Covariant Representations of Subproduct Systems

From harmonic analysis of Hilbert space contractions to modern research

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Contractions

Notation

Let $\mathcal{H}$ denote a complex Hilbert space throughout.

Definition

A contraction is an operator $T \in B(\mathcal{H})$ with $\|T\| \leq 1$.

There are plentiful contractions!

Examples

Examples

1. Every (orthogonal) projection is of norm 1
2. $K \subseteq \overline{\mathbb{D}}$, $\mathcal{H} := L^2(K)$, $T : f(z) \mapsto zf(z)$
3. Direct sums: if $T_\alpha \in B(H_\alpha)$ is a contraction for all $\alpha \in I$, then $\bigoplus_{\alpha \in I} T_\alpha$ is a contraction over $\bigoplus_{\alpha \in I} \mathcal{H}_\alpha$.

Important class of contractions—isometries

An operator $V \in B(\mathcal{H})$ is a *isometry* if $\|Vx\| = \|x\|$ for all $x \in \mathcal{H}$;

- equivalently: $(Vx, Vy) = (x, y)$ for all $x, y \in \mathcal{H}$;
- equivalently: $V^*V = I$

A surjective isometry is called a *unitary*

- equivalently: $V^*V = I = VV^*$

Isometries

Constructing isometries is easy

Let $(e_\alpha)_{\alpha\in I}$ be an orthonormal base of $\mathcal{H}$,
and let $(f_\alpha)_{\alpha\in I}$ be an orthonormal system in $\mathcal{H}$ (*not necessarily a base*!).
There exists a unique bounded operator $V \in B(\mathcal{H})$ with

$$V : e_\alpha \mapsto f_\alpha.$$

This operator is an isometry.

Example

Let $\mathcal{H} := \ell_2(\mathbb{N}) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \dots$
Consider the standard base $(b_n)_{\mathbb{N}}$ where $b_n = (0, \dots, 0, 1, 0, \dots)$. The *(unilateral) shift operator* $S \in B(\mathcal{H})$ is defined by

$$S : b_n \mapsto b_{n+1}.$$

S is evidently not unitary!

Operator algebras & C^*-algebras

Definition

A (concrete) *operator algebra* is a norm-closed subalgebra of some $B(\mathcal{H})$.

Definition

A *C^*-algebra* is a Banach algebra with involution such that

$$\|A^*A\| = \|A\|^2 .$$

Theorem (Gelfand-Naimark-Segal)

Every C^*-algebra "sits" in $B(\mathcal{H})$ for a suitable Hilbert space $\mathcal{H}$

The universality of the shift

Let $V \in B(\mathcal{H})$ be an isometry.

The Wold decomposition (von Neumann, 1929; Halmos, 1961)

There exist a unitary $U \in B(\mathcal{H}_1)$ and a cardinal $\mathfrak{n}$ such that

$$V \cong S^{(\mathfrak{n})} \oplus U.$$

Particularly, $\mathcal{H} \cong \bigoplus_{0 \le \mathfrak{m} < \mathfrak{n}} \ell_2(\mathbb{N}) \oplus \mathcal{H}_1$.

The universality of the shift (cont.)

Theorem (Von Neumann's inequality (1951); later Sz.-Nagy–Foiaș)

Let T be a contraction.
For every polynomial $p = p(z)$ we have

$$\|p(T)\| \leq \|p(S)\|.$$

`Could you guess what is` $\|p(S)\|$ `?`

Theorem

Let V be an isometry.
For every polynomial $p = p(z, w)$ in two noncommutative variables we have

$$\|p(V, V^*)\| \leq \|p(S, S^*)\|.$$

The universality of the shift (cont.)

Consider $\mathrm{Alg}(S), C^*(S) \subseteq B(\ell_2(\mathbb{N}))$.

$T \in B(\mathcal{H})$ a contraction	$V \in B(\mathcal{H})$ an isometry
$\mathrm{Alg}\, T \subseteq B(\mathcal{H})$	$C^*(V) \subseteq B(\mathcal{H})$
Question: $\exists ? \pi : \mathrm{Alg}\, S \to \mathrm{Alg}\, T$ with $S \mapsto T$?	Question: $\exists ? \pi : C^*(S) \to C^*(V)$ with $S \mapsto V$?
Yes! Consider $p(S) \mapsto p(T)$ for every polynomial $p(z)$	Yes! Consider $p(S, S^*) \mapsto p(V, V^*)$ for every polynomial $p(z, w)$
Since $\|p(T)\| \leq \|p(S)\|$, this map is well defined, and it extends to a norm-decreasing unital homomorphism from $\mathrm{Alg}\, S$ to $\mathrm{Alg}\, T$.	Since $\|p(V, V^*)\| \leq \|p(S, S^*)\|$, this map is well defined, and it extends to a $*$-homomorphism from $C^*(S)$ to $C^*(V)$.

$C^*(S)$ is called the *Toeplitz algebra*

$\mathrm{Alg}(S) \cong$ the disc algebra $A(\mathbb{D})$ ($\subseteq C(\overline{\mathbb{D}})$).

Tensor products of Hilbert spaces

Let $\mathcal{H}, \mathcal{K}$ be Hilbert spaces.

Definition

The *tensor product* $\mathcal{H} \otimes \mathcal{K}$ is the completion of the algebraic tensor product of $\mathcal{H}$ and $\mathcal{K}$ (over $\mathbb{C}$) with the inner product

$$(x_1 \otimes y_1, x_2 \otimes y_2)_{\mathcal{H}\otimes\mathcal{K}} = (x_1, x_2)_{\mathcal{H}} \cdot (y_1, y_2)_{\mathcal{K}}.$$

If $(e_\alpha)_{\alpha \in I}$, $(f_\beta)_{\beta \in J}$ are bases for $\mathcal{H}$, $\mathcal{K}$, respectively, then $(e_\alpha \otimes f_\beta)_{(\alpha,\beta)\in I \times J}$ is a base for $\mathcal{H} \otimes \mathcal{K}$.

If $C \in B(\mathcal{H})$ and $D \in B(\mathcal{K})$, there exists a unique operator $C \otimes D \in B(\mathcal{H} \otimes \mathcal{K})$ with

$$(C \otimes D)(x \otimes y) = (Cx) \otimes (Dy).$$

Multidimensional shift operators

Let $d \in \mathbb{N}$ be given. Consider $\mathbb{C}^d$ with the standard base $\{e_1, \ldots, e_d\}$.

- The *Fock space* is the Hilbert space

$$\mathcal{F}_d := \bigoplus_{n \in \mathbb{Z}_+} (\mathbb{C}^d)^{\otimes n} = \mathbb{C} \oplus \mathbb{C}^d \oplus (\mathbb{C}^d \otimes \mathbb{C}^d) \oplus (\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d) \oplus \ldots$$

- For $1 \leq i \leq d$, consider the shift operator

$$S_i \in B(\mathcal{F}_d)$$

  of "left tensoring by e_i": it maps an element $x \in (\mathbb{C}^d)^{\otimes n}$ to $e_i \otimes x \in (\mathbb{C}^d)^{\otimes (n+1)}$.
- The operators $S_1, \ldots, S_d$ are isometries with orthogonal ranges.
- The sum $S_1 S_1^* + \ldots + S_d S_d^*$ is a projection onto the subspace $\bigoplus_{n \in \mathbb{N}} (\mathbb{C}^d)^{\otimes n}$ of $\mathcal{F}_d$.
  In particular, $S_1 S_1^* + \ldots + S_d S_d^*$ is a contraction.

The universality of the multidimensional shift

Recall that the "simple" shift S was the "universal" isometry.

- A *row contraction* is a family $T_1, \ldots, T_d \in B(\mathcal{H})$ such that $T_1 T_1^* + \ldots + T_d T_d^*$ is a contraction.

Consider a row contraction of isometries $V_1, \ldots, V_d$ [1].

Theorem (G. Popescu, 1989)

There exist isometries $U_1, \ldots, U_d \in B(\mathcal{H}_1)$ with $U_1 U_1^* + \ldots + U_d U_d^* = I_{\mathcal{H}_1}$ and a cardinal $\mathfrak{n}$ such that

$$V_i \cong S_i^{(\mathfrak{n})} \oplus U_i.$$

Particularly, $\mathcal{H} \cong \bigoplus_{0 \leq \mathfrak{m} < \mathfrak{n}} \mathcal{F}_d \oplus \mathcal{H}_1$.

d=1

[1] $V_1, \ldots, V_d$ necessarily have orthogonal ranges

The universality of the multidimensional shift (cont.)

Theorem (G. Popescu, 1991)

Let $T_1, \ldots, T_d$ be a row contraction.
For every polynomial $p = p(z_1, \ldots, z_d)$ in d noncommutative variables we have

$$\|p(T_1, \ldots, T_d)\| \le \|p(S_1, \ldots, S_d)\| .$$

Theorem (G. Popescu, 1995)

Let $V_1, \ldots, V_d$ be a row contraction of isometries.
For every polynomial $p = p(z_1, w_1, \ldots, z_d, w_d)$ in $2d$ noncommutative variables we have

$$\|p(V_1, V_1^*, \ldots, V_d, V_d^*)\| \le \|p(S_1, S_1^*, \ldots, S_d, S_d^*)\| .$$

d=1

The universality of the multidimensional shift (cont.)

Corollary

The map

$$S_i \mapsto T_i \qquad (1 \leq i \leq d)$$

extends to a norm-decreasing unital homomorphism $\mathrm{Alg}(S_1, \ldots, S_d) \to \mathrm{Alg}(T_1, \ldots, T_d)$.

Corollary

The map

$$S_i \mapsto V_i \qquad (1 \leq i \leq d)$$

extends to a $*$*-homomorphism* $C^*(S_1, \ldots, S_d) \to C^*(V_1, \ldots, V_d)$. *The algebra* $C^*(S_1, \ldots, S_d)$ *is called the* d*-Toeplitz algebra.*

What about the converse?

If $\pi : C^*(S_1, \ldots, S_d) \to B(\mathcal{H})$ is a $*$-homomorphism, define $V_i := \pi(S_i)$. Then $V_1, \ldots, V_d$ is a row contraction of isometries.

Constrained row contractions

Again, $d \in \mathbb{N}$ and $T_1, \dots, T_d$ is a row contraction.

- We wish to find a "universal object" for *commuting* row contractions:

$$T_i T_j = T_j T_i.$$

- More generally: if $\mathcal{Q}$ is a set of homogeneous polynomials of d noncommuting variables, which object is "universal" for row contractions with

$$q(T_1, \dots, T_d) = 0 \qquad \text{for all } q \in \mathcal{Q}?$$

  (take $\mathcal{Q} = \{z_i z_j - z_j z_i : 1 \le i, j \le d\}$ for the commuting example).
- The shifts $S_1, \dots, S_d$ are no good—for example, they don't commute.
- So how can we *make* them commute? That is, how can we make

$$e_1 \otimes e_2 \otimes x \text{ be "equal" to } e_2 \otimes e_1 \otimes x \text{ for all } x \in (\mathbb{C}^d)^{\otimes n} \text{ ?}$$

The symmetric subproduct system

- For all $n \geq 2$, let $Y(n) \subseteq (\mathbb{C}^d)^{\otimes n}$ be generated by all differences of the form
$$z_1 \otimes \cdots \otimes z_n - z_{\pi(1)} \otimes \cdots \otimes z_{\pi(n)}$$
where $z_1 \ldots, z_n \in \mathbb{C}^d$ and π is a permutation of $\{1, \ldots, n\}$.
- Set $X(n) := Y(n)^{\perp}$. Write p_n for the projection of $(\mathbb{C}^d)^{\otimes n}$ onto $X(n)$. Ex.: $d = 2 \Longrightarrow X(2) = \mathrm{span}\left\{e_1 \otimes e_1, e_2 \otimes e_2, e_1 \otimes e_2 + e_2 \otimes e_1\right\}$.
- Consider the *symmetric Fock space*
$$\mathcal{F}_d^{\mathrm{Symm}} := \mathbb{C} \oplus \mathbb{C}^d \oplus X(2) \oplus X(3) \oplus \ldots \subseteq \mathcal{F}_d.$$
- For $1 \leq i \leq d$, consider also the *symmetric shift*
$$S_i^{\mathrm{Symm}} \in B(\mathcal{F}_d^{\mathrm{Symm}})$$
defined by "left tensoring by e_i" and then "projecting": it maps an element $x \in X(n)$ to $p_{n+1}(e_i \otimes x) \in X(n+1)$.
- Now the shifts *do* commute: $S_i^{\mathrm{Symm}} S_j^{\mathrm{Symm}} = S_j^{\mathrm{Symm}} S_i^{\mathrm{Symm}}$ for all i, j!

Universality of the symmetric shifts

Theorem (Arveson, 1998)

Let $T_1, \dots, T_d$ be a row contraction of commuting operators. Then the map

$$S_i^{\mathrm{Symm}} \mapsto T_i$$

extends to a norm-decreasing unital homomorphism $\mathrm{Alg}(S_1^{\mathrm{Symm}}, \dots, S_d^{\mathrm{Symm}}) \to \mathrm{Alg}(T_1, \dots, T_d)$.

non comm

What about $*$-homomorphisms of the C^*-algebra $C^*(S_1^{\mathrm{Symm}}, \dots, S_d^{\mathrm{Symm}})$?

Universality of the symmetric shifts (cont.)

Theorem (Arveson, 1998)

Let $V_1, \ldots, V_d$ *be row contraction of commuting operators in* $B(\mathcal{H})$.

There exists a $*$*-homomorphism* $\pi : C^*(S_1^{\mathrm{Symm}}, \ldots, S_d^{\mathrm{Symm}}) \to B(\mathcal{H})$ *with* $\pi(S_i^{\mathrm{Symm}}) = V_i$

if and only if

there exist normal commuting operators $U_1, \ldots, U_d \in B(\mathcal{H}_1)$ *with* $U_1 U_1^* + \ldots + U_d U_d^* = \mathcal{I}_{\mathcal{H}_1}$ *and a cardinal* $\mathfrak{n}$ *such that*

$$V_i \cong (S_i^{\mathrm{Symm}})^{(\mathfrak{n})} \oplus U_i.$$

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General subproduct systems

Definition

A *subproduct system* is a sequence $X = (X(n))_{n \in \mathbb{Z}_+}$ such that:

- $X(0) = \mathbb{C}$, $X(1) = \mathbb{C}^d$ and $X(n)$ is a subspace of $(\mathbb{C}^d)^{\otimes n}$, $n \geq 2$
- $X(n+m) \subseteq X(n) \otimes X(m)$

- The *X-Fock* space: $\mathcal{F}_X := X(0) \oplus X(1) \oplus X(2) \oplus \ldots \subseteq \mathcal{F}_d$.
- For $1 \leq i \leq d$, the *X-shift*

$$S_i^X \in B(\mathcal{F}_X)$$

is defined by "*left tensoring by e_i*" and then "projecting": it maps an element $x \in X(n)$ to $p_{n+1}(e_i \otimes x) \in X(n+1)$.

Trivial example: $X(n) = (\mathbb{C}^d)^{\otimes n}$ (*product* system).
Another example: the *symmetric* subproduct system from before

General subproduct systems (cont.)

- Given a set $\mathcal{Q}$ of homogeneous polynomials of d noncommuting variables, there exists a subproduct system $X = X_{\mathcal{Q}}$ such that

$$q(S_1^X, \ldots, S_d^X) = 0$$

for all $q \in \mathcal{Q}$, and—

- this equality holds "only" for $q \in \mathcal{Q}$.

Definition

Let X be as above. A row contraction $T_1, \ldots, T_d$ which satisfies

$$q(T_1, \ldots, T_d) = 0$$

for all $q \in \mathcal{Q}$ is called a *contractive covariant representation* of X.

Universality of the X-shift

Theorem (G. Popescu, 2006)

Let $T_1, \ldots, T_d$ be a contractive covariant representation of X. Then the map

$$S_i^X \mapsto T_i$$

extends to a norm-decreasing unital homomorphism $\mathrm{Alg}(S_1^X, \ldots, S_d^X) \to \mathrm{Alg}(T_1 \ldots, T_d)$.

What about $C^*(S_1^X, \ldots, S_d^X)$?

Universality of the X-shift (cont.)

Let $V_1, \ldots, V_d$ be a contractive covariant representation of X.

- Define a linear map $V(\cdot) : \left(\mathbb{C}^d\right)^{\otimes n} \to B(\mathcal{H})$ (suppressing the n) by

$$V(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}) := V_{\alpha_1} \cdots V_{\alpha_n}.$$

- Let

$$A_n := \sum_{(\alpha_1,\ldots,\alpha_n)\in\{1,\ldots,d\}^n} V(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n}) V(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n})^*.$$

  Then $\{A_n\}_{n=1}^{\infty}$ is a decreasing sequence of positive operators (?!). It thus admits a *strong* limit, A.

Call $(V_1, \ldots, V_d)$ a *relative isometry* if for every $n \in \mathbb{N}$:

1. A_n is a projection
2. $(I - A_1)V(x)^* V(x)(I - A_1) = \|p_n(x)\| (I - A_1)$ for all $x \in \left(\mathbb{C}^d\right)^{\otimes n}$.

Universality of the X-shift (cont.)

- Fix $n \in \mathbb{N}$, and choose a base $x_1, \ldots, x_{k_n}$ for $X(n) \subseteq \left(\mathbb{C}^d\right)^{\otimes n}$.
- Define $B_n : \mathcal{H} \to X(n) \otimes \mathcal{H}$ by $B_n h := \sum_{k=1}^{k_n} x_k \otimes V(x_k)^* h$.

Theorem (V., 2010)

Let X be a subproduct system and $V_1 \ldots, V_d$ be a contractive covariant representation of X. Assume that

1. *$(V_1, \ldots, V_d)$ is relatively isometric.*
2. *For all $n \in \mathbb{N}$, $x \in X(n)$ and $h \in \mathcal{H}$,*

$$\lim_{\ell \to \infty} \|(p_\ell \otimes A)(x \otimes B_{\ell-n} h)\|_{X(\ell) \otimes \mathcal{H}} = \|V(x)h\|_{\mathcal{H}}.$$

Then there exists a $*$-homomorphism $\pi : C^*(S_1^X, \ldots, S_d^X) \to B(\mathcal{H})$ with $\pi(S_i^X) = V_i$.

Universality of the X-shift (cont.)

Theorem (cont.)

Moreover, there exist a $*$-homomorphism $\pi_1 : C^*(S_1^X, \ldots, S_d^X) \to B(\mathcal{H}_1)$ and a cardinal $\mathfrak{n}$ such that—upon defining $U_i := \pi_1(S_i^X)$—we have $U_1 U_1^* + \ldots + U_d U_d^* = I_{\mathcal{H}_1}$ and

$$V_i \cong (S_i^X)^{(\mathfrak{n})} \oplus U_i.$$

What about necessity?

- The relative isometricity condition is always necessary
- The second condition is necessary in the two prototype cases—
  - row contraction of isometries $V_1, \ldots, V_d$
  - the symmetric case ▸ Arveson
- In general?

Construction of subproduct systems

General notation:

- For $1 \leq \alpha_1, \ldots, a_n \leq d$, let $e_\alpha := e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_n} \in (\mathbb{C}^d)^{\otimes n}$.
- For $q \in \mathbb{C}\langle z_1, \ldots, z_d \rangle$, say $q(z) = \sum c_\alpha z^\alpha$, let $q(e) := \sum c_\alpha e_\alpha \in \mathcal{F}_d$.

Fix a subset $\mathcal{Q} \subseteq \mathbb{C}\langle z_1, \ldots, z_d \rangle$ of homogeneous polynomials.

- Define $\mathcal{I} := \langle \mathcal{Q} \rangle \trianglelefteq \mathbb{C}\langle z_1, \ldots, z_d \rangle$, and let $\mathcal{I}^{(n)}$ denote the set of all homogeneous polynomials of degree n in $\mathcal{I}$.
- The subproduct system is constructed as follows:

$$Y_{\mathcal{I}}(n) := \left\{ q(e) : q \in \mathcal{I}^{(n)} \right\} \quad \text{and} \quad X_{\mathcal{I}}(n) := (\mathbb{C}^d)^{\otimes n} \ominus Y_{\mathcal{I}}(n).$$

Proposition (O. M. Shalit and B. Solel, 2009)

1. The mapping $\mathcal{I} \mapsto X_{\mathcal{I}}$ is a bijection between all (proper) homogeneous ideals and all subproduct systems.
2. As promised: given $q \in \mathbb{C}\langle z_1, \ldots, z_d \rangle$, we have

$$q(S_1^X, \ldots, S_d^X) = 0 \iff q \in \mathcal{I}.$$

Epilogue: the general setting

- Let $\mathscr{M}$ be a C^*-algebra. A *Hilbert C^*-module* over $\mathscr{M}$ is a (complete) right $\mathscr{M}$-module with an $\mathscr{M}$-valued "inner product" (rigging).
- A Hilbert C^*-module over $\mathscr{M}$ with a certain type of left $\mathscr{M}$-action is called a *C^*-correspondence*.
- Let E be a C^*-correspondence. The "full" Fock space is

$$\mathcal{F}(E) = \mathscr{M} \oplus E \oplus E^{\otimes 2} \oplus E^{\otimes 3} \oplus \dots$$

- (Full) shifts are defined by "left tensoring".
- The universality of the full shifts was established by M. V. Pimsner (1995) and P. S. Muhly & B. Solel (1998).
- Everything else can also be defined in this context: subproduct systems, covariant representations, the shift operators, the Toeplitz algebra, ...
- Universality properties of the subproduct system shifts (V.).

Thank you for listening!