SPECTRAL REPRESENTATION OF LOCAL SYMMETRIC SEMIGROUPS OF OPERATORS OVER TOPOLOGICAL GROUPS

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Abstract. We consider local symmetric semigroups of Hilbert space operators. For an open semigroup $\mathfrak{S}$ in some topological group and a dense subsemigroup $\mathfrak{S}'$ of $\mathfrak{S}$, these are semigroups of unbounded selfadjoint operators $(H(t))_{t\in\mathfrak{S}'}$ that admit local continuous extensions to open subsets of $\mathfrak{S}$. We study the possibility to continuously extend $H(\cdot)$ to a semigroup of selfadjoint operators defined for all $t \in \mathfrak{S}$ in several settings. Integral representation formulae for the extended semigroups $(H(t))_{t\in\mathfrak{S}}$ by means of real characters of $\mathfrak{S}$ are established. Our proofs rely on graph limits of selfadjoint operators, commutativity of unbounded operators and semigroup techniques, among others.

Introduction

The research of semigroups of operators in Hilbert space began with the pioneering work of Hille [16, 17] and Sz.-Nagy [27] on the spectral representation of real-indexed semigroups of bounded selfadjoint operators. They proved that such a semigroup $(T(t))_{t>0}$ possessed a positive selfadjoint operator A so that $T(t) = A^t$ for all $t > 0$. The natural generalization to unbounded selfadjoint operators, due to Devinatz [5], followed, stimulating a series of further advances. From this point, however, the road split, and the theory advanced in two, rather distinct, directions.

Several works dealt with semigroups of more general operators, symmetric in some sense, but with indices still in $\mathbb{R}$. Nussbaum [32] analyzed semigroups of densely defined symmetric operators, for which the semigroup property holds on a common, dense domain. Fröhlich [12] and Klein and Landau [24] considered *local* semigroups $(T(t))_{t\geq 0}$ of symmetric operators that need not be densely defined. This means that for every x in a dense linear subspace, $T(t)x$ is defined only when t belongs to a small enough interval $[0, \varepsilon(x))$, depending upon x. They proved the spectral representation $T(t)x = e^{tH}x$ was satisfied by a suitable selfadjoint

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operator H. These results had many applications, and in particular, an impact on mathematical physics. Further progress in this direction includes extensions to multidimensional Euclidean spaces (see Nussbaum [33] and Shucker [39]) and to semigroups whose indices do not necessarily belong to a neighborhood of zero (see the author's recent paper [44]).

Other works were concerned with semigroups of unbounded operators with indices in a more general set. The foundations were laid by Phillips [34], who demonstrated how the theory of Abelian von Neumann algebras might be employed to obtain the Stone Representation Theorem for groups of unitary operators when the group involved was any locally compact Abelian one. Following in his footsteps, Nussbaum utilized this method to prove a representation theorem for semigroups of selfadjoint operators with indices in a locally compact group, first for bounded operators [29], then for unbounded ones [30]. Given a full semigroup $\mathfrak{S}$ in a locally compact group and a semigroup $(T(t))_{t\in\mathfrak{S}}$ of selfadjoint operators, Nussbaum proved the existence of a spectral measure $E(\cdot)$ over the space $\hat{\mathfrak{S}}$ of real continuous characters of $\mathfrak{S}$ satisfying $T(t) = \int_{\hat{\mathfrak{S}}} \chi(t)E(d\chi)$. Ionescu Tulcea [21] showed that a similar theorem holds for semigroups of unbounded normal operators. Related results include papers of Ressel and Ricker [37, 38], where an integral representation is established for semigroups of normal operators over non-topological semigroups; and a recent memoir of Glöckner [14], in which pertinent questions are addressed.

In this paper we study semigroups of operators that are general in both aspects. Let $\mathfrak{S}$ be an open semigroup in a topological group $\mathfrak{G}$. Our basic assumption is that $(H(t))$ is a semigroup of unbounded selfadjoint operators defined for t in a dense subsemigroup $\mathfrak{S}'$ of $\mathfrak{S}$, which is additionally "locally defined"; that is, for each x in a dense linear subspace $\mathcal{D}$ of $\mathcal{H}$, the function $t \mapsto H(t)x$ extends continuously to a suitable arbitrarily small open subset $U(x)$ of $\mathfrak{S}$. Our objective is to examine the possibility to extend $(H(t))$ to a semigroup of selfadjoint operators defined for all $t \in \mathfrak{S}$ that is continuous as a function of t in some sense, and to establish for the extended semigroup a spectral representation, as close as possible to the ones mentioned above.

When tackling such a problem, it is but natural to consult the theory of convergence of nets of operators. The operators we consider are, nonetheless, unbounded, in which case convergence is a far more complicated and subtle issue. Of all contributions to the subject that were known to us, we found useful the theory of graph limits of selfadjoint operators developed by Glimm and Jaffe [13]. Our fundamental idea is to amalgamate graph limits with commutativity of operators. This, in combination with the Abelian von Neumann algebras approach of Phillips and Nussbaum, together with other techniques, enables us to provide an answer to the proposed question in several contexts.

The analysis is carried out under two Standing Hypotheses. The first is the most general we could put together to meet our requirements. It considers semigroups of operators with indices ranging over an open semigroup in an arbitrary topological group $\mathfrak{G}$. In the second one, a stronger integral representation is obtained by means of restricting $\mathfrak{G}$ to the class of locally compact groups, and the incorporation of an additional analytic condition. The last section is dedicated to a comparison of our results to similar ones in the literature.

Spectral commutativity (abbreviated to "commutativity") of unbounded normal operators in Hilbert space is a key component of our discussion. The readers are assumed to be familiar with the elements of this field, which may be found in [31, 35, 43].

1. Preliminaries

Throughout this paper $\mathcal{H}$ denotes an arbitrary complex Hilbert space, and $B(\mathcal{H})$ stands for the algebra of bounded linear operators on $\mathcal{H}$.

We begin with a brief self-contained account of the theory of Abelian von Neumann algebras and algebras of unbounded operators. The interested reader is referred to the original papers listed below and to [22, Ch. 5] for more details.

Let $\mathcal{R}$ be an Abelian C^*-algebra of bounded operators over $\mathcal{H}$. The well-known Gelfand-Neumark Theorem states that $\mathcal{R}$ is isometrically isomorphic to $C(\mathfrak{M})$, the algebra of all complex valued continuous functions over the structure space $\mathfrak{M}$ of $\mathcal{R}$, which is a compact Hausdorff space. For $A \in \mathcal{R}$, let $\hat{A}(\cdot)$ denote its corresponding element of $C(\mathfrak{M})$.

In [41], Stone investigates the space $C_{\mathbb{R}}(X)$ of continuous real functions over a completely regular topological space X that satisfies the property that every non-void subset of $C_{\mathbb{R}}(X)$ which has an upper bound has a *least* upper bound. Suppose now that $\mathcal{R}$ is an Abelian von Neumann algebra. Then this property is indeed satisfied by $X = \mathfrak{M}$ in view of [11, Theorem 1.1], the Gelfand-Neumark Theorem and the fact that if $A, B \in \mathcal{R}$ are selfadjoint, then $A \leq B$ if and only if $\hat{A} \leq \hat{B}$. Since every compact Hausdorff space is completely regular, we infer that $\mathfrak{M}$ has some important properties, including (cf. [41, Theorem 12]):

(1) The clopen subsets of $\mathfrak{M}$ form a basis for it (i.e., $\mathfrak{M}$ is zero-dimensional).
(2) The closure of every open set in $\mathfrak{M}$ is clopen (i.e., $\mathfrak{M}$ is extremally disconnected).

Hence, every Borel subset B of $\mathfrak{M}$ possesses a unique clopen set B^* that differs from B by a set of first category (cf., for example, [11, §2]). Therefore, we can construct a regular spectral measure over the Borel algebra of $\mathfrak{M}$ in the following manner: for each Borel set B in $\mathfrak{M}$, let $\hat{E}_B$ denote the characteristic function of B^*. Since B^* is clopen, $\hat{E}_B \in C(\mathfrak{M})$. Let E_B be its corresponding operator in $\mathcal{R}$. E_B is evidently an orthogonal projection. The family (E_B) forms a regular spectral measure over $\mathfrak{M}$, and for each $A \in \mathcal{R}$, $A = \int_{\mathfrak{M}} \hat{A}(M)E_{dM}$, the integral converging in the norm topology (cf. [11, §4]).

In [11], Fell and Kelley expand the algebra $\mathcal{R}$ to an algebra of unbounded operators $\overline{\mathcal{R}}$, by means of expanding first the algebra $C(\mathfrak{M})$ to unbounded functions. We proceed to describe their method. Let $\overline{C}(\mathfrak{M})$ be the set of all continuous functions from $\mathfrak{M}$ to $\mathbb{C} \cup \{\infty\}$ that are ∞ only on a nowhere dense set.

Theorem 1.1 ([11, p. 594], [28, p. 219])**.** *$\overline{C}(\mathfrak{M})$ is an algebra when we define, for $f, g \in \overline{C}(\mathfrak{M})$, $f + g$ and fg to be the unique continuous functions in $\overline{C}(\mathfrak{M})$ that agree with the sum and product of f and g, respectively, except on a set of first category (see* [11, Theorem 2.2]*).*

It is easily seen that the equations $(f+g)(M) = f(M)+g(M)$ and $(fg)(M) = f(M)g(M)$ hold whenever the right side makes sense by usual arithmetics.

Given $f \in \overline{C}(\mathfrak{M})$ we define the normal (perhaps unbounded) operator $N_f := \int_{\mathfrak{M}} f(M)E_{dM}$ (as usual, $D(N_f) = \left\{x \in \mathcal{H} : \int_{\mathfrak{M}} |f(M)|^2 (E_{dM}x, x) < \infty\right\}$). Let $\overline{\mathcal{R}} := \left\{N_f : f \in \overline{C}(\mathfrak{M})\right\}$. So $\mathcal{R} \subseteq \overline{\mathcal{R}}$ and $C(\mathfrak{M}) \subseteq \overline{C}(\mathfrak{M})$. The following theorem demonstrates the connection between the two expansions:

Theorem 1.2 ([11, Theorems 3.4 and 3.5])**.** *The algebra $\overline{C}(\mathfrak{M})$ is isomorphic to $\overline{\mathcal{R}}$ in the sense that $N_f^* = N_{\overline{f}}$ and N_{f+g}, N_{fg} equal $\overline{N_f + N_g}$, $\overline{N_f N_g}$, respectively, for all $f, g \in \overline{C}(\mathfrak{M})$. Moreover, $N_f \in \overline{\mathcal{R}}$ is invertible if and only if $f^{-1}(0)$ is nowhere dense, and in this case $(N_f)^{-1} = N_{f^{-1}}$.*

As a direct corollary, every two operators in $\overline{\mathcal{R}}$ commute (spectrally) as unbounded normal operators, that is, their spectral measures commute.

Remark 1.3. Suppose that $f \in \overline{C}(\mathfrak{M})$ and N_f is selfadjoint (equivalently, $f(M) \in \mathbb{R}$ for every $M \in \mathfrak{M}$ for which $f(M)$ is finite). By applying [11, Theorem 2.2] on f with the compact space $Y := \overline{\mathbb{R}} = \mathbb{R} \uplus \{+\infty, -\infty\}$ (the extended real numbers) instead of $\mathbb{C} \cup \{\infty\}$, we deduce that f can be regarded as a continuous function from $\mathfrak{M}$ to $\overline{\mathbb{R}}$, that is, when the two infinities are topologically distinguished. To avoid confusion, we shall always use the signed version $+\infty$ of the positive infinity when in the context of $\overline{\mathbb{R}}$.

Proposition 1.4. *If $f, g \in \overline{C}(\mathfrak{M})$ are such that $f^{-1}(0) \cap g^{-1}(\infty) = \emptyset$, then $N_{fg} = N_f N_g$.*

The proof follows routinely from Theorems 1.1 and 1.2, and so its details are omitted.

We continue by introducing the topic of graph limits (see [13, 35, 36]). Various theories have been developed regarding convergence of unbounded selfadjoint operators (e.g. [9, 23, 35]). The deficiency in most theorems that fall into this category is that they *require* the limit operator to be selfadjoint in advance. This is problematic when we do not have a preliminary grasp of this operator. A partial "solution" is given below in the form of graph limits. The importance of the following Proposition 1.6 is that it provides (relatively easily verified) conditions under which the *graph limit of a net of selfadjoint operators exists, and is itself selfadjoint*. Moreover, it supplies us with information on its resolvent.

Definition 1.5. Let $(A_j)_{\mathcal{J}}$ be a net of operators over $\mathcal{H}$. We say that $(x, y) \in \mathcal{H} \times \mathcal{H}$ is in the *(strong) graph limit* Γ^s_∞ of $(A_j)_{\mathcal{J}}$ if there exists a net $(x_j)_{\mathcal{J}}$, $x_j \in D(A_j)$ for all $j \in \mathcal{J}$, such that $x_j \underset{\mathcal{J}}{\to} x$ and $A_j x_j \underset{\mathcal{J}}{\to} y$. If Γ^s_∞ is the graph of an operator A, we say that A is the strong graph limit of the net $(A_j)_{\mathcal{J}}$, and write $A = \mathrm{s.\,g\text{-}lim}_{\mathcal{J}} A_j$.

As indicated in the definition, the graph limit of a net of operators need not necessarily be the graph of an operator. The next result is the strong graph limits version of [36, Theorem X.63] for nets instead of sequences.

Proposition 1.6. *Let $(A_j)_{\mathcal{J}}$ be a net of selfadjoint operators that satisfy:*

(1) $D^s_\infty := \{x \in \mathcal{H} : (\exists y \in \mathcal{H}) \quad (x, y) \in \Gamma^s_\infty\}$ *is dense in $\mathcal{H}$.*
(2) *There exist (bounded) operators $R_\pm$ such that $R(\pm i; A_j) \to R_\pm$ strongly.*

Then Γ^s_∞ is the graph of a (densely defined) selfadjoint operator $A = \mathrm{s.\,g\text{-}lim}_{\mathcal{J}} A_j$, which satisfies $R(\pm i; A) = R_\pm$.

Proof. The methods of [35, Theorem VIII.27] can be employed almost verbatim to establish that Γ^s_∞ is the graph of a (densely defined) symmetric operator A.

It is therefore sufficient to prove that $\pm i \in \rho(A)$ and $R(\pm i; A) = R_\pm$. Let $x \in \mathcal{H}$. For all $j \in \mathcal{J}$, denote $y_j := R(\pm i; A_j)x$, that is, $x = (\pm iI - A_j)y_j$. Since $y_j \to R_\pm x$ by (2), we have $A_j y_j \to \pm i R_\pm x - x$. Thus $R_\pm x \in D(A)$ and $AR_\pm x = \pm i R_\pm x - x$, hence $(\pm iI - A)R_\pm = I$.

As A is symmetric, the operators $\pm iI - A$ are injective. From the last equality, they are also surjective, and so $R(\pm i; A) = R_\pm$, as desired. □

The most fundamental case in which D^s_∞ is dense is when $(A_j)_\mathcal{J}$ converges strongly on a dense linear subspace of $\mathcal{H}$. But even this is generally not sufficient for Condition (2) of Proposition 1.6 to hold. However, if one additionally assumes that the operators $(A_j)_\mathcal{J}$ commute pairwise, then this condition is satisfied indeed.

Theorem 1.7. *Let $(A_j)_\mathcal{J}$ be a net of pairwise commuting selfadjoint operators that admit a dense linear subspace $\mathcal{D}$ of $\mathcal{H}$ such that $\mathcal{D} \subseteq \bigcap_{j\in\mathcal{J}} D(A_j)$, and $\lim_\mathcal{J} A_j x$ exists for every x in $\mathcal{D}$. Then the conditions of Proposition 1.6 are satisfied, and $\operatorname{s.g-lim}_\mathcal{J} A_j$ commutes with A_j for every $j \in \mathcal{J}$.*

Proof. We assert that the conditions of Proposition 1.6 hold. As mentioned above, the theorem's assumptions yield easily that Condition (1) is satisfied. As for Condition (2), let $j_1, j_2 \in \mathcal{J}$ be given. Since A_{j_1}, A_{j_2} commute, we have

$$R(i; A_{j_1})x - R(i; A_{j_2})x = R(i; A_{j_1})R(i; A_{j_2})(A_{j_2} - A_{j_1})x \tag{1.1}$$

for all $x \in D(A_{j_1}) \cap D(A_{j_2})$. This is true, in particular, when $x \in \mathcal{D}$. Fix such x. Since $\lim_\mathcal{J} A_j x$ exists, the net $(A_j x)_{j\in\mathcal{J}}$ is a Cauchy net. Thus, since $\|R(i; A_j)\| \le 1$ for all $j \in \mathcal{J}$ (the operators $(A_j)_\mathcal{J}$ being selfadjoint), the net $(R(i; A_j)x)_{j\in\mathcal{J}}$ is also Cauchy by virtue of (1.1). As already stated, the family $\{R(i; A_j) : j \in \mathcal{J}\}$ is uniformly bounded and $\mathcal{D}$ is dense, whence $(R(i; A_j)y)_{j\in\mathcal{J}}$ is a Cauchy net for *all* $y \in \mathcal{H}$. Therefore, the net $(R(i; A_j))_{j\in\mathcal{J}}$ converges strongly to a bounded operator R_+. Similarly, $(R(-i; A_j))_{j\in\mathcal{J}}$ converges strongly to R_-. From Proposition 1.6, $A := \operatorname{s.g-lim}_\mathcal{J} A_j$ is a (well-defined) selfadjoint operator, and $R(\pm i; A) = R_\pm$. For every two indices $j, j' \in \mathcal{J}$, $R(i; A_j)R(i; A_{j'}) = R(i; A_{j'})R(i; A_j)$ due to the commutativity of A_j, $A_{j'}$. Hence $R(i; A_j)R(i; A) = R(i; A)R(i; A_j)$, thus A commutes with A_j. □

2. First hypothesis

The term "semigroup of unbounded operators" can have several different meanings. The following is our definition. See §5 for comparison with others.

Definition 2.1. Let $(\mathfrak{S}, \cdot)$ be a semigroup. A family $(H(t))_{t\in\mathfrak{S}}$ of (generally unbounded) selfadjoint operators is called a *semigroup of selfadjoint operators over* $\mathfrak{S}$ if for every $t, s \in \mathfrak{S}$, the operators $H(t)$ and $H(s)$ commute, and

$$\overline{H(t)H(s)} = H(ts). \tag{2.1}$$

An immediate implication of the definition is that

$$H(ts) = \overline{H(t)H(s)} = \overline{H(s)H(t)} = H(st) \tag{2.2}$$

for all $t, s \in \mathfrak{S}$, even though $\mathfrak{S}$ is not required to be Abelian.

Let $\mathfrak{G}$ be a topological group with identity e (all topological spaces are tacitly assumed to be Hausdorff throughout). We shall henceforth assume that $\mathfrak{S}$ is an open semigroup in $\mathfrak{G}$ that admits a dense subsemigroup $\mathfrak{S}'$, which may be written as $\mathfrak{S}' = \mathfrak{S} \cap \mathfrak{G}'$ for a suitable subgroup $\mathfrak{G}'$ of $\mathfrak{G}$.

The purpose of this work is to study various settings in which a semigroup of selfadjoint operators over $\mathfrak{S}'$ that is also locally defined in $\mathfrak{S}$ may be extended "continuously" to a semigroup of selfadjoint operators over $\mathfrak{S}$. Our first such setting is the next hypothesis, where $\mathfrak{A}'$ is to be intuitively understood as topologically "small" (on first reading, take $\mathfrak{A}' = \{e\}$).

Standing Hypothesis 2.2. Suppose that $\mathfrak{A}' \subseteq \overline{\mathfrak{S}} \cap \mathfrak{G}'$ is a semigroup, and $(H(t))_{\mathfrak{S}' \cup \mathfrak{A}'}$ is a semigroup of selfadjoint operators that is "locally defined" in the sense that there exists a dense linear subspace $\mathcal{D}$ of $\mathcal{H}$ satisfying:

- (I) For every $x \in \mathcal{D}$ there exists an open neighborhood $V(x)$ of $\mathfrak{A}'$ in $\mathfrak{G}$, such that if $U(x) := V(x) \cap \mathfrak{S}$, then $x \in D(H(t))$ for all $t \in U(x) \cap \mathfrak{S}'$ and the function $U(x) \cap \mathfrak{S}' \ni t \mapsto H(t)x$ extends to a continuous function on $U(x)$.

Assume moreover that:

- (II) There exists a countable subset $\mathfrak{D}$ of $\mathfrak{S}' \cup \mathfrak{A}'$ for which the equality $\bigcap_{t \in \mathfrak{S}' \cup \mathfrak{A}'} D(H(t)) = \bigcap_{t \in \mathfrak{D}} D(H(t))$ is satisfied.

If $e \notin \mathfrak{A}'$ then we require, in addition, that there exist a pairwise orthogonal family $(P_k)_{k \in \mathcal{K}}$ of orthogonal projections, such that:

- (III) P_k and $H(t)$ commute for every $k \in \mathcal{K}$ and $t \in \mathfrak{S}' \cup \mathfrak{A}'$.
- (IV) For each $k \in \mathcal{K}$ there exists $a_k \in \mathfrak{A}'$ such that the restriction of $H(a_k)$ to its reducing subspace $P_k \mathcal{H}$ has a bounded inverse.
- (V) $H(t)(I - \sum_{k \in \mathcal{K}} P_k) = 0$ for all $t \in \mathfrak{S}'$.

Remark 2.3.

(1) Note that $\mathfrak{S}' \cup \mathfrak{A}'$ is indeed a semigroup: $\mathfrak{S}$ is an open semigroup, and so $\overline{\mathfrak{S}} \cdot \mathfrak{S} = \mathfrak{S} \cdot \mathfrak{S} \subseteq \mathfrak{S}$. As a result, if $t \in \mathfrak{S}'$ and $a \in \mathfrak{A}'$, then $ta \in \mathfrak{S}$. Since also $ta \in \mathfrak{G}'$, we obtain $ta \in \mathfrak{S} \cap \mathfrak{G}' = \mathfrak{S}'$. The same is true for at.
(2) Condition (II) is essentially [21, (13)]. It is satisfied, e.g., when $\mathfrak{G}'$ is countable.
(3) $\mathcal{K}$ is not assumed to be countable.

Of special interest is the case in which $\mathfrak{A}' = \{e\}$ (this can only happen if $e \in \overline{\mathfrak{S}}$). The family $H(\cdot)$ is then required to be locally defined only close to e. Nevertheless, the majority of the proof of Theorem 2.6 is still indispensable.

We introduce some notation to be used in the sequel. Given $x \in \mathcal{D}$, denote the extension of $t \mapsto H(t)x$ to $U(x)$ by $t \mapsto T(t)x$ (see (I)). In doing so we obtain a family $(T(t))_{t\in\mathfrak{S}}$ of linear operators, such that if $x \in \mathcal{D}$, then the function $t \mapsto T(t)x$ is (well-defined and) continuous on $U(x)$, and $T(t)x = H(t)x$ for all $t \in U(x) \cap \mathfrak{S}'$. However, for general $t \in \mathfrak{S}$, the domain $D(T(t))$ *need not be dense in* $\mathcal{H}$, and it might even be $\{0\}$.

Definition 2.4 (compare with Definitions 3.1, 3.2 below)**.** An extended real valued function χ over $\mathfrak{S}$ is called an *extended real character* of $\mathfrak{S}$ if $\chi(t)\chi(s) = \chi(ts)$ for all $t, s \in \mathfrak{S}$ for which this multiplication makes sense by usual arithmetics (i.e., if $\chi(t), \chi(s)$ are either both finite or both nonzero). The set of all extended real characters of $\mathfrak{S}$, endowed with the topology of pointwise convergence, will be denoted by $\mathfrak{S}^{*,\mathrm{nc}}_{\infty}$.

Remark 2.5. Elements of $\mathfrak{S}^{*,\mathrm{nc}}_{\infty}$ are *not* required to be continuous over $\mathfrak{S}$ (the "nc" superscript stands for "not necessarily continuous"). Furthermore, $\mathfrak{S}^{*,\mathrm{nc}}_{\infty}$ is a compact Hausdorff space, for it may be identified with a closed subspace of the topological space $\overline{\mathbb{R}}^{\mathfrak{S}}$, which is compact owing to Tychonoff's Theorem.

We are ready to state the main theorem of this section.

Theorem 2.6. *Under Standing Hypothesis 2.2, the semigroup* $(H(t))_{t\in\mathfrak{S}'}$ *extends to a semigroup* $(H(t))_{t\in\mathfrak{S}}$ *of selfadjoint operators (see Definition 2.1), in such a way that*

$$T(t)x = H(t)x \tag{2.3}$$

for all $x \in \mathcal{D}, t \in U(x)$*. There exists a unique regular spectral measure* $E(\cdot)$ *over* $\mathfrak{S}^{*,\mathrm{nc}}_{\infty}$ *such that* $\chi(t)$ *is finite* $E(d\chi)$*-a.e. and*

$$H(t) = \int_{\mathfrak{S}^{*,\mathrm{nc}}_{\infty}} \chi(t)E(d\chi) \tag{2.4}$$

for all $t \in \mathfrak{S}$. Moreover, $(H(t))_{t\in\mathfrak{S}}$ is "continuous" in the following sense: there exists a dense linear subspace $\mathcal{D}'$ of $\mathcal{H}$ such that $\mathcal{D}' \subseteq \bigcap_{t\in\mathfrak{S}} D(H(t))$ and $t \mapsto H(t)x$ is continuous over $\mathfrak{S}$ for all $x \in \mathcal{D}'$.

Remark 2.7. It is perhaps more convenient for some purposes that conditions (III)-(V) (and their preface) may be reformulated to guarantee the existence of a family $(P'_k)_{k\in\mathcal{K}}$ of pairwise commuting orthogonal projections, not necessarily pairwise orthogonal, satisfying the mentioned conditions with P_k replaced by P'_k and $\sum_{k\in\mathcal{K}} P_k$ replaced by $\bigvee_{k\in\mathcal{K}} P'_k$. The two formulations are equivalent (to derive the first from the second, cast a well-order relation on $\mathcal{K}$, and define $(P_k)_{k\in\mathcal{K}}$ by $P_k := P'_k(I - \bigvee_{l<k} P'_l)$).

A few words are in order about the theorem's assumptions, being mostly not only sufficient, but also necessary in a sense, for the stated results to hold. Condition (I) is natural in light of the extended operator semigroup $(H(t))_{t\in\mathfrak{S}}$ satisfying (2.3). As for (III)-(V), assume that (2.4) holds. Denote by χ_0 the zero character on $\mathfrak{S}$. Let $\mathcal{L} := \{(\chi, t) : \chi \in \mathfrak{S}^{*,\mathrm{nc}}_\infty, t \in \mathfrak{S}, \chi(t) \neq 0\}$, and for $(\chi, t) \in \mathcal{L}$ write

$$A_{(\chi,t)} := \begin{cases} \{\xi \in \mathfrak{S}^{*,\mathrm{nc}}_\infty : |\xi(t)| > |\chi(t)|/2\}, & \text{if } |\chi(t)| \neq \infty \\ \{\xi \in \mathfrak{S}^{*,\mathrm{nc}}_\infty : |\xi(t)| > 1\}, & \text{if } |\chi(t)| = \infty. \end{cases}$$

Each of the sets $A_{(\chi,t)}$ is open in $\mathfrak{S}^{*,\mathrm{nc}}_\infty$ and disjoint from $\{\chi_0\}$, and their union equals $\mathfrak{S}^{*,\mathrm{nc}}_\infty \backslash \{\chi_0\}$. For any subset $\mathcal{K}$ of $\mathcal{L}$ with the property that the union $\bigcup_{(\chi,t)\in\mathcal{K}} A_{(\chi,t)}$ is also $\mathfrak{S}^{*,\mathrm{nc}}_\infty \backslash \{\chi_0\}$, define $P'_k := E(A_k)$ for $k \in \mathcal{K}$ and $Q := E(\{\chi_0\})$. Then the orthogonal projections $(P'_k)_{k\in\mathcal{K}}$ commute pairwise and also with $H(t)$ for all $t \in \mathfrak{S}$. The regularity of $E(\cdot)$ implies that $Q = I - \bigvee_{k\in\mathcal{K}} P'_k$. Surely $H(t)Q = 0$, and for every $k = (\chi_k, a_k) \in \mathcal{K}$ with $P'_k \neq 0$, the operator $H(a_k)_{|P'_k\mathcal{H}}$ possesses a bounded inverse. Observe that we chose our elements a_k from $\mathfrak{S}$, while in Standing Hypothesis 2.2 it was permitted to choose them from $\overline{\mathfrak{S}}$.

Let us outline the course of the proof of Theorem 2.6 before going into the details. We construct an Abelian von Neumann algebra $\mathcal{R}$ that contains the spectral projections of all the operators $H(t)$, $t \in \mathfrak{S}' \cup \mathfrak{A}'$. This algebra is extended to an algebra $\overline{\mathcal{R}}$ of unbounded normal operators, as described in §1. $\mathcal{R}$ is used to procure a family of orthogonal projections with least upper bound I, each of which reduces each of the operators $H(t)$ to a bounded one. Using these projections and Theorem 1.7, we extend $(H(t))_{t\in\mathfrak{S}'\cup\mathfrak{A}'}$ to a family of commuting selfadjoint operators $(H(t))_{t\in\mathfrak{S}\mathfrak{A}'\cup\mathfrak{A}'\mathfrak{S}}$, which satisfies the semigroup property, equation (2.3) and the "strong continuity" property mentioned in the statement of the theorem. Conditions (III)-(V) are then employed to further extend $(H(t))_{t\in\mathfrak{S}\mathfrak{A}'\cup\mathfrak{A}'\mathfrak{S}}$ to a semigroup of selfadjoint operators $(H(t))_{t\in\mathfrak{S}}$. The spectral representation (2.4) is obtained as a consequence of the integral representation formula of elements of $\overline{\mathcal{R}}$ and the semigroup property.

The first extension of $(H(t))_{t\in\mathfrak{S}'}$, namely to the open subset $\mathfrak{S}\mathfrak{A}'\cup\mathfrak{A}'\mathfrak{S}$ of $\mathfrak{S}$, is only feasible by virtue of the family $H(\cdot)$ being locally defined in $\mathfrak{S}$ close to elements of $\mathfrak{A}'$ (Condition (I)). This is the principal role of $\mathfrak{A}'$ in Standing Hypothesis 2.2. The definition of $H(t)$ is thus technically confined to the case when t is a multiplication of an element of $\mathfrak{A}'$ by an element of $\mathfrak{S}$, which is the reason we need to go through $\mathfrak{S}\mathfrak{A}'\cup\mathfrak{A}'\mathfrak{S}$ in our way to eventually extend $H(\cdot)$ to all of $\mathfrak{S}$.

Proof of Theorem 2.6. Following [34, 29, 30], we denote by $F_t(\cdot)$ the spectral measure of $H(t)$ for $t\in\mathfrak{S}'\cup\mathfrak{A}'$. Let $\mathcal{R}$ be any Abelian von Neumann algebra in $B(\mathcal{H})$ that contains $F_t(C)$ for all Borel subsets C of $\mathbb{R}$ and $t\in\mathfrak{S}'\cup\mathfrak{A}'$. Such $\mathcal{R}$ necessarily exists since $F_t(\cdot)$ commutes with $F_s(\cdot)$ when $t,s\in\mathfrak{S}'\cup\mathfrak{A}'$ (see Definition 2.1), and the results of §1 apply to it. In particular, $\overline{\mathcal{R}}\cong\overline{C}(\mathfrak{M})$ where $\mathfrak{M}$ is the structure space of $\mathcal{R}$ (cf. Theorems 1.1 and 1.2). As proved in [30, Theorem 4], $H(t)\in\overline{\mathcal{R}}$ for all $t\in\mathfrak{S}'\cup\mathfrak{A}'$. Let f_t be the element of $\overline{C}(\mathfrak{M})$ corresponding to $H(t)$ (i.e., $N_{f_t}=H(t)$), and write $S_t:=f_t^{-1}(\infty)$ (which is nowhere dense in $\mathfrak{M}$). Consider the set $S:=\bigcup_{t\in\mathfrak{D}}S_t$. $\mathfrak{D}$ is countable by our assumption, hence S is of the first category. Accordingly, S is in fact *nowhere dense* in $\mathfrak{M}$ (see [30, Theorem 5], originally from [3, p. 65, b], [42, pp. 187-188]). Fix $M\in S^c$. There exists a clopen subset σ of $\mathfrak{M}$ such that $M\in\sigma$ and $\sigma\cap S=\emptyset$. For this σ we have, using (II), $E_\sigma\mathcal{H}\subseteq\bigcap_{t\in\mathfrak{D}}D(H(t))=\bigcap_{t\in\mathfrak{S}'\cup\mathfrak{A}'}D(H(t))$. Consequently, $\sigma\subseteq\bigcap_{t\in\mathfrak{S}'\cup\mathfrak{A}'}S_t^c$. Since $M\in\sigma$, we infer that $S=\bigcup_{t\in\mathfrak{S}'\cup\mathfrak{A}'}S_t$. Let

$$\mathcal{O}:=\{\sigma:\sigma\text{ is clopen in }\mathfrak{M}\text{ and }\sigma\cap S=\emptyset\}.$$

If $\sigma\in\mathcal{O}$, then by the foregoing, $H(t)E_\sigma$ is bounded for all $t\in\mathfrak{S}'\cup\mathfrak{A}'$. Thus $(H(t)E_\sigma)_{t\in\mathfrak{S}'\cup\mathfrak{A}'}$ is a semigroup of bounded selfadjoint operators by virtue of (2.1) (as $H(t)E_\sigma H(s)E_\sigma\subseteq H(ts)E_\sigma$, and equality holds since both sides are everywhere defined).

We have already commented (see Remark 2.3) that $\mathfrak{S}\mathfrak{A}',\mathfrak{A}'\mathfrak{S}\subseteq\mathfrak{S}$. Let $s\in\mathfrak{S}\mathfrak{A}'$, $x\in\mathcal{D}$ and $\sigma\in\mathcal{O}$ be given. Fix $t\in\mathfrak{S}$, $a\in\mathfrak{A}'$ so that $s=ta$. Since $a\in\overline{\mathfrak{S}}$ and $U(x)$ is the intersection of $\mathfrak{S}$ with a neighborhood of a (cf. (I)), we have $a\in\overline{U(x)}$. Consequently, t belongs to the closure of the open set $taU(x)^{-1}$. In addition, $t\in\mathfrak{S}$ which is open, therefore $(taU(x)^{-1})\cap\mathfrak{S}\neq\emptyset$. As a result, it is possible to select an element t_0 of $\mathfrak{S}'$ that belongs to $taU(x)^{-1}$ (recall that $\mathfrak{S}'$ is dense in $\mathfrak{S}$), that is, $t_0^{-1}ta\in U(x)$. Hence $H(t_0)E_\sigma$ is bounded and $x \in D(T(t_0^{-1}ta))$, so that one may intuitively define

$$\tilde{H}(s)E_\sigma x := H(t_0)E_\sigma T(t_0^{-1}ta)x. \tag{2.5}$$

In particular, if $s = ta$ itself belongs to $\mathfrak{S}'$, so does $t_0^{-1}ta$ (seeing that $t_0^{-1}ta \in \mathfrak{S}$, and the equality $\mathfrak{S}' = \mathfrak{S} \cap \mathfrak{G}'$ holds for the subgroup $\mathfrak{G}'$ of $\mathfrak{G}$). Thus (I) and the definition of $T(\cdot)$ yield that

$$\begin{aligned} H(t_0)E_\sigma T(t_0^{-1}ta)x &= H(t_0)E_\sigma H(t_0^{-1}ta)x \\ &= H(t_0)H(t_0^{-1}ta)E_\sigma x = H(s)E_\sigma x. \end{aligned} \tag{2.6}$$

The above definition does not depend on the specifically chosen t, a or t_0, since by virtue of (I) we have

$$\begin{aligned} H(t_0)E_\sigma T(t_0^{-1}ta)x &= H(t_0)E_\sigma(\lim_{\substack{t'\to t\\ t'\in\mathfrak{S}'}} T(t_0^{-1}t'a)x) \\ &= \lim H(t_0)E_\sigma T(t_0^{-1}t'a)x \\ &= \lim H(t_0)E_\sigma H(t_0^{-1}t'a)x \\ &= \lim H(t_0)H(t_0^{-1}t'a)E_\sigma x \\ &= \lim H(t'a)E_\sigma x, \end{aligned} \tag{2.7}$$

all limits being the same as the first one. This limit is well-defined since $\mathfrak{S}'$ is dense in $\mathfrak{S}$. We used here the fact that $t_0^{-1}t'a$ belongs to $U(x) \cap \mathfrak{S}'$ for $t' \in \mathfrak{S}'$ close enough to t (as $\mathfrak{S}' = \mathfrak{S} \cap \mathfrak{G}'$). Notice that in particular, the limits in (2.7) necessarily exist. In a similar fashion, an element of $\mathfrak{S}'$ is "close" to s if and only if it may be expressed as $t'a$, for t' "close" to t in $\mathfrak{S}'$. Hence (2.7) becomes

$$\tilde{H}(s)E_\sigma x = \lim_{\substack{s'\to s\\ s'\in\mathfrak{S}'}} H(s')E_\sigma x, \tag{2.8}$$

which is valid for all $s \in \mathfrak{S}\mathfrak{A}'$, $x \in \mathcal{D}$ and $\sigma \in \mathcal{O}$.

We wish to extend our definition in (2.5) to a selfadjoint operator $\tilde{H}(s)$. Let us define, for $s \in \mathfrak{S}\mathfrak{A}'$,

$$\tilde{H}(s) := \operatorname*{s.\,g\text{-}lim}_{\substack{s'\to s\\ s'\in\mathfrak{S}'}} H(s'). \tag{2.9}$$

The fact that S is nowhere dense in $\mathfrak{M}$, [11, Theorem 1.1] and the Gelfand-Neumark Theorem imply that $E_\sigma \xrightarrow[\sigma\in\mathcal{O}]{} I$ in the strong operator topology, thus

$$\bigcup_{\sigma\in\mathcal{O}} E_\sigma\mathcal{D} \text{ is dense in } \mathcal{H}. \tag{2.10}$$

The requirements of Theorem 1.7 are consequently satisfied by (2.8) and (2.10). The operator $\tilde{H}(s)$, as defined in (2.9), is therefore a well-defined selfadjoint operator, and $R(\pm i;\tilde{H}(s)) = \text{s-lim}_{\substack{s'\to s\\ s'\in\mathfrak{S}'}} R(\pm i; H(s'))$. Since $R(\pm i; H(s')) \in \mathcal{R}$ for all $s' \in \mathfrak{S}'$, and since $\mathcal{R}$ is strongly closed, we deduce that $R(\pm i;\tilde{H}(s)) \in \mathcal{R}$. Consequently, $\tilde{H}(s) \in \overline{\mathcal{R}}$ for all $s \in \mathfrak{S}\mathfrak{A}'$ by Theorem 1.2, and the unbounded selfadjoint operators $\{\tilde{H}(s) : s \in \mathfrak{S}\mathfrak{A}'\}$ commute pairwise. We have already proved (cf. (2.6)) that for $s \in \mathfrak{S}\mathfrak{A}' \cap \mathfrak{S}'$, $\tilde{H}(s)$ and $H(s)$ agree on the dense linear subspace defined in (2.10). Since they both belong to $\overline{\mathcal{R}}$, they commute, and so, by [32, Proposition 1], they are equal. Henceforth, we shall consequently write $H(s)$ instead of $\tilde{H}(s)$ for all $s \in \mathfrak{S}\mathfrak{A}'$, and denote by $f_s(\cdot)$ the element of $\overline{C}(\mathfrak{M})$ that corresponds to $H(s) \in \overline{\mathcal{R}}$.

Having $H(s)$ defined for $s \in \mathfrak{S}\mathfrak{A}'$, we assert that the strong graph limit in (2.9) is not restricted to nets in $\mathfrak{S}'$. Indeed, fix $s \in \mathfrak{S}\mathfrak{A}'$, and let $(s_j)_{\mathcal{J}}$ be any net in $\mathfrak{S}\mathfrak{A}'$ that converges to s. The operators $(H(s_j))_{\mathcal{J}}$ commute pairwise. Additionally, (2.5) implies that $H(s)E_\sigma x$ is a *continuous* function of s in $\mathfrak{S}\mathfrak{A}'$ for fixed $x \in \mathcal{D}$ and $\sigma \in \mathcal{O}$. Therefore, from Theorem 1.7, $\text{s. g-lim}_{\mathcal{J}} H(s_j)$ is a well-defined selfadjoint operator, which agrees with $H(s)$ on a dense linear subspace. As in the preceding paragraph, $\text{s. g-lim}_{\mathcal{J}} H(s_j) \in \overline{\mathcal{R}}$. [32, Proposition 1] consequently yields that $\text{s. g-lim}_{\mathcal{J}} H(s_j) = H(s)$.

The foregoing construction may be repeated for elements $s \in \mathfrak{A}'\mathfrak{S}$, in such a way that $H(s)$ possess all of the features that are proved above for elements of $\mathfrak{S}\mathfrak{A}'$. Moreover, by (2.2) and (2.9) (see also the passage from (2.7) to (2.8)),

$$H(ta) = H(at) \tag{2.11}$$

for all $t \in \mathfrak{S}$ and $a \in \mathfrak{A}'$.

Our next task is to prove that the family $(H(s))_{s\in\mathfrak{S}\mathfrak{A}'\cup\mathfrak{A}'\mathfrak{S}}$ satisfies the following version of the semigroup law. Let $s_1 \in \mathfrak{A}'\mathfrak{S}$, $s_2 \in \mathfrak{S}\mathfrak{A}'$ be given. Since $H(s_1)$ and $H(s_2)$ are commuting selfadjoint operators in $\overline{\mathcal{R}}$, the operator $\overline{H(s_1)H(s_2)}$ is selfadjoint and belongs to $\overline{\mathcal{R}}$ (cf. Theorem 1.2). On the other hand, $s_1s_2 \in \mathfrak{A}'\mathfrak{S}$ and $H(s_1s_2)$ also belongs to $\overline{\mathcal{R}}$, thus it commutes with $\overline{H(s_1)H(s_2)}$. To prove that they are equal, we use [32, Proposition 1]. Indeed, for $x \in \mathcal{D}$ and $\sigma \in \mathcal{O}$, from $(H(t)E_\sigma)_{t\in\mathfrak{S}'\cup\mathfrak{A}'}$ being a semigroup of bounded operators we infer that

$$H(s_1s_2)E_\sigma x \overset{(1)}{=} \lim_{\substack{s_1'\to s_1\\ s_1'\in\mathfrak{S}'}} \lim_{\substack{s_2'\to s_2\\ s_2'\in\mathfrak{S}'}} H(s_1's_2')E_\sigma x \qquad = \lim_{\substack{s_1'\to s_1\\ s_1'\in\mathfrak{S}'}} \lim_{\substack{s_2'\to s_2\\ s_2'\in\mathfrak{S}'}} H(s_1')E_\sigma H(s_2')E_\sigma x$$

$$= \lim_{\substack{s_1' \to s_1 \\ s_1' \in \mathfrak{S}'}} H(s_1')E_\sigma(\lim_{\substack{s_2' \to s_2 \\ s_2' \in \mathfrak{S}'}} H(s_2')E_\sigma x) \overset{(2)}{=} \lim_{\substack{s_1' \to s_1 \\ s_1' \in \mathfrak{S}'}} H(s_1')E_\sigma H(s_2)E_\sigma x$$

$$= \lim_{\substack{s_1' \to s_1 \\ s_1' \in \mathfrak{S}'}} H(s_1')H(s_2)E_\sigma x \qquad \overset{(3)}{=} H(s_1)H(s_2)E_\sigma x.$$

Note that equalities (1) and (2) are true since the function $s \mapsto H(s)E_\sigma x$ is continuous on $\mathfrak{S}\mathfrak{A}' \cup \mathfrak{A}'\mathfrak{S}$, and equality (3) holds in view of (2.9). By (2.10) we conclude that

$$(\forall s_1 \in \mathfrak{A}'\mathfrak{S}, s_2 \in \mathfrak{S}\mathfrak{A}') \quad H(s_1 s_2) = \overline{H(s_1)H(s_2)}. \tag{2.12}$$

Very similarly, if $a \in \mathfrak{A}'$, $s \in \mathfrak{A}'\mathfrak{S}$, $x \in \mathcal{D}$ and $\sigma \in \mathcal{O}$, we have

$$H(sa)E_\sigma x = \lim_{\substack{s' \to s \\ s' \in \mathfrak{S}'}} H(s'a)E_\sigma x = \lim_{\substack{s' \to s \\ s' \in \mathfrak{S}'}} H(s')H(a)E_\sigma x = H(s)H(a)E_\sigma x,$$

so we obtain

$$(\forall a \in \mathfrak{A}', s \in \mathfrak{A}'\mathfrak{S}) \quad H(sa) = \overline{H(s)H(a)} = \overline{H(a)H(s)}. \tag{2.13}$$

If $e \in \mathfrak{A}'$ then $\mathfrak{S}\mathfrak{A}' = \mathfrak{S} = \mathfrak{A}'\mathfrak{S}$, and one may take $\mathcal{D}'$ to be the linear subspace of $\mathcal{H}$ that appears in (2.10).

Suppose otherwise. Then (III)-(V) hold. From (III), $\mathcal{R}$ could have been chosen such that $\{P_k\}_{k \in \mathcal{K}} \subseteq \mathcal{R}$, and so will be assumed hereinafter. Write $Q := I - \sum_{k \in \mathcal{K}} P_k$ (which necessarily belongs to $\mathcal{R}$ as the latter is a von Neumann algebra). For $k \in \mathcal{K}$, let σ_k be the clopen set in $\mathfrak{M}$ so that the indicator I_{σ_k} is the element of $C(\mathfrak{M})$ corresponding to P_k. Similarly, let σ_* be the clopen set in $\mathfrak{M}$ such that I_{σ_*} corresponds to Q. Then $(\sigma_k)_{\mathcal{K} \uplus \{*\}}$ is a family of mutually disjoint clopen subsets of $\mathfrak{M}$, and one verifies easily that (IV) is equivalent to the function f_{a_k} being nonzero on the (compact) subset σ_k of $\mathfrak{M}$ for every $k \in \mathcal{K}$. By (V), (2.9) and (2.11) we infer that $H(s)Q = 0$ for all $s \in \mathfrak{S}\mathfrak{A}' \cup \mathfrak{A}'\mathfrak{S}$, that is, f_s is identically zero on σ_* for all such s. Let S_1 denote the complement in $\mathfrak{M}$ of the open set $\biguplus_{k \in \mathcal{K} \uplus \{*\}} \sigma_k$. Then [11, Theorem 1.1], the extremal disconnectedness of $\mathfrak{M}$ and the definition of Q imply that S_1 is nowhere dense in $\mathfrak{M}$. Consequently, $S_2 := S \cup S_1$ is also nowhere dense. Fix $t \in \mathfrak{S}$. We shall construct a function $f_t \in \overline{C}(\mathfrak{M})$ in the following manner. Define $f_t(M)$ to be zero when $M \in \sigma_*$. For every $M \in S_2^c \backslash \sigma_*$ there exists a unique $k \in \mathcal{K}$ such that $M \in \sigma_k$. Moreover, $f_{a_k}(M) \notin \{0, \infty\}$, and we may define

$$f_t(M) := f_{ta_k}(M) f_{a_k}(M)^{-1} = f_{a_k t}(M) f_{a_k}(M)^{-1} \tag{2.14}$$

(cf. (2.11)). Since $f_{ta_k} \in \overline{C}(\mathfrak{M})$ and σ_k is open for every $k \in \mathcal{K} \uplus \{*\}$, f_t is a continuous function from S_2^c to $\mathbb{R} \cup \{\infty\}$. We shall require the following simple lemma.

Lemma 2.8. *If* $(\theta_l)_{l\in\mathcal{L}}$ *is a family of nowhere dense subsets of a topological space* X, *and* $(\tau_l)_{l\in\mathcal{L}}$ *is a family of mutually disjoint open subsets of* X, *then* $\upsilon := \biguplus_{l\in\mathcal{L}}(\theta_l \cap \tau_l)$ *is also nowhere dense in* X.

Proof. Put $\tau := \biguplus_{l\in\mathcal{L}} \tau_l$. Since the sets τ_l are open, $\overline{\upsilon} \subseteq \biguplus_{l\in\mathcal{L}}(\overline{\theta_l} \cap \tau_l) \uplus (\overline{\tau}\backslash\tau)$. τ is open, thus $\overline{\tau}\backslash\tau$ is nowhere dense. As the sets θ_l are nowhere dense we infer that $\operatorname{int}\overline{\upsilon} = \emptyset$. □

By Lemma 2.8, $f_t^{-1}(\infty) = \bigcup_{k\in\mathcal{K}}(f_{ta_k}^{-1}(\infty) \cap \sigma_k)$ is nowhere dense in $\mathfrak{M}$. Using [11, Theorem 2.2] and Baire's theorem, f_t may be extended to an element of $\overline{C}(\mathfrak{M})$, also denoted by f_t. Let $H(t)$ denote the selfadjoint operator in $\overline{\mathcal{R}}$ that corresponds to f_t. If $t \in \mathfrak{S}' \cup \mathfrak{S}\mathfrak{A}' \cup \mathfrak{A}'\mathfrak{S}$ then the operator $H(t)$ (thus $f_t(\cdot)$) is already defined. These notions are consistent with the new ones: indeed, (2.1), (2.13), (2.14) and Theorem 1.2 indicate that the new and the old functions $f_t(\cdot)$ agree on S_2^c, which is dense in $\mathfrak{M}$. Since both belong to $\overline{C}(\mathfrak{M})$, they are equal. The consistency in the definition of the corresponding operators follows at once.

From $f_t(\cdot)$ being identically zero on σ_* we deduce that $H(t)Q = 0$ for all $t \in \mathfrak{S}$. Fix $t_1, t_2 \in \mathfrak{S}$ and $M \in \left[S_2 \cup \biguplus_{k\in\mathcal{K}}\big((S_{a_k t_1} \cup S_{a_k t_2} \cup S_{a_k t_1 t_2}) \cap \sigma_k\big)\right]^c$. If $M \notin \sigma_*$ and k is such that $M \in \sigma_k$, then

$$
\begin{aligned}
f_{t_1}(M) f_{t_2}(M) &= f_{a_k t_1}(M) f_{a_k}(M)^{-1} f_{t_2 a_k}(M) f_{a_k}(M)^{-1} \\
&= f_{a_k t_1 t_2 a_k}(M) f_{a_k}(M)^{-2} \\
&= f_{a_k t_1 t_2}(M) f_{a_k}(M) f_{a_k}(M)^{-2} \\
&= f_{a_k t_1 t_2}(M) f_{a_k}(M)^{-1} = f_{t_1 t_2}(M)
\end{aligned}
$$

(cf. (2.12), (2.13), (2.14) and Theorem 1.2). The equality $f_{t_1}(M) f_{t_2}(M) = f_{t_1 t_2}(M)$ surely holds when $M \in \sigma_*$. Therefore $f_{t_1} f_{t_2} = f_{t_1 t_2}$ in $\overline{C}(\mathfrak{M})$, that is,

$$
(\forall t_1, t_2 \in \mathfrak{S}) \quad \overline{H(t_1)H(t_2)} = H(t_1 t_2). \tag{2.15}
$$

Given $k \in \mathcal{K}$, consider the function that takes $M \in \sigma_k$ to $f_{a_k}(M)^{-1}$ and $M \in \sigma_k^c$ to 0. This function is an element of $C(\mathfrak{M})$. Denote by B_k its corresponding bounded operator in $\mathcal{R}$. Then (2.14) may be written as $H(t)P_k = H(ta_k)B_k$ (the commutativity of the involved operators was used). Hence, if $x \in \mathcal{D}$ and $\sigma \in \mathcal{O}$, then

$$
H(t)P_k E_\sigma x = H(ta_k) B_k E_\sigma x = B_k H(ta_k) E_\sigma x,
$$

thus $t \mapsto H(t)P_k E_\sigma x$ is continuous over $\mathfrak{S}$. Therefore, $t \mapsto H(t)y$ is continuous over $\mathfrak{S}$ for every y in the linear subspace

$$\mathcal{D}' := \operatorname{span}\{P_k E_\sigma x : x \in \mathcal{D}, \sigma \in \mathcal{O}, k \in \mathcal{K}\} \oplus Q\mathcal{H},$$

which is dense in $\mathcal{H}$ by virtue of (2.10) and the definition of Q. By mimicking earlier parts of the proof, one confirms that

$$\operatorname{s.g-lim}_{\mathcal{J}} H(t_j) = H(t) \tag{2.16}$$

for every $t \in \mathfrak{S}$ and every net $(t_j)_\mathcal{J}$ in $\mathfrak{S}$ that converges to t.

To ascertain (2.3), let $x \in \mathcal{D}$ and $t \in U(x)$ be given. Then $T(t)x = \lim_{\substack{t' \to t \\ t' \in \mathfrak{S}'}} T(t')x = \lim_{\substack{t' \to t \\ t' \in \mathfrak{S}'}} H(t')x = H(t)x$ by virtue of (I) and (2.16).

The establishment of (2.4) is now easy in light of the integral spectral representation that operators in $\overline{\mathcal{R}}$ admit. For every $t \in \mathfrak{S}$, we have

$$H(t) = \int_{\mathfrak{M}} f_t(M) E_{dM}. \tag{2.17}$$

f_t is continuous as a function from $\mathfrak{M}$ to $\mathbb{C} \cup \{\infty\}$, but it may also be viewed as a continuous extended real valued function (cf. Remark 1.3). For fixed $M \in \mathfrak{M}$, the function $t \mapsto f_t(M)$ is consequently an extended real valued function over $\mathfrak{S}$. Moreover, $(H(t))_{t\in\mathfrak{S}}$ satisfying the semigroup law (2.15) makes this function an extended real character of $\mathfrak{S}$ in the sense of Definition 2.4. Thus it is natural to define a function $\varphi : \mathfrak{M} \to \mathfrak{S}_\infty^{*,\mathrm{nc}}$ by $M \mapsto (t \mapsto f_t(M))$. φ is continuous on account of the continuity of f_t for all $t \in \mathfrak{S}$. We may therefore define a spectral measure over the algebra of Borel sets in $\mathfrak{S}_\infty^{*,\mathrm{nc}}$ by

$$E(\Omega) := E_{\varphi^{-1}(\Omega)}$$

for every such Borel set Ω. $(E(\Omega))$ is inner regular since (E_B), as a spectral measure over $\mathfrak{M}$, is inner regular and φ is continuous. By complementation, $(E(\Omega))$ is outer regular as well. Finally, a simple change of parameter in (2.17) yields (2.4).

To prove the uniqueness of $E(\cdot)$, suppose that $E'(\cdot)$ is a regular spectral measure over $\mathfrak{S}_\infty^{*,\mathrm{nc}}$ satisfying (2.4). Using standard functional calculus arguments, it is readily seen that

$$\int_{\mathfrak{S}_\infty^{*,\mathrm{nc}}} \arctan(\chi(t)) E(d\chi) = \arctan(H(t)) = \int_{\mathfrak{S}_\infty^{*,\mathrm{nc}}} \arctan(\chi(t)) E'(d\chi) \tag{2.18}$$

for all $t \in \mathfrak{S}$. Let $\mathcal{A}$ denote the set of all continuous real valued functions f over $\mathfrak{S}_\infty^{*,\mathrm{nc}}$ that satisfy the equality $\int_{\mathfrak{S}_\infty^{*,\mathrm{nc}}} f(\chi)E(d\chi) = \int_{\mathfrak{S}_\infty^{*,\mathrm{nc}}} f(\chi)E'(d\chi)$. Then $\mathcal{A}$ is a closed subalgebra of $C_\mathbb{R}(\mathfrak{S}_\infty^{*,\mathrm{nc}})$ and $1 \in \mathcal{A}$. From (2.18), the set $\mathcal{B} := \{\chi \mapsto \arctan(\chi(t)) : t \in \mathfrak{S}\}$ is contained in $\mathcal{A}$. As arctan is injective from $\overline{\mathbb{R}}$ to $[-\pi/2, \pi/2]$, the family $\mathcal{B}$ (and consequently also $\mathcal{A}$) separates points on $\mathfrak{S}^{*,\mathrm{nc}}_{\infty}$. The Stone-Weierstrass Theorem implies that $\mathcal{A} = C_{\mathbb{R}}(\mathfrak{S}^{*,\mathrm{nc}}_{\infty})$. Both $E(\cdot)$ and $E'(\cdot)$ are regular, thus they are equal. □

3. Second hypothesis

Definition 3.1 (compare with [29, Definitions 2,3], [38, p. 220] and Definition 3.2)**.** A real valued function χ over a (topological) semigroup $\mathfrak{S}$ is called a *real character* of $\mathfrak{S}$ if $\chi(t)\chi(s) = \chi(ts)$ for all $t, s \in \mathfrak{S}$. As in the case of $\mathfrak{S}^{*,\mathrm{nc}}_{\infty}$, real characters are not required to be continuous. The semigroup of all real characters of $\mathfrak{S}$, with the topology of pointwise convergence, will be denoted by $\mathfrak{S}^{*,\mathrm{nc}}$.

Definition 3.2 ([30, Definitions 2,3])**.** We denote by $\hat{\mathfrak{S}}$ the semigroup of all *continuous* real characters of a topological semigroup $\mathfrak{S}$, together with the topology of uniform convergence on compact subsets of $\mathfrak{S}$.

Both $\mathfrak{S}^{*,\mathrm{nc}}$ and $\hat{\mathfrak{S}}$ are plainly topological semigroups.

In this section we assume that $\mathfrak{G}$ is a locally compact group. As in §2, $\mathfrak{G}'$ is a subgroup of $\mathfrak{G}$ and $\mathfrak{S}$ is an open semigroup in $\mathfrak{G}$ such that $\mathfrak{S}' = \mathfrak{S} \cap \mathfrak{G}'$ is dense in $\mathfrak{S}$.

Standing Hypothesis 3.3. Suppose that the assumptions of Standing Hypothesis 2.2 are satisfied and, in addition, the following condition holds:

(UBB) There exist an open subset $\mathcal{U}$ of $\mathfrak{S}$ and a constant $m > 0$ such that if $t \in \mathcal{U}' := \mathcal{U} \cap \mathfrak{S}'$ then $\|H(t)x\| \geq m\,\|x\|$ for all $x \in D(H(t))$.

In other words, the family $(H(t))_{t \in \mathcal{U}'}$ is uniformly bounded from below.

Theorem 3.4. *Under Standing Hypothesis 3.3, there exists a unique regular spectral measure $F(\cdot)$ over $\hat{\mathfrak{S}}$ such that*

$$H(t) = \int_{\hat{\mathfrak{S}}} \chi(t) F(d\chi) \tag{3.1}$$

for all $t \in \mathfrak{S}$, where $(H(t))_{t\in\mathfrak{S}}$ is the extended semigroup discussed in Theorem 2.6.

We shall require the following lemma, which is a generalization of [18, Lemma 10.2.1] to the case of semigroups of normal operators with indices in an open subsemigroup of a locally compact group. A similar result is [7, §4, Lemma 1] (where it is assumed that $e \in \overline{\mathfrak{S}}$).

Lemma 3.5. *Let $\mathfrak{G}$ be a locally compact group, $\mathfrak{S}$ an open subsemigroup of $\mathfrak{G}$, and let $T(\cdot) : \mathfrak{S} \to B(\mathcal{H})$ be a semigroup of bounded normal operators such that $\|T(\cdot)x\|$ is Borel measurable over $\mathfrak{S}$ for every $x \in \mathcal{H}$. Then $T(\cdot)$ is bounded on every compact subset of $\mathfrak{S}$.*

Proof. Let μ denote the left Haar measure of $\mathfrak{G}$. Fix a compact subset K of $\mathfrak{S}$. Assume that $T(\cdot)$ is not bounded on K. Since the operators $T(t)$ are normal, $\|T(t^2)\| = \|T(t)^2\| = \|T(t)\|^2$ for all $t \in \mathfrak{S}$. Hence, $T(\cdot)$ is not bounded on $K^2 := \{t^2 : t \in K\}$. By the Uniform Boundedness Theorem, there exists $x \in \mathcal{H}$ such that $T(\cdot)x$ is unbounded on K^2. Therefore, there exists a sequence $(\zeta_n)_{\mathbb{N}}$ in K such that

$$\left\|T(\zeta_n^2)x\right\| \geq n \text{ for all } n \in \mathbb{N}. \tag{3.2}$$

As K is compact, $(\zeta_n)_{\mathbb{N}}$ possesses a cluster point $\zeta \in K$. Since $\mathfrak{G}$ is a locally compact group and $\mathfrak{S}$ is open, there exists an open neighborhood U of the identity of $\mathfrak{G}$ such that $\overline{U}$ is compact, and $\zeta U, \zeta^2 UUU^{-1}\zeta^{-1} \subseteq \mathfrak{S}$. Write $V := \zeta U$ and $W := \zeta^2 UU$. Then V, W are open neighborhoods of ζ, ζ^2, respectively, and $V, WV^{-1} \subseteq \mathfrak{S}$. Additionally, WV^{-1} is of finite measure, being a subset of the compact set $\zeta^2 \cdot \overline{U} \cdot \overline{U} \cdot (\overline{U})^{-1} \cdot \zeta^{-1}$. As $(\zeta_n)_{\mathbb{N}}$ clusters at ζ we may assume, by passing to a subsequence if necessary, that $\zeta_n \in V$ and $\zeta_n^2 \in W$ for every $n \in \mathbb{N}$. Note that (3.2) is retained.

On account of the measurability of $\|T(\cdot)x\|$, there exist a measurable subset $F \subseteq V$ and a constant $M < \infty$ such that $\mu(F^{-1}) > 0$ and $\|T(t)x\| \leq M$ for every $t \in F$. Indeed, V may be expressed as the union of measurable sets $\cup_{n=1}^{\infty} A_n$ where $A_n := \{t \in V : \|T(t)x\| \leq n\}$ for all $n \in \mathbb{N}$; since $\mu(V^{-1}) > 0$, we must have $\mu(A_{n_0}^{-1}) > 0$ for some $n_0 \in \mathbb{N}$, so we may take $F := A_{n_0}$ and $M := n_0$. Set $E_n := \zeta_n^2 F^{-1}$ for all $n \in \mathbb{N}$. Then $E_n \subseteq WV^{-1} \subseteq \mathfrak{S}$. $T(\cdot)$ is a semigroup of bounded operators, and so, from (3.2),

$$(\forall n \in \mathbb{N}, t \in F) \quad n \leq \left\|T(\zeta_n^2)x\right\| \leq \left\|T(\zeta_n^2 t^{-1})\right\| \|T(t)x\| \leq M \left\|T(\zeta_n^2 t^{-1})\right\|.$$

Consequently, $\|T(s)\| \geq n/M$ for all $s \in E_n$. Hence, if $s \in E := \limsup E_n$, then $\|T(s)\| = \infty$, which is impossible. Thus $E = \emptyset$. But μ is left invariant, therefore

$$\mu(E) = \lim_{n\to\infty} \mu\left(\bigcup_{k=n}^{\infty} E_k\right) \geq \mu(F^{-1}) > 0, \tag{3.3}$$

and so $E \neq \emptyset$, a contradiction. Equation (3.3) holds since for every n, E_n is a subset of the set WV^{-1}, which is of finite measure. □

Proof of Theorem 3.4. Throughout the present proof, the notations of the proof of Theorem 2.6 will be freely used. It is routine to check that (UBB) and (2.16) yield that

$$\|H(u)x\| \geq m\,\|x\|$$

for all $u \in \mathcal{U}$ and $x \in D(H(u))$. Consequently, $f_u^{-1}(0) = \emptyset$ for all $u \in \mathcal{U}$. Given $\sigma \in \mathcal{O}$ and $t \in \mathfrak{S}$, the openness of $\mathcal{U}$ and the density of $\mathfrak{S}'$ in $\mathfrak{S}$ furnish the existence of $u \in \mathcal{U}$ so that $ut \in \mathfrak{S}'$. With this u at hand we have, by virtue of Proposition 1.4,

$$H(u)H(t) = \overline{H(u)H(t)} = H(ut) \quad \Longrightarrow \quad H(u)H(t)E_\sigma = H(ut)E_\sigma. \tag{3.4}$$

The operator $H(ut)E_\sigma$ is bounded, and the Closed Graph Theorem thus implies that $H(t)E_\sigma$ is bounded as well. We conclude that $(H(t)E_\sigma)_{t\in\mathfrak{S}}$ is a semigroup of bounded selfadjoint operators.

Let $x \in \mathcal{H}$. Fix a sequence $(x_n)_{\mathbb{N}}$ in $\mathcal{D}'$ that converges to x, and consider the functions $\rho_n(t) := \|H(t)E_\sigma x_n\| = \|E_\sigma H(t)x_n\|$, $n \in \mathbb{N}$, as nonnegative real functions over $\mathfrak{S}$. As proved in Theorem 2.6, the function $\rho_n(t)$ is continuous for every n. By the foregoing, these functions converge pointwise to the function $\rho : t \mapsto \|H(t)E_\sigma x\|$ (also defined over $\mathfrak{S}$). Consequently, ρ is Borel measurable. This being true for all $x \in \mathcal{H}$, Lemma 3.5 implies that $H(t)E_\sigma$ is uniformly bounded on every compact subset of $\mathfrak{S}$. It is now easy to prove the continuity of $t \mapsto H(t)E_\sigma x$ over $\mathfrak{S}$ (either using ε type arguments, or directly, from (2.16)).

To complete the proof, it would now be sufficient to employ [30, Theorem 6]. In this theorem, Nussbaum requires that the equation $H(t+s) = H(t)H(s)$ hold for every $t, s \in \mathfrak{S}$, in contrast to our $(H(t))_{t\in\mathfrak{S}}$ satisfying (2.1), which is more general (see §5). However, this requirement is needed only to guarantee that $H(\cdot)E_\sigma$ is a strongly continuous semigroup of *bounded* operators for every $\sigma \in \mathcal{O}$ (using our notation; see [30, Theorem 5] and its proof), and that being already established, (2.1) is sufficient.

Alternatively, an idea of [21, Theorem 3] (see also their §6) can be used. We describe it succinctly. Choose a maximal family $(\sigma_l)_{\mathcal{L}}$ of disjoint clopen subsets of $\mathfrak{M}$ in $\mathcal{O}$. For $l \in \mathcal{L}$, $(H(t)_{|E_{\sigma_l}\mathcal{H}})_{t\in\mathfrak{S}}$ is a strongly continuous semigroup of bounded selfadjoint operators (acting on the Hilbert space $E_{\sigma_l}\mathcal{H}$) that satisfies the requirements of [30, Theorem 6]. It thus admits a representing (inner regular, thus) regular spectral measure $F_{\sigma_l}(\cdot)$ over $\hat{\mathfrak{S}}$ in the sense of (3.1). We now obtain the desired regular spectral measure by defining $F(\cdot) := \bigoplus_{\mathcal{L}} F_{\sigma_l}(\cdot)$.

The uniqueness of $F(\cdot)$ is a consequence of the uniqueness part of Theorem 2.6 and the regularity of the measure. To see this, notice that a compact subset K of $\hat{\mathfrak{S}}$ is compact in the topology of $\mathfrak{S}_\infty^{*,\mathrm{nc}}$ as well. $\square$

Remark 3.6. Condition (UBB) is evidently not necessary for (3.1) to hold. We required it in the proof of Theorem 3.4 to verify that $H(t)E_\sigma$ was bounded for $\sigma \in \mathcal{O}$ and $t \in \mathfrak{S}$. Nevertheless, (3.4) could have been established directly, using simpler boundedness from below arguments, instead of the more general Proposition 1.4. The latter was nonetheless integrated into the proof to demonstrate the possibility to relax Condition (UBB). The difficulty would be that the theorem's assumptions cannot include any restriction on the extended semigroup $(H(t))_{t\in\mathfrak{S}}$, which is constructed during the course of the proof.

Open Question 3.7. Can Condition (UBB) be substituted by a weaker one?

4. Application to convolution semigroups

Let $\mathfrak{G}$ be a topological group and let $\mathfrak{S}$ be an open semigroup in $\mathfrak{G}$ with $e \in \overline{\mathfrak{S}}$. Suppose that $\mathfrak{S}'$ is a countable dense subsemigroup of $\mathfrak{S}$ that equals the intersection of $\mathfrak{S}$ with some subgroup of $\mathfrak{G}$. The following application of Theorems 2.6 and 3.4 to (local) convolution semigroups with indices in $\mathfrak{S}'$ is in the spirit of [32, §4].

We employ the theory of unbounded convolution operators of [25], with whose results the reader is assumed to be familiar. Fix a locally compact Abelian group G and a Haar measure on it. If f, g are (complex valued) measurable functions over G, we say that their convolution $f * g$ exists if $\int_G |f(x-y)g(y)|dy$ is locally integrable; and in this case, we define $(f * g)(x) := \int_G f(x-y)g(y)dy$. Fix a measurable function f over G. The operator L'_f over the Hilbert space $\mathcal{H} := L^2(G)$ is the (possibly unbounded) operator defined by $L'_f(g) := f * g$ with domain consisting of all functions $g \in L^1(G) \cap L^2(G)$ such that $f * g$ exists in the above sense and belongs to $L^2(G)$. We say that f satisfies condition (K) if L'_f is densely defined and closable. In this case its closure, denoted by L_f, is automatically normal. Condition (K) is satisfied if $D(L'_f)$ contains a subset of functions with compact supports whose set of translates is total (in the sense that its span is dense in $L^2(G)$). This happens, e.g., when $f \in L^p(G)$ for some $1 \le p \le 2$. Moreover, upon setting $f^*(x) := \overline{f(-x)}$, we have $L_f^* = L_{f^*}$. Thus, if f is *symmetric* in the sense that $f = f^*$ then L_f is selfadjoint.

Our assumptions are therefore the following:

(1) $(f_t)_{t\in\mathfrak{S}'}$ is a family of symmetric measurable functions over G that satisfy condition (K).
(2) For all $t, s \in \mathfrak{S}'$, $f_t * f_s$ exists and equals f_{ts}, and there is a total set $\mathcal{C}$ of functions in $D(L'_{f_s})$ with compact supports such that $f_s * \mathcal{C} \subseteq D(L'_{f_t})$.
(3) There exists a dense subspace $\mathcal{D}$ of $L^2(G)$ with the property that for every $g \in \mathcal{D}$ there is an open neighborhood $V(g)$ of e in $\mathfrak{G}$, such that $g \in D(L_{f_t})$ for $t \in \mathfrak{S}' \cap V(g)$ and the function $t \mapsto L_{f_t}(g)$ extends to a continuous function on $\mathfrak{S} \cap V(g)$.

Defining $H(t) := L_{f_t}$, the family $(H(t))_{t\in\mathfrak{S}'}$ is a semigroup of selfadjoint operators that meets the requirements of Theorem 2.6 with $\mathfrak{A}' = \{e\}$ (see especially [25, Corollary 3.3.5]). It therefore extends to a semigroup of selfadjoint operators $(H(t))_{t\in\mathfrak{S}}$ having all the properties mentioned in the theorem. In particular, it admits the integral representation (2.4).

Suppose that in addition, $\mathfrak{G}$ is locally compact and the following condition holds:

(4) There exists an open subset $\mathcal{U}$ of $\mathfrak{S}$ such that the operators L'_{f_t}, $t \in \mathcal{U} \cap \mathfrak{S}'$, are uniformly bounded from below.

Then Theorem 3.4 is applicable, hence $(H(t))_{t\in\mathfrak{S}}$ admits the stronger representation (3.1). Condition (4) is not far-fetched on account of [25, Corollaries 3.1.1, 3.3.4 and 3.3.5].

This example can presumably be generalized further to the case in which G is not necessarily Abelian and convolution by functions is replaced by convolution by measures (as suggested in [32]) using [10]. In this connection see also [15, §20].

5. Comparison with other similar results

In this section, we compare our Theorems 2.6 and 3.4 to various past results concerning the spectral representation of semigroups of (unbounded) selfadjoint or normal operators, with indices in some general semigroup. Our focus will be chiefly, but not only, on Definition 2.1. For the general theory of semigroups of unbounded operators in Banach space, consult [19, 20]. We start by commenting on the possibility to generalize Theorem 2.6 to semigroups of *normal* operators. The obstacle would be Proposition 1.6, which does not hold for normal operators. This is demonstrated by the following result:

Theorem ([2, Theorem 3.3], [4])**.** *A bounded operator is subnormal if and only if it is the limit, in the strong operator topology, of a net of bounded normal operators.*

Let us continue by comparing the conditions related to commutativity and the semigroup law in [21, 30, 38] to Definition 2.1. A thorough comparison of those three papers can be found in [38, §2].

Nussbaum [30] considers families $(H(t))_{t\in\mathfrak{S}}$ of unbounded selfadjoint operators, and requires that the following semigroup law hold (cf. *ibid.* p. 134, (2)):

(Nuss) $H(ts) \subseteq H(t)H(s)$ for every $t, s \in \mathfrak{S}$.

As commented in [30], this implies (cf. [8, Theorem 1 and Corollary 1]) that for all $t, s \in \mathfrak{S}$, $H(s)$ commutes with $H(t)$ and $H(ts) = H(t)H(s) = H(s)H(t)$. This semigroup law is by far more restrictive than (2.1) (for example, if A is an unbounded normal operator and $0 \in \rho(A)$, then $AA^{-1} = I$ but $A^{-1}A = I_{|D(A)} \subsetneqq I$; one might come across this situation when $\mathfrak{S}$ is in fact a group, and $A = H(t)$ for some $t \in \mathfrak{S}$).

Ionescu Tulcea [21] discusses the analogous case for families of unbounded normal operators $(H(t))_{t\in\mathfrak{S}}$, and it is assumed there that (cf. *ibid.* (10), (11)):

(IT1) The semigroup operators commute pairwise.

(IT2) The semigroup law (Nuss) is satisfied (apart from the selfadjointness requirement).

Commutativity must be explicitly required, since for unbounded normal operators A, B, C, the fact that $C \subseteq AB$ does not automatically imply that A, B commute (in contrast to selfadjoint operators; compare [6, Theorem 1]). It is mentioned in [21] that both the commutativity and the semigroup law are implied by the condition that $H(t)H(s) = H(ts) = H(st) = H(s)H(t)$ for all $t, s \in \mathfrak{S}$ (cf. [6]). The converse implication is, of course, also true: if (IT1) and (IT2) are satisfied, then $H(ts) \subseteq H(t)H(s) \subseteq \overline{H(t)H(s)} = \overline{H(s)H(t)}$, and since both $H(ts)$ and $\overline{H(t)H(s)}$ are normal (by virtue of the commutativity), we infer that they are equal, thus $H(ts) = H(t)H(s) = H(s)H(t)$. We conclude that {(IT1), (IT2)} is the natural extension of (Nuss) to normal operators, therefore the former and the latter compare to our requirements equally.

The indices semigroups $\mathfrak{S}$ treated in [21, 30] are locally compact full semigroups (cf. [29, 30]). Our discussion was restricted to open semigroups, which are a special case of full semigroups when embeddible in a locally compact group, as is the case in Theorem 3.4. It is interesting to note that every locally compact full semigroup contains an open subsemigroup (cf. [1, Theorem 1]).

While in the previous results the semigroup $\mathfrak{S}$ is a topological one, Ressel and Ricker [38] consider Abelian unital $*$-semigroups $\mathfrak{S}$ that need not have a topology. In their setting, as in [21], the family $(H(t))_{t\in\mathfrak{S}}$ consists of unbounded normal operators, and additionally, for all $t, s \in \mathfrak{S}$ (cf. *ibid.* Definition 1.1, (iii) and (iv)):

(RR1) $H(t)H(s) \subseteq H(ts)$ and $D(H(t)H(s)) = D(H(ts)) \cap D(H(s))$.

(RR2) $\overline{H(t)H(s)} = H(ts)$.

Commutativity of the operators is not required, but follows as a consequence of (RR1), (RR2) and the rest of the conditions in [38, Definition 1.1] (especially (v), which is the "substitute" for the lacking topological requirements, and does not appear in our hypotheses). This framework is therefore closer to being a "normal operator" version of our Definition 2.1 than the above-mentioned one. Note that the pair {(RR1), (RR2)} is implied by both (Nuss) and the pair {(IT1), (IT2)} individually. Stochel and Szafraniec obtained in [40, §5] a similar spectral representation for the shift operators related to positive definite forms over Abelian unital $*$-semigroups.

All results establish the existence of a representing spectral measure for the family $(H(t))_{t\in\mathfrak{S}}$, but over different character spaces: in Theorem 2.6 it is $\mathfrak{S}^{*;\mathrm{nc}}_{\infty}$, in [38, Theorem 1.2] and [40, Theorem 4] it is an analog of $\mathfrak{S}^{*,\mathrm{nc}}$, in Theorem 3.4 and [30, Theorem 6] it is $\hat{\mathfrak{S}}$, and in [21, Theorem 3] a complex version of $\hat{\mathfrak{S}}$ is used.

As a concluding remark, we wish to point out that in the recent paper [26] a problem related to ours is considered: the extendability of a weakly continuous semigroup of bounded operators with indices in a dense subsemigroup of $\mathbb{R}_+$ to a C_0-semigroup. Their context, however, is completely different (the operators are bounded, the indices are real and the operators act on a separable reflexive Banach space), and thus so are the methods involved.

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