Cuntz-Pimsner algebras for subproduct systems

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Function Theory and Operator Theory: Infinite Dimensional and Free Settings

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Outline

1. We start with the classical constructions of the Toeplitz algebra and its two multidimensional versions: the non-commutative and the commutative.
2. For each, a certain quotient of this algebra will be shown to admit interesting virtues.
3. Possible generalizations will be discussed, leading to our construction:
   Cuntz-Pimsner algebras in the setting of subproduct systems.
4. We will demonstrate the construction by examples, including the original Cuntz-Pimsner algebra.
5. Some features of it will be presented.

The classical Toeplitz algebra

- Consider the Hilbert space $\ell_2 = \ell_2(\mathbb{Z}_+)$ and the unilateral shift $S \in B(\ell_2)$ which maps $(x_1, x_2, x_3, \ldots)$ to $(0, x_1, x_2, x_3, \ldots)$.
- S is an isometry. The classical Toeplitz algebra is $\mathcal{T} := C^*(S) \subseteq B(\ell_2)$.

An alternative approach:

- Recall that $L^2(\mathbb{T})$ is generated by $\{z^n : n \in \mathbb{Z}\}$, and that $H^2(\mathbb{T}) \subseteq L^2(\mathbb{T})$ is generated by $\{z^n : n \in \mathbb{Z}_+\}$
- $H^2(\mathbb{T}) \cong \ell_2$ by $z^n \leftrightarrow e_n$
- For $f \in C(\mathbb{T})$, consider the Toeplitz operator $T_f \in B(H^2(\mathbb{T}))$ defined by $g \mapsto \mathrm{Proj}_{H^2(\mathbb{T})}(fg)$
- Then S is (unitarily equivalent) to T_z, and so $\mathcal{T}$ is (unitarily equivalent) to $C^*(T_z) \subseteq B(H^2(\mathbb{T}))$

The classical Toeplitz algebra

Theorem

Denote by $\mathbb{K}$ *the compacts over* $H^2(\mathbb{T})$*. Then*

$$\mathcal{T} = \{T_f : f \in C(\mathbb{T})\} \oplus \mathbb{K}.$$

Corollary

$$\mathcal{T}/\mathbb{K} \cong C(\mathbb{T})$$

Recall that $C(\mathbb{T})$ is the "universal C^*-algebra generated by a unitary": if U is an arbitrary unitary, there is a representation $\pi : C(\mathbb{T}) \to C^*(U)$ with $z \mapsto U$.

The multidimensional setting

Fix $d \in \mathbb{N}$.

- Instead of $\ell_2(\mathbb{Z}_+) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \ldots$, consider the Fock space

$$\mathcal{F}_d := \bigoplus_{n \in \mathbb{Z}_+} \left(\mathbb{C}^d\right)^{\otimes n} = \mathbb{C} \oplus \mathbb{C}^d \oplus \left(\mathbb{C}^d\right)^{\otimes 2} \oplus \ldots$$

- Instead of S, consider the shifts $S_1, \ldots, S_d \in B(\mathcal{F}_d)$ defined by

$$S_i : \left(\mathbb{C}^d\right)^{\otimes n} \ni \eta \mapsto e_i \otimes \eta \in \left(\mathbb{C}^d\right)^{\otimes (n+1)}.$$

  These are isometries with orthogonal ranges.
- Instead of $\mathcal{T}$, consider the algebra $\mathcal{T}_d := C^*(S_1, \ldots, S_d) \subseteq B(\mathcal{F}_d)$.
- $S_1 S_1^* + \ldots + S_d S_d^*$ is the projection onto $\mathcal{F}_d \ominus \mathbb{C}$. In particular, $I - (S_1 S_1^* + \ldots + S_d S_d^*)$ is compact.

The multidimensional setting

Definition

The Cuntz algebra $\mathcal{O}_d$ is the universal C^*-algebra generated by d isometries $V_1, \ldots, V_d$ with orthogonal ranges such that $V_1 V_1^* + \ldots + V_d V_d^* = I$.

So if $W_1, \ldots, W_d$ are isometries with orthogonal ranges (over an arbitrary Hilbert space) such that $W_1 W_1^* + \ldots + W_d W_d^* = I$, then there is a representation $\pi : \mathcal{O}_d \to C^*(W_1, \ldots, W_d)$ with $\pi(V_i) = W_i$ for all i.

Theorem

Denote by $\mathbb{K}$ the compacts over $\mathcal{F}_d$. Then $\mathbb{K} \subseteq \mathcal{T}_d$ and

$$\mathcal{T}_d / \mathbb{K} \cong \mathcal{O}_d.$$

The commutative multidimensional setting

Goal

Change last construction to yield $C(\partial B_d)$.
The problem: the shifts $S_1, \ldots, S_d$ don't commute.

Definition

If $\mathcal{H}$ is a Hilbert space, its 2-fold symmetric tensor product is

$$\mathcal{H}^{\circledS 2} := \mathcal{H}^{\otimes 2} \ominus \overline{\operatorname{span}} \{x \otimes y - y \otimes x : x, y \in \mathcal{H}\}.$$

$\mathcal{H}^{\circledS n}$ is defined similarly for all n.

The characterizing property of $\mathcal{H}^{\circledS n}$ is that for all $x_1, \ldots, x_n \in \mathcal{H}$ and every permutation $\sigma \in S_n$,

$$\mathrm{Proj}_{\mathcal{H}^{\circledS n}}(x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes \cdots \otimes x_{\sigma(n)}) = \mathrm{Proj}_{\mathcal{H}^{\circledS n}}(x_1 \otimes x_2 \otimes \cdots \otimes x_n).$$

The commutative multidimensional setting

We now choose $\mathcal{H} := \mathbb{C}^d$.

- Instead of $\mathcal{F}_d := \bigoplus_{n\in\mathbb{Z}_+} \left(\mathbb{C}^d\right)^{\otimes n}$ consider the Drury-Arveson space

$$\mathcal{F}_d^{\mathrm{Sym}} := \bigoplus_{n\in\mathbb{Z}_+} \left(\mathbb{C}^d\right)^{ⓢ n} = \mathbb{C} \oplus \mathbb{C}^d \oplus \left(\mathbb{C}^d\right)^{ⓢ 2} \oplus \dots$$

- Instead of $S_1, \dots, S_d \in B(\mathcal{F}_d)$, consider the symmetric shifts $S_1^{\mathrm{Sym}}, \dots, S_d^{\mathrm{Sym}} \in B(\mathcal{F}_d^{\mathrm{Sym}})$ defined by

$$S_i^{\mathrm{Sym}} : \left(\mathbb{C}^d\right)^{ⓢ n} \ni \eta \mapsto \mathrm{Proj}_{(\mathbb{C}^d)^{ⓢ(n+1)}}(e_i \otimes \eta) \in \left(\mathbb{C}^d\right)^{ⓢ(n+1)}.$$

  These are *not* isometries, but they commute: $S_i^{\mathrm{Sym}} S_j^{\mathrm{Sym}} = S_j^{\mathrm{Sym}} S_i^{\mathrm{Sym}}$.

- Instead of $\mathcal{T}_d$, consider the algebra $\mathcal{T}_d^{\mathrm{Sym}} := C^*(S_1^{\mathrm{Sym}}, \dots, S_d^{\mathrm{Sym}}) \subseteq B(\mathcal{F}_d^{\mathrm{Sym}})$.

The commutative multidimensional setting

Theorem (W. Arveson, 1998)

Denote by $\mathbb{K}$ *the compacts over* $\mathcal{F}_d^{\mathrm{Sym}}$. *Then* $\mathbb{K} \subseteq \mathcal{T}_d^{\mathrm{Sym}}$ *and*

$$\mathcal{T}_d^{\mathrm{Sym}} / \mathbb{K} \cong C(\partial B_d).$$

Universality:
$C(\partial B_d)$ is the universal C^*-algebra generated by d normal commuting operators $V_1 (= M_{z_1}), \ldots, V_d (= M_{z_d})$ such that $V_1 V_1^* + \ldots + V_d V_d^* = I$.

More general settings?

The process

Fock space $\rightsquigarrow$ shifts $\rightsquigarrow$ Toeplitz algebra $\rightsquigarrow$ quotient of Toeplitz with an interesting universal property.

Possible generalizations

1. The infinite-dimensional version of the last example
    - problem: $\mathbb{K} \not\subseteq \mathcal{T}_\infty^{\mathrm{Sym}}$!
2. Replace the commutation relation by other polynomial constraints
    - not difficult to construct (along the lines of the commutative setting)
    - but very difficult to “handle”
3. Fock spaces whose direct summands are not Hilbert spaces
    - Hilbert C^*-modules

A toy case: the infinite-dim. commutative setting

1. Construct $\mathcal{F}_\infty^{\mathrm{Sym}}$ by replacing $\mathbb{C}^d$ by ℓ_2 in $\mathcal{F}_d^{\mathrm{Sym}} = \mathbb{C} \oplus \mathbb{C}^d \oplus \left(\mathbb{C}^d\right)^{ⓢ 2} \oplus \ldots$.
2. Define the shifts $S_1^{\mathrm{Sym}}, S_2^{\mathrm{Sym}}, S_3^{\mathrm{Sym}}, \ldots$ appropriately. They commute!
3. $\mathcal{T}_\infty^{\mathrm{Sym}}$ is generated by $S_1^{\mathrm{Sym}}, S_2^{\mathrm{Sym}}, S_3^{\mathrm{Sym}}, \ldots$ in $B(\mathcal{F}_\infty^{\mathrm{Sym}})$.

What could replace $\mathbb{K}$ as the ideal the we “mod out”?

A toy case: the infinite-dim. commutative setting

- Write Q_n for the projection of $\mathcal{F}_\infty^{\mathrm{Sym}}$ onto its nth direct summand.
- Let
$$\mathcal{I} := \left\{ S \in \mathcal{T}_\infty^{\mathrm{Sym}} : \lim_{n\to\infty} \|SQ_n\| = 0 \right\}.$$
- $\mathcal{I}$ is an ideal!
- The generalized Cuntz-Pimsner algebra $\mathcal{O}_\infty^{\mathrm{Sym}}$ is $\mathcal{T}_\infty^{\mathrm{Sym}}/\mathcal{I}$.

Theorem (V.)

$\mathcal{O}_\infty^{\mathrm{Sym}} \cong C(B)$, where B is the closed unit ball of ℓ_2 with the Tychonoff topology. (Question: where did the "∂" go?)

As before, $C(B)$ is universal in the suitable sense.

Hilbert C^*-modules and C^*-correspondences[1]

Intuition

Hilbert C^*-modules generalize Hilbert spaces by replacing the scalars $\mathbb{C}$ by an arbitrary C^*-algebra.

Definitions

Let $\mathscr{M}$ denote a C^*-algebra. A (right) Hilbert C^*-module over $\mathscr{M}$ is a right $\mathscr{M}$-module E with a map $\langle \cdot, \cdot \rangle : E \times E \to \mathscr{M}$ ("rigging") such that:

1. $\langle \zeta, \zeta \rangle \geq 0$ in $\mathscr{M}$, with equality $\Leftrightarrow \zeta = 0$
2. $\langle \cdot, \cdot \rangle$ is linear in the second variable and $\langle \zeta, \eta \cdot a \rangle = \langle \zeta, \eta \rangle a$, $a \in \mathscr{M}$
3. $\langle \zeta, \eta \rangle^* = \langle \eta, \zeta \rangle$

and such that E is *complete* w.r.t the norm $\|\zeta\| := \|\langle \zeta, \zeta \rangle_{\mathscr{M}}\|^{1/2}$.

We will call E a C^*-correspondence if it is also a left $\mathscr{M}$-module s.t.

$$\langle a \cdot \zeta, \eta \rangle = \langle \zeta, a^* \cdot \eta \rangle .$$

[1] Henceforth, we omit some details for convenience...

Examples

1. $\mathscr{M} = \mathbb{C}$, $E = \mathcal{H}$ (in particular, $E = \mathbb{C}^d$)
2. $\mathscr{M}$ is any C^*-algebra, α is an endomorphism of $\mathscr{M}$, $E = \mathscr{M}$ as sets, $\langle a, b \rangle = a^* b$, right multiplication is standard multiplication, and left given by
$$a \cdot \zeta = \alpha(a)\zeta$$
3. X is a compact Hausdorff space, $\mathscr{M} = C(X)$, $E = C(X, \mathcal{H})$,
$$\langle f, g \rangle (x) := \langle f(x), g(x) \rangle_{\mathcal{H}} \qquad (\forall f, g \in E, x \in X)$$
4. Every quiver (directed graph) possesses an associated C^*-correspondence

Tensor products

Suppose that E, F are C^*-correspondences over $\mathscr{M}$.

The bimodule structure enables one to define the (internal) tensor product C^*-correspondence

$$E \otimes F$$

that is $\mathscr{M}$-balanced:

$$(\zeta \cdot a) \otimes \eta = \zeta \otimes (a \cdot \eta)$$

and with rigging given by

$$\langle \zeta_1 \otimes \eta_1, \zeta_2 \otimes \eta_2 \rangle_{E \otimes F} := \langle \eta_1, \langle \zeta_1, \zeta_2 \rangle_E \cdot \eta_2 \rangle_F .$$

Subproduct systems

The direct summands of our generalized Fock space will constitute a subproduct system.

Easy fact

If $\mathcal{H}$ is a Hilbert space, then $\mathcal{H}^{\circledS(n+m)} \subseteq \mathcal{H}^{\circledS n} \otimes \mathcal{H}^{\circledS m}$ for all n, m.

Definition (O. M. Shalit and B. Solel, 2009)

A subproduct system over $\mathscr{M}$ is a sequence $X = (X(n))_{n \in \mathbb{Z}_+}$ of C^*-correspondences over $\mathscr{M} = X(0)$ s.t.

$$X(n+m) \subseteq X(n) \otimes X(m)$$

for all n, m.

Examples

Product systems

$X(n) = E^{\otimes n}$ for some C^*-correspondence E over $\mathscr{M}$.

- The prototype: $\mathscr{M} = \mathbb{C}$, $E = \mathbb{C}^d$

SSP_d (the symmetric subproduct system), $d \in \mathbb{N}$

$X(n) = \left(\mathbb{C}^d\right)^{\textcircled{s} n}$.

SSP_∞ (the infinite-dimensional symmetric subproduct system)

$X(n) = \left(\ell_2\right)^{\textcircled{s} n}$. Here $\dim X(n)$ is infinite for all $n \in \mathbb{N}$.

$P \in M_d$, $P_{ij} \geq 0$ for all i, j, no all-zero columns

$X(n)$ is the C^*-correspondence of the "support" *quiver* of the matrix P^n.

The Toeplitz algebra for subproduct systems

Let $X = (X(n))_{n\in\mathbb{Z}_+}$ be a subproduct system.

- Setting $E := X(1)$, we have $X(n) \subseteq E^{\otimes n}$.
- Instead of $\mathcal{F}_d = \bigoplus_{n\in\mathbb{Z}_+} \left(\mathbb{C}^d\right)^{\otimes n}$ or $\mathcal{F}_d^{\mathrm{Sym}} = \bigoplus_{n\in\mathbb{Z}_+} \left(\mathbb{C}^d\right)^{ⓢ n}$, consider

$$\mathcal{F}_X := \bigoplus_{n\in\mathbb{Z}_+} X(n) = \mathscr{M} \oplus X(1) \oplus X(2) \oplus X(3) \oplus \dots$$

- Instead of $S_1, \dots, S_d \in B(\mathcal{F}_d)$ or $S_1^{\mathrm{Sym}}, \dots, S_d^{\mathrm{Sym}} \in B(\mathcal{F}_d^{\mathrm{Sym}})$, consider

$$\begin{aligned} \varphi_\infty(a) \in B(\mathcal{F}_X), &\qquad \eta \mapsto a \cdot \eta &\qquad (a \in \mathscr{M}, \eta \in X(n)), \\ S(\zeta) \in B(\mathcal{F}_X), &\qquad \eta \mapsto \mathrm{Proj}_{X(n+1)}(\zeta \otimes \eta) &\qquad (\zeta \in X(1), \eta \in X(n)). \end{aligned}$$

- The Toeplitz algebra $\mathcal{T}(X)$ is the C^*-subalgebra of $B(\mathcal{F}_X)$ generated by the operators $\varphi_\infty(\cdot)$, $S(\cdot)$.

What could replace $\mathbb{K}$ as the ideal the we "mod out"?

- Write $Q_n \in B(\mathcal{F}_X)$ for the projection onto the nth direct summand, $X(n)$.
- Define
$$\mathcal{I} := \left\{ S \in \mathcal{T}(X) : \lim_{n\to\infty} \|SQ_n\| = 0 \right\}.$$
Then $\mathcal{I} \trianglelefteq \mathcal{T}(X)$ and $\mathcal{T}(X) \cap \mathcal{K}(\mathcal{F}_X) \subseteq \mathcal{I}$ ("generalized compacts").

Definition (V.)

The generalized Cuntz-Pimsner algebra $\mathcal{O}(X)$ of X is $\mathcal{T}(X)/\mathcal{I}$.

Specific case: product systems

Consider first the case $X(n) = E^{\otimes n}$.

- It reduces to the well-known construction of Pimsner (1995):
  - $\mathcal{I} = \mathcal{T}(E) \cap \mathcal{K}(\mathcal{F}_E)$
- This algebra has many interesting universal properties:

Examples

- Of course, $\mathcal{O}(\mathbb{C}^d) = \mathcal{O}_d$, $d \in \mathbb{N}$.
- $\mathscr{M}$ is a unital C^*-algebra, $\alpha \in \operatorname{Aut} \mathscr{M}$, $E := {}_\alpha\mathscr{M} \rightsquigarrow \mathcal{O}(E) \cong \mathscr{M} \rtimes_\alpha \mathbb{Z}$.
- This could be generalized further to crossed products of Hilbert bimodules.
- G is a finite graph of d vertices, E is the graph correspondence of G (with $\mathscr{M} = \mathbb{C}^d$) $\rightsquigarrow \mathcal{O}(E)$ is the Cuntz-Krieger algebra of G.

Specific case: product systems — gauge invariance

We are still in the case $X(n) = E^{\otimes n}$.

The Toeplitz algebra $\mathcal{T}(E)$ has a gauge action: for $\lambda \in \mathbb{T}$ there is $\alpha_\lambda \in \mathrm{Aut}(\mathcal{T}(E))$ with

$$\varphi_\infty(a) \mapsto \varphi_\infty(a) \qquad S(\zeta) \mapsto \lambda S(\zeta).$$

An ideal $\mathcal{J} \trianglelefteq \mathcal{T}(E)$ is called *gauge invariant* if $\alpha_\lambda(\mathcal{J}) = \mathcal{J}$ for all λ.

The gauge-invariant uniqueness theorem (Katsura, 2007)

The ideal $\mathcal{I} = \mathcal{T}(E) \cap \mathcal{K}(\mathcal{F}_E)$ is the largest among ideals $\mathcal{J}$ of $\mathcal{T}(E)$ s.t.:

1. $\varphi_\infty(\mathcal{M}) \cap \mathcal{J} = \{0\}$.
2. $\mathcal{J}$ is gauge invariant.

Gauge-invariant uniqueness theorem? No!

Unfortunately, we were not able to extend the gauge-invariant uniqueness theorem to general subproduct systems.

Example

The Toeplitz algebra of SSP_2 does not admit a largest ideal which does not contain the unit I, and which is gauge invariant.

Sketch of proof.

Suppose that such ideal $\mathcal{P} \trianglelefteq \mathcal{T}(\mathrm{SSP}_2)$ exists.

1. $\mathcal{P}$ is largest $\rightsquigarrow \mathbb{K} \subseteq \mathcal{P}$
2. $0 \to \mathbb{K} \to \mathcal{T}(\mathrm{SSP}_2) \to C(\partial B_2)$
3. $\mathcal{P}/\mathbb{K}$ has a clear structure as an ideal of $C(\partial B_2)$

Now it is easy to find a larger ideal with the desired properties. □

We do have a partial substitute in terms of essential representations of $\mathcal{T}(X)$.

Families of examples

1. The $X(n)$'s are all finite-dimensional Hilbert spaces $\Rightarrow \mathcal{I} = \mathbb{K}$
   - $X(n) = \left(\mathbb{C}^d\right)^{\otimes n}$
   - $X(n) = \left(\mathbb{C}^d\right)^{\circledS n}$
   - For every subshift Λ, a subproduct system X_Λ can be associated so that $\mathcal{O}(X_\Lambda)$ is the C^*-algebra attached to Λ by K. Matsumoto
   - Generally: subproduct systems associated with polynomial constraints
2. $Q_n \in \mathcal{T}(X)$ for all $n \Rightarrow \mathcal{I} = \langle Q_n : n \in \mathbb{Z}_+ \rangle$
   Example: the subproduct system of $P \in M_d$ with $P_{ij} \geq 0$ for all i, j and no all-zero columns

Morita equivalence

Definition (P. S. Muhly and B. Solel (2000))

Let E, F be C^*-correspondences over $\mathscr{A}, \mathscr{B}$. E is *strongly Morita equivalent* to F if $\mathscr{A}$ is ME to $\mathscr{B}$ via an equivalence bimodule M, and there exists an isomorphism $W : \mathrm{M} \otimes F \to E \otimes \mathrm{M}$. Notation: $E \overset{\mathrm{SME}}{\sim}_{\mathrm{M}} F$.

If $E \overset{\mathrm{SME}}{\sim}_{\mathrm{M}} F$, define isomorphisms $W_n : \mathrm{M} \otimes F^{\otimes n} \to E^{\otimes n} \otimes \mathrm{M}$ by $W_1 := W$ and $W_n := (I_E \otimes W_{n-1})(W \otimes I_{F^{\otimes(n-1)}})$.

Definition (V.)

Subproduct systems X, Y are *strongly Morita equivalent* if $X(1) \overset{\mathrm{SME}}{\sim}_{\mathrm{M}} Y(1)$ and

$$W_n(\mathrm{M} \otimes Y(n)) = X(n) \otimes \mathrm{M}$$

for all n.

Morita equivalence (cont.)

The following generalizes a theorem of Muhly and Solel (2000) for *product systems*.

Theorem (V.)

If X is strongly Morita equivalent to Y, then:

1. $\mathcal{T}(X)$ *is Morita equivalent to* $\mathcal{T}(Y)$ *as* C^**-algebras*
2. *The Rieffel correspondence of* $\mathcal{T}(X) \sim \mathcal{T}(Y)$ *carries* $\mathcal{I}(X)$ *to* $\mathcal{I}(Y)$*. Therefore* $\mathcal{O}(X)$ *is Morita equivalent to* $\mathcal{O}(Y)$*.*

This is another evidence that our definition of the Cuntz-Pimsner algebra for subproduct systems is “natural”.

More open questions

1. Is there a "strong" universality characterization of $\mathcal{O}(X)$?
2. Determining the ideal structure, nuclearity and exactness of $\mathcal{O}(X)$.
3. In the spirit of Cuntz (1977), Pimsner used an "extension of scalars" method to find a C^*-algebra that is naturally isomorphic to $\mathcal{O}(E)$, and for which there is a *semi-split* exact sequence with the Toeplitz algebra[2].
   Could this be done in our context?
4. Is there a relation between $\mathcal{O}(X)$ and $C^*_{\mathrm{env}}(\mathcal{T}_+(X))$?
   Different cases have very different answers:

$$C^*_{\mathrm{env}}(\mathcal{T}_+(E)) = \mathcal{O}(E),$$

   but

$$C^*_{\mathrm{env}}(\mathcal{T}_+(\mathrm{SSP}_d)) = \mathcal{T}(\mathrm{SSP}_d) \qquad (d \in \mathbb{N}).$$

   We do not know what $C^*_{\mathrm{env}}(\mathcal{T}_+(\mathrm{SSP}_\infty))$ is.

[2] Pimsner used this to obtain a *KK*-theoretical six-term exact sequence.

Thank you for listening!