oplus X(3) \oplus \dots$$

Definition (The creation operators (X-shifts))

Given $n \in \mathbb{Z}_+$ and $\zeta \in X(n)$, define an operator $S_n^X(\zeta) \in \mathcal{L}(\mathcal{F}_X)$ by

$$(\forall m \in \mathbb{Z}_+, \eta \in X(m)) \qquad S_n^X(\zeta)\eta := p_{n+m}(\zeta \otimes \eta) \in X(n+m)$$

The Toeplitz and tensor algebras

Definitions

1. The *Toeplitz algebra* of X is the C^*-subalgebra of $\mathcal{L}(\mathcal{F}_X)$ generated by $\left\{S_n^X(\zeta) : n \in \mathbb{Z}_+, \zeta \in X(n)\right\}$. Denoted by $\mathcal{T}(X)$.
2. The *tensor algebra* of X is the non-selfadjoint subalgebra of $\mathcal{L}(\mathcal{F}_X)$ generated by the same operators. Denoted by $\mathcal{T}_+(X)$.

Question

How do the representations of $\mathcal{T}(X)$ and $\mathcal{T}_+(X)$ look like?

Covariant representations of C^*-correspondences

Fix a C^*-correspondence F over $\mathscr{M}$.

Definition

A pair (T, σ) is called a *covariant representation* of F on $\mathcal{H}$ if:

1. σ is a nondegenerate C^*-representation of $\mathscr{M}$ on $\mathcal{H}$.
2. $T : F \to B(\mathcal{H})$ is a linear bimodule map with respect to σ, that is: $T(a\zeta) = \sigma(a)T(\zeta)$, $T(\zeta a) = T(\zeta)\sigma(a)$ for all $\zeta \in F$ and $a \in \mathscr{M}$.

(T, σ) is called *completely contractive* in case T is completely contractive with respect to the structure of the "linking algebra" of $\mathscr{M}$ and F.

Covariant representations of subproduct systems

Let $X = (X(n))_{n \in \mathbb{Z}_+}$ be a fixed subproduct system.

Definition

A family $T = (T_n)_{n \in \mathbb{Z}_+}$ is called a *covariant representation* of X if the following conditions hold with $\sigma := T_0$:

1. For every $n \in \mathbb{Z}_+$, (T_n, σ) is a covariant representation of the C^*-correspondence $X(n)$.
2. For every $n, m \in \mathbb{Z}_+$, $\zeta \in X(n)$ and $\eta \in X(m)$,

$$T_{n+m}(p_{n+m}(\zeta \otimes \eta)) = T_n(\zeta) T_m(\eta).$$

The covariant representation is called *completely contractive* if T_n is completely contractive for all n.

Representations vs. covariant representations

If π is a completely contractive representation of $\mathcal{T}_+(X)$ then upon defining

$$T_n(\zeta) := \pi\big(S_n^X(\zeta)\big)$$

one obtains a completely contractive, covariant representation of X.

Does the converse hold?
When does π extend to a C^*-representation of $\mathcal{T}(X)$?

The tilde operators

Definition

Let F be a C^*-correspondence. Given a completely contractive, covariant representation (T, σ) of F on $\mathcal{H}$, define an operator

$$\widetilde{T} : F \otimes_\sigma \mathcal{H} \to \mathcal{H}$$

by

$$\widetilde{T}(\zeta \otimes h) := T(\zeta)h.$$

$\widetilde{T}$ is convenient to use since it is an operator between two Hilbert *spaces*. It can be shown to be well-defined and contractive.

The tilde operators (cont.)

Suppose that $X = (X(n))_{n\in\mathbb{Z}_+}$ is a subproduct system and $T = (T_n)_{n\in\mathbb{Z}_+}$ is a completely contractive, covariant representation of X on $\mathcal{H}$.

- For $n \in \mathbb{Z}_+$, $T_n : X(n) \to B(\mathcal{H})$ is a completely contractive, covariant representation of $X(n)$ on $\mathcal{H}$. Hence $\widetilde{T}_n : X(n) \otimes_\sigma \mathcal{H} \to \mathcal{H}$ is a well-defined contraction.
- The sequence $\left\{\widetilde{T}_n \widetilde{T}_n^*\right\}_{n\in\mathbb{Z}_+}$ is a decreasing sequence of positive contractions in $B(\mathcal{H})$. It thus possesses a strong limit, Q.
- T is called *pure* if $Q = 0$.
- T is said to be *fully coisometric* in case $\widetilde{T}_n \widetilde{T}_n^* = I_{\mathcal{H}}$ for all $n \in \mathbb{Z}_+$.

The C^*-representability question

Definition

A completely contractive, covariant representation T of a subproduct system X *on* $\mathcal{H}$ *extends to a* C^**-representation* if there exists a C^*-representation π of $\mathcal{T}(X)$ on $\mathcal{H}$ such that

$$\pi(S_n^X(\zeta)) = T_n(\zeta).$$

When does a completely contractive, covariant representation extend to a C^*-representation?

Motivated by a Wold decomposition-like dilation theorem, we divide the problem to two cases: the *pure* and the *fully coisometric*.

Definition

A completely contractive, covariant representation T of a subproduct system X on $\mathcal{H}$ is called *relatively isometric* if:

1. The maps $\widetilde{T}_n$, $n \in \mathbb{Z}_+$, are all *partial isometries*. Denote by Δ_* the projection $I_{\mathcal{H}} - \widetilde{T}_1 \widetilde{T}_1^*$.
2. For all $n \in \mathbb{Z}_+$ and $\zeta \in X(n)$,

$$\Delta_* T_n(\zeta)^* T_n(\zeta) \Delta_* = \sigma\left(\langle \zeta, \zeta \rangle\right) \Delta_*.$$

The pure case (cont.)

Theorem (V., 2010)

Let T be a completely contractive, covariant representation of the subproduct system X. The following are equivalent:

1. *T is relatively isometric.*
2. *There exist Hilbert spaces $\mathcal{U}, \mathcal{D}$ and a fully coisometric, covariant representation Z of X on $\mathcal{U}$ such that*

$$T_n(\zeta) = (S_n^X(\zeta) \otimes I_{\mathcal{D}}) \oplus Z_n(\zeta).$$

Corollary

If T is relatively isometric and pure, then $T_n(\zeta) = S_n^X(\zeta) \otimes I_{\mathcal{D}}$, i.e., T is an induced representation. It therefore extends to a C^*-representation ($\pi : \mathcal{T}(X) \to \mathcal{L}(\mathcal{F}_X \otimes_\sigma \mathcal{D})$ is defined by $\pi(A) = A \otimes I_{\mathcal{D}}$).

The fully coisometric case

Theorem (V., 2010)

Let T be a fully coisometric, covariant representation of the subproduct system X on $\mathcal{H}$ that satisfies

$$\lim_{\ell\to\infty}\left\|(p_\ell \otimes I_{\mathcal{H}})(\eta \otimes \widetilde{T}^*_{\ell-m}h)\right\|_{X(\ell)\otimes_\sigma \mathcal{H}} = \|T_m(\eta)h\|_{\mathcal{H}}$$

for all $m \in \mathbb{N}$, $\eta \in X(m)$ and $h \in \mathcal{H}$. Then T extends to a C^*-representation.

Remark

The sequence $\left\{\left\|(p_\ell \otimes I_{\mathcal{H}})(\eta \otimes \widetilde{T}^*_{\ell-m}h)\right\|_{X(\ell)\otimes_\sigma \mathcal{H}}\right\}_{\ell\geq m}$ is decreasing, so that the its limit always exists, and it is greater than or equal to $\|T_m(\eta)h\|_{\mathcal{H}}$.

General covariant representations

Recall the notation $Q := \text{s-lim}_{n\to\infty} \widetilde{T}_n \widetilde{T}_n^*$.

Theorem

If T is a completely contractive, covariant representation of X on $\mathcal{H}$, such that T is relatively isometric, and

$$\lim_{\ell\to\infty} \left\| (p_\ell \otimes Q)(\eta \otimes \widetilde{T}_{\ell-m}^* h) \right\|_{X(\ell)\otimes_\sigma \mathcal{H}} = \| T_m(\eta) Q h \|_{\mathcal{H}}$$

for all suitable m, η, h.
Then there exist Hilbert spaces $\mathcal{U}, \mathcal{D}$ and a fully coisometric, covariant representation Z of X on $\mathcal{U}$, which extends to a C^*-representation, such that

$$T_n(\zeta) = (S_n^X(\zeta) \otimes I_{\mathcal{D}}) \oplus Z_n(\zeta).$$

In particular, T extends to a C^*-representation.

Necessity?

We presented sufficient conditions for the C^*-extendability of T. Are they also necessary?

The general answer is unknown. However, it is positive in (at least) a few important special cases:

1. If X is a product system. Our condition coincides with the *isometricity* condition of M. V. Pimsner (1997).
2. If X consists of finite dimensional Hilbert spaces and T is pure.
   - Our conditions are very easy to check
   - Related to the work of G. Popescu
3. If $X = \mathrm{SSP}_d$ and T is fully coisometric
   - Our conditions are equivalent to T being spherical in the sense of W. Arveson (1998).

Universality of the tensor algebra

Theorem (V., 2009)

If T is a completely contractive, covariant representation of X on $\mathcal{H}$, then there exists a (completely contractive) representation π of $\mathcal{T}_+(X)$ on $\mathcal{H}$, such that $\pi(S_n^X(\zeta)) = T_n(\zeta)$ for all $n \in \mathbb{Z}_+$, $\zeta \in X(n)$.

In other words: there is a bijection $T \leftrightarrow \pi$ between completely contractive, covariant representations of X and completely contractive representations of $\mathcal{T}_+(X)$.

In other words (2): the tensor algebra is the universal non-selfadjoint algebra generated by a completely contractive, covariant representation of X.

This is a generalization of a theorem of P. S. Muhly and B. Solel (1998) about *product* systems.

A von Neumann inequality

From the last theorem we derive the following von Neumann inequality:

Corollary

If T is a completely contractive, covariant representation of X on $\mathcal{H}$, then

$$\|p(T)\|_{B(\mathcal{H})} \leq \|p(S^X)\|_{\mathcal{L}(\mathcal{F}_X)}$$

for every "polynomial" p over X.

Thank you for listening!

Questions?