Generalized Widder Theorem via fractional moments

This talk is about the first chapter of my thesis.

The classical problem of representability as the Laplace transform: when may a function $f : (0, \infty) \to \mathbb{R}$ be represented in the form

$$(\forall x > 0) \quad f(x) = \int_{-\infty}^{\infty} e^{-xt} d\mu(t),$$

where μ is a positive Borel measure over $\mathbb{R}$?

Widder's Theorem: if and only if f is continuous and of positive type.

Definition 1. A function $f : (0, \infty) \to \mathbb{R}$ is of positive type if for every sequence $x_1, \ldots, x_n \in (0, \infty)$ and a sequence $c_1, \ldots, c_n$ of complex numbers,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} f(x_i + x_j) \geq 0.$$

Widder's result was generalized to the multidimensional case by Akhiezer (1965), Devinatz (1955) and Shucker (1984).

Suppose now that we want to limit the support of the representing measure.
Devinatz characterized the functions representable as the multidimensional Laplace transform, when the support of the representing measure is contained in some multidimensional box.
The Paley-Wiener-Schwartz Theorem enables one to characterize those functions whose representing measure's support lies in a given convex set.

Question: What about more general sets?

The moment problem: given a sequence $(\gamma_n)_{n=0}^{\infty}$ of real numbers, to find a necessary and sufficient condition for the existence of a Borel measure μ such that

$$(n = 0, 1, 2, \ldots) \quad \gamma_n = \int t^n d\mu.$$

The domain of integration might be either $\mathbb{R}$ (the Hamburger moment problem), $[0, \infty)$ (Stieltjes) or a finite interval (Hausdorff).

Definition 2. A sequence $(\gamma_n)_{n=0}^\infty$ is called *positive semi-definite* if for every finite sequence $c_0, \ldots, c_n$ of complex numbers,

$$\sum_{i=0}^{n}\sum_{j=0}^{n} c_i \overline{c_j} \gamma_{i+j} \geq 0.$$

Theorem 3. *The sequence $(\gamma_n)_{n=0}^\infty$ is a Hamburger moment sequence iff it is positive semi-definite.*

Sketch of proof of sufficiency: we define a semi-inner product space over the space of all finite complex sequences by

$$((\alpha_n)_{n=0}^\infty, (\beta_n)_{n=0}^\infty) := \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha_i \overline{\beta_j} \gamma_{i+j}.$$

It is indeed a semi-inner product since γ is positive semi-definite. We make this space an inner product space by dividing by the "null space"

$$\mathcal{N} := \left\{(\alpha_n)_{n=0}^\infty : (\alpha, \alpha) = 0\right\}.$$

This inner product space is, in turn, completed, and we obtain a complex Hilbert space. Then, the operator of right shift

$$T(\alpha_0, \alpha_1, \alpha_2, \ldots) := (0, \alpha_0, \alpha_1, \alpha_2, \ldots)$$

is an (unbounded) symmetric operator, with a selfadjoint extension $S = \int_{\mathbb{R}} tE(dt)$. Finally, if $e = (1, 0, 0, \ldots\ldots)$, then for every $n = 0, 1, 2, \ldots$,

$$\gamma_n = (T^n e, e) = (S^n e, e) = \int_{\mathbb{R}} t^n (E(dt)e, e)$$

and one may take $(E(\cdot)e, e)$ to be μ. $\square$

In the n-dimensional moment problem, γ is now a multi-sequence $(\gamma_\alpha)_{\alpha \in \mathbb{Z}_+^n}$, and the method we have just described doesn't work.

<ins>The solution</ins>: Putinar and Vasilescu used a method of dimensional extension to solve the multidimensional moment problem. In their paper, the moment problem is translated to the problem of representation of a certain linear functional over an algebra of functions.

The bonus in their method is that it enables them to characterize moment sequences, whose representing measures support lies in a given semi-algebraic set.

Fractional moments:

Fix $n \in \mathbb{N}$. Denote by $\mathcal{P}_n$ the algebra of all complex polynomials with n real variables, and by $\mathcal{Q}_n$ the the complex algebra of all "fractional polynomials" of positive rational exponents and n variables. That is, $\mathcal{Q}_n$ is the set of all of the functions in the form $\mathbb{R}_+^n \ni t \mapsto \sum_{\alpha \in \mathbb{Q}_+^n} a_\alpha t^\alpha$, where the a_α's are complex, and differ from zero only for a finite number of indices α.

Let $\mathcal{R}$ be an algebra of complex functions, such that $\overline{f} \in \mathcal{R}$ for all $f \in \mathcal{R}$ (that is, $\mathcal{R}$ is selfadjoint). We say that a linear functional Λ over $\mathcal{R}$ is *positive semi-definite* if $\Lambda(|f|^2) \geq 0$ for each $f \in \mathcal{R}$. When this is the case, one can define the semi-inner product $(f, g) := \Lambda(f\overline{g})$. Thus, if $\mathcal{N} = \{f \in \mathcal{R} : \Lambda(|f|^2) = 0\}$, then $\mathcal{R}/\mathcal{N}$ is an inner-product space. Hence, its completion, $\mathcal{H}$, is a complex Hilbert space.

Let $\mathbb{A}$ be a subsemigroup of $\mathbb{Q}_+^n$. A family of complex numbers $\delta = (\delta_\alpha)_{\mathbb{A}}$ induces the linear functional L_δ over the subalgebra of $\mathcal{Q}_n$ generated by $\{t^\alpha : \alpha \in \mathbb{A}\}$, defined by $L_\delta(t^\alpha) = \delta_\alpha$ for all $\alpha \in \mathbb{A}$. We say that δ is positive semi-definite if the functional L_δ is positive semi-definite.

Fix $p_1, \ldots, p_m \in \mathcal{Q}_n$. For this fixed set of polynomials, let $\theta_p : \mathbb{R}_+^n \to \mathbb{C}$ be defined as

$$\theta_p(t) := (1 + t_1^2 + \ldots + t_n^2 + p_1(t)^2 + \ldots + p_m(t)^2)^{-1}.$$

We denote by $\mathcal{R}$ the complex algebra generated by $\mathcal{Q}_n$ and the function θ_p.

Theorem 4. *Let Λ be a positive semi-definite functional over $\mathcal{R}$. Then there exists a unique representing measure for Λ, whose support is contained in $\mathbb{R}_+^n$. Moreover, if $\Lambda(p_k |r|^2) \geq 0$ for all $r \in \mathcal{R}$, $1 \leq k \leq m$, then the support of that (unique) measure is a subset of $\bigcap_{k=1}^m p_k^{-1}(\mathbb{R}_+)$.*

A set of the form $\bigcap_{k=1}^m p_k^{-1}(\mathbb{R}_+)$ is called a semi-algebraic set.

Sketch of proof: Let $\mathcal{H}$ be the Hilbert generated by Λ. For $1 \le i \le n$, $1 \le j \le m$, we define the operators T_i, P_j over $\mathcal{R}/\mathcal{N}$ by

$$T_i : r + \mathcal{N} \mapsto t_i r + \mathcal{N}, \quad P_j : r + \mathcal{N} \mapsto p_j r + \mathcal{N}.$$

Let B be the operator $B := T_1^2 + \ldots T_n^2 + P_1^2 + \ldots + P_m^2$. Then $B : \mathcal{R}/\mathcal{N} \to \mathcal{R}/\mathcal{N}$ is a positive operator, since for all $r \in \mathcal{R}$, $(Br, r) = \sum_{i=1}^n \Lambda(|t_i r|^2) + \sum_{j=1}^m \Lambda(|p_j r|^2) \ge 0$, by the positivity of Λ. Moreover, $I + B$ is bijective, since for all $r \in \mathcal{R}$, $(I + B)u = r$ for some $u \in \mathcal{R}$ if and only if $u = \theta_p r$. Therefore, B is essentially selfadjoint. Thus, the operators T_i and P_j are essentially selfadjoint for all $1 \le i \le n$, $1 \le j \le m$. Moreover, the selfadjoint operators $A_1 := \overline{T_1}, \ldots, A_n := \overline{T_n}$ commute, and thus have a joint resolution of the identity, E. We used here the following propositions:

Proposition 5. *Let A be a positive densely defined operator in $\mathcal{H}$, such that $AD(A) \subseteq D(A)$. Suppose that $I + A$ is bijective on $D(A)$. Then A is essentially selfadjoint.*

Proposition 6. *Let $T_1, \ldots, T_n$ be symmetric operators in $\mathcal{H}$. Assume that there exist a dense linear space $\mathcal{D} \subseteq \cap_{j,k=1}^n D(T_j T_k)$ such that $T_j T_k x = T_k T_j x$ for all $x \in \mathcal{D}$, $j \ne k$, $j, k = 1, \ldots, n$. If the operator $(T_1^2 + \cdots + T_n^2)_{|\mathcal{D}}$ is essentially selfadjoint, then the operators $T_1, \ldots, T_n$ are essentially selfadjoint, and their canonical closures $\overline{T_1}, \ldots, \overline{T_n}$ commute.*

We return to the proof. As a result, for all $r \in \mathcal{R}$,

$$\Lambda(r) = (r + \mathcal{N}, 1 + \mathcal{N}) = \int_{\mathbb{R}_+^n} r(t) \underbrace{(E(dt)(1 + \mathcal{N}), 1 + \mathcal{N})}_{\mu}.$$

Assume now that $\Lambda(p_k |r|^2) \ge 0$ for all $r \in \mathcal{R}$ and $1 \le k \le m$. This condition is equivalent to the operators $P_1, \ldots, P_m$ being positive. Thus, for all such k, $\overline{P_k}$ is a positive selfadjoint operator. Hence, E is supported by $p_k^{-1}(\mathbb{R}_+)$. Consequently, E is supported by $\cap_{k=1}^m p_k^{-1}(\mathbb{R}_+)$

Definition 7. Let $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{R}_+^n}$ be a family of non-negative numbers.

(1) We say that γ is continuous if the function $\alpha \mapsto \gamma_\alpha$ is continuous (as a function from $\mathbb{R}_+^n$ to $\mathbb{R}_+$).

(2) We say that γ is an *(n-dimensional) fractional moments family* if there exists a positive Borel measure, μ, over $\mathbb{R}_+^n$, such that

$$(\forall \alpha \in \mathbb{R}_+^n) \quad \gamma_\alpha = \int_{\mathbb{R}_+^n} t^\alpha d\mu.$$

Note that this is equivalent to the (multidimensional) Laplace representation

$$(\forall \alpha \in \mathbb{R}_+^n) \quad \gamma_\alpha = \int_{\mathbb{R}^n} e^{-\alpha \cdot s} d\nu(s)$$

obtained by the change of variable $t_i = e^{-s_i}$.

Theorem 8. *Let $\gamma = (\gamma_\alpha)_{\alpha \in \mathbb{R}_+^n}$ be a continuous family of non-negative numbers. Let $p_1, \ldots, p_m \in \mathcal{Q}_n$, $p_k(t) = \sum_{\xi \in I_k} a_{k\xi} t^\xi$ ($I_k \subseteq \mathbb{Q}_+^n$ is finite) for $k = 1, 2, \ldots, m$. Then γ is a fractional moments family with a representing measure whose support is a subset of $\cap_{k=1}^m p_k^{-1}(\mathbb{R}_+)$ if and only if there exists a positive semi-definite family*

$$\delta = (\delta_{(\alpha,\beta)})_{(\alpha,\beta) \in \mathbb{Q}_+^n \times \mathbb{Z}_+}$$

that satisfies:

(1) $\delta_{(\alpha,0)} = \gamma_\alpha$ for all $\alpha \in \mathbb{Q}_+^n$.
(2) $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+1)} + \sum_{j=1}^n \delta_{(\alpha+2e_j,\beta+1)} + \sum_{k=1}^m \sum_{\xi,\eta \in I_k} a_{k\xi} a_{k\eta} \delta_{(\alpha+\xi+\eta,\beta+1)}$ for all $(\alpha,\beta) \in \mathbb{Q}_+^n \times \mathbb{Z}_+$.
(3) The families $\left(\sum_{\xi \in I_k} a_{k\xi} \delta_{(\alpha+\xi,\beta)}\right)_{(\alpha,\beta) \in \mathbb{Q}_+^n \times \mathbb{Z}_+}$ are positive semi-definite for all $k = 1, \ldots, m$.

Moreover, the representing measure of γ (with the properties mentioned above) is unique if and only if the family δ is unique.

Sketch of proof:

Necessity. Assume that γ is a fractional moments family with a representing measure μ, whose support is a subset of $E := \cap_{k=1}^m p_k^{-1}(\mathbb{R}_+)$. We define the family δ by

$$(\forall (\alpha,\beta) \in \mathbb{Q}_+^n \times \mathbb{Z}_+) \quad \delta_{(\alpha,\beta)} := \int_E t^\alpha \theta_p(t)^\beta d\mu.$$

Then δ is a positive semi-definite family, that satisfies (1). (2) is a result of the obvious equality

$$\int_E \left(\theta_p(t)(1+t_1^2+\ldots+t_n^2+p_1(t)^2+\ldots p_m(t)^2)-1\right) t^\alpha \theta_p(t)^\beta d\mu = 0,$$

which is true for all $\alpha \in \mathbb{Q}_+^n$, $\beta \in \mathbb{Z}_+$. Finally, (3) is true since

$$\int_E p_k(t) \left|p(t,\theta_p(t))\right|^2 d\mu \geq 0$$

for all $p \in \widetilde{\mathcal{Q}}_n :=< \mathcal{Q}_n, \mathcal{P} >$, $1 \leq k \leq m$.

Sufficiency. Let δ be as in the theorem's statement, and the algebra $\mathcal{R}$ be the one defined before the last theorem. We define the linear functional Λ over $\mathcal{R}$ by

$$\Lambda(r) = L_\delta(p)$$

for all $r \in \mathcal{R}$, where L_δ is the linear functional induced by δ over $\widetilde{\mathcal{Q}}_n$, and $p \in \widetilde{\mathcal{Q}}_n$ is such that $r(t) = p(t,\theta_p(t))$ for all $t \in \mathbb{R}_+^n$.

Then Λ is well-defined by (2), and it is positive semi-definite, and $\Lambda(p_k\left|r\right|^2) \geq 0$ for all $r \in \mathcal{R}$, $1 \leq k \leq m$ by (3). We then use Theorem 4 and Lebesgue's Dominated Convergence Theorem.

As a concrete demonstration, we have the following immediate corollary of Theorem 8.

Corollary 9. *Denote $F := \{t \in \mathbb{R}_+^2 : t_1^2 \leq t_2\}$. In order for a continuous 2-dimensional family $(\gamma_\alpha)_{\alpha\in\mathbb{R}_+^2}$ of non-negative numbers to be representable in the form*

$$\gamma_\alpha = \int_F t^\alpha d\mu$$

where μ is a non-negative measure over F, it is necessary and sufficient that there exist a positive semi-definite family $(\delta_{(\alpha,\beta)})_{(\alpha,\beta)\in\mathbb{Q}_+^2\times\mathbb{Z}_+}$, such that the following conditions hold:

(1) $\delta_{(\alpha,0)} = \gamma_\alpha$ for all $\alpha \in \mathbb{Q}_+^2$.

(2) $\delta_{(\alpha,\beta)} = \delta_{(\alpha,\beta+1)} + \delta_{(\alpha+2e_1,\beta+1)} + 2\delta_{(\alpha+2e_2,\beta+1)} + \delta_{(\alpha+4e_1,\beta+1)} - 2\delta_{(\alpha+2e_1+e_2,\beta+1)}$ for all $(\alpha,\beta) \in \mathbb{Q}_+^2 \times \mathbb{Z}_+$.

(3) The family $\left(\delta_{(\alpha+e_2,\beta)} - \delta_{(\alpha+2e_1,\beta)}\right)_{(\alpha,\beta)\in\mathbb{Q}_+^2\times\mathbb{Z}_+}$ is positive semi-definite.