SPECTRAL REPRESENTATION OF LOCAL SYMMETRIC SEMIGROUPS OF OPERATORS

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Abstract. A spectral integral representation is established for locally defined symmetric semigroups of operators, with indices which are not restricted to a neighborhood of zero. This extends the well-known results of Fröhlich [Unbounded, symmetric semigroups on a separable Hilbert space are essentially selfadjoint, Adv. in Appl. Math. 1 (1980), 237-256] and Klein and Landau [Construction of a unique selfadjoint generator for a symmetric local semigroup, J. Funct. Anal. 44 (1981), 121-137].

Introduction

Semigroups of selfadjoint and symmetric operators with real indices have been studied by many authors over the years. The classical results, concerning the spectral representation of semigroups of selfadjoint operators, are those of Nagy [11, XI.2] and Hille [4], [5, §4.1] for bounded operators, and Devinatz [1] for unbounded operators. Nussbaum [12] generalized these results to semigroups of densely-defined symmetric operators, for which the semigroup property holds on a common, dense domain.

Fröhlich [2] and Klein and Landau [10] proved an even stronger theorem. They considered local semigroups $T(\cdot)$ of symmetric operators, that need not be densely-defined. The term "local" means that for each x in some dense set, $T(t)x$ is defined for all $t > 0$ small enough, depending upon x.

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The results of both papers are essentially similar, and yield the spectral representation $T(t)x = e^{tH}x$ for a suitable selfadjoint operator H.

Other consequences in the same spirit followed. Examples are the corresponding generalizations to multidimensional Euclidean spaces [13, 17]; local cosine families of symmetric operators [8]; and local semigroups over settings other than Hilbert space [7, 15, 16].

In the present paper, we establish a generalization of [2, 10] to a broader class of semigroups. The semigroups considered in these results all have indices in right neighborhoods of 0. The same is true in the multidimensional cases [13, 17], in which 0 must either belong to the indices set, or to its closure in $\mathbb{R}^n$. In Theorem 2.8 we obtain a similar spectral representation, for local symmetric semigroups of operators whose set of indices is a right neighborhood (conforming to some restrictions) of any positive real semigroup. This generalization is needed in a forthcoming paper of the author on spectral representation of local symmetric operator semigroups over subsemigroups of locally compact Abelian groups. In our basic result, Theorem 2.3, the considered "pre-semigroups" of operators do not encompass any algebraic structure.

1. Preliminaries

We begin with a brief survey on commuting selfadjoint operators. Throughout this paper, $\mathcal{H}$ will denote a complex Hilbert space.

Definition 1.1 ([11, VIII.1, p. 50])**.** Let A, B be (possibly unbounded) normal operators, with spectral measures E and F respectively. We say that A and B *commute* if their spectral measures commute, i.e., if for every two Borel sets in $\mathbb{C}$, σ_1 and σ_2, $E(\sigma_1)F(\sigma_2) = F(\sigma_2)E(\sigma_1)$.

If A and B commute, then $f(A)$ and $g(B)$ commute for every Borel functions $f, g : \mathbb{C} \to \mathbb{C}$. If B is bounded, then A and B commute if and only if $BA \subseteq AB$ (c.f. [3, Theorem I]). In this case, $\overline{BA} = AB$.

Proposition 1.2. *Let A and B be selfadjoint operators. Then the following statements are equivalent:*

(1) A and B commute.
(2) For all $s, t \in \mathbb{R}$, $e^{itA}e^{isB} = e^{isB}e^{itA}$.
(3) For all $t \in \mathbb{R}$, $e^{itA}B \subseteq Be^{itA}$.

Proof. The equivalence (1) $\Leftrightarrow$ (2) is taken from [14, Theorem VIII.13]. (1) $\Rightarrow$ (3) is simple as B is selfadjoint, and for (3) $\Rightarrow$ (2), see [3, Theorem I]. □

Proposition 1.3. *Let A and B be commuting selfadjoint operators. Then $\overline{AB}$ is a selfadjoint operator, that commutes with both A and B.*

Proof. This proposition is widely known, and its proof is elementary. We omit the details.

Corollary 1.4. *Let A, B, C be pairwise commuting selfadjoint operators. Assume that $x, y \in \mathcal{H}$, $n_0 \in \mathbb{N}$ are such that $x \in D(B)$ and*

$$(\forall n \geq n_0) \quad \overline{BA^n}x = \overline{CA^n}y$$

(in particular, $x \in D(\overline{BA^n})$ and $y \in D(\overline{CA^n})$). If $E(\cdot)$ is the resolution of the identity of A, then $E(\mathbb{R}\backslash\{0\})x \in D(B)$, $E(\mathbb{R}\backslash\{0\})y \in D(C)$ and

$$BE(\mathbb{R}\backslash\{0\})x = CE(\mathbb{R}\backslash\{0\})y.$$

Proof. Let $F(\cdot), G(\cdot)$ be the resolutions of the identity of B, C respectively. Fix $j, k, \ell \in \mathbb{N}$, denote $\sigma_j := [-j, -\frac{1}{j}] \cup [\frac{1}{j}, j]$, and write

$$H_{j,k,\ell} := E(\sigma_j)F([-k,k])G([-\ell,\ell])$$

(in short, H). By the commutativity of A, B, C (which will be used implicitly throughout the proof), $BA^nHx = CA^nHy$ for all $n \geq n_0$. Consequently,

$$Bp(A)Hx = Cp(A)Hy \tag{1.1}$$

for every polynomial p of the form $p(t) = t^{n_0}q(t)$ (where q is some other polynomial).

By virtue of the Weierstrass Approximation Theorem, there exists a sequence of polynomials $\{q_n\}_{n=1}^{\infty}$ such that $q_n(t) \to t^{-n_0}$ uniformly on σ_j. Define $p_n(t) := t^{n_0} q_n(t)$. Then $\{p_n\}_{n=1}^{\infty}$ converges uniformly to 1 on σ_j. In particular, $p_n(A)E(\sigma_j)z \to E(\sigma_j)z$ for all $z \in \mathcal{H}$. Thus, replacing p by p_n in (1.1) and letting $n \to \infty$, one has

$$BH_{j,k,\ell}x = CH_{j,k,\ell}y$$

for all $j, k, \ell \in \mathbb{N}$. Letting $j \to \infty$ yields

$$G([-\ell,\ell])BF([-k,k])E(\mathbb{R}\backslash\{0\})x = F([-k,k])CG([-\ell,\ell])E(\mathbb{R}\backslash\{0\})y \tag{1.2}$$

for all k, ℓ. Since $x \in D(B)$, so does $E(\mathbb{R}\backslash\{0\})x$. As a result, letting $k \to \infty$ in (1.2) gives $G([-\ell,\ell])BE(\mathbb{R}\backslash\{0\})x = CG([-\ell,\ell])E(\mathbb{R}\backslash\{0\})y$. Finally, let $\ell \to \infty$. Since C is closed, $E(\mathbb{R}\backslash\{0\})y \in D(C)$ and $BE(\mathbb{R}\backslash\{0\})x = CE(\mathbb{R}\backslash\{0\})y$. □

Lemma 1.5. *Suppose that T, B are selfadjoint operators, and B is bounded. If BT is symmetric, then T, B commute.*

Proof. Fix $y \in D(T)$. Then for all $x \in D(T)$, $(Tx, By) = (BTx, y) = (x, BTy)$ by the symmetry of B, BT. Therefore, as T is selfadjoint, $By \in D(T)$, and $TBy = BTy$. In conclusion, $BT \subseteq TB$, thus T, B commute. □

2. Main results

Our first objective is to generalize [2, Theorem I.1] to the case described by the following Definition 2.1, that should be compared against [2, §I.1, (1)-(3)]. In particular, 0 is replaced by an arbitrary index set $\mathcal{A}$. Only later, in Definition 2.6, will $\mathcal{A}$ become a semigroup of positive real numbers.

Definition 2.1. Let $\mathcal{A} \neq \emptyset$ be an index set. A family of linear operators $(T_a(t))_{a\in\mathcal{A},t\geq 0}$ is called a *local symmetric pre-semigroup of operators* (over $\mathbb{R}$) if there exists a (not necessarily dense) subspace $\mathcal{D}$ of $\mathcal{H}$, such that:

(P1) Domain: for every $a \in \mathcal{A}$, $x \in \mathcal{D}$, there exists $\varepsilon_a(x) > 0$ such that $x \in D(T_a(t))$ for every $0 \leq t < \varepsilon_a(x)$; $t \mapsto \|T_a(t)x\|$ is either measurable over some nonempty subinterval, or bounded on a measurable subset of positive measure, of $(0, \varepsilon_a(x))$; and $\lim_{t\to 0^+} T_a(t)x = T_a(0)x$.

(P2) Semigroup symmetry: for every $a, b \in \mathcal{A}$ and $x, y \in \mathcal{D}$, if $t, s, u \geq 0$ satisfy $t + u < \varepsilon_a(x)$ and $s + u < \varepsilon_b(y)$, then the following equality holds:

$$(T_a(t+u)x, T_b(s)y) = (T_a(t)x, T_b(s+u)y).$$

You may observe that according to (2.1) in the following Theorem 2.3, the terminology is in accordance with the terminology in [6, page 75] (with the adjective "local" applied in the usual sense), except that we do not require the operators $T_a(0)$, $a \in \mathcal{A}$, to be injective.

Remark 2.2. It is important to note that even in case $\mathcal{D}$ is dense, $T_a(t)$ need not be densely defined for general $a \in \mathcal{A}$, $t > 0$. Its domain may even be $\{0\}$. Only for $a \in \mathcal{A}$ it is implied by Postulate (P1) that $\mathcal{D} \subseteq D(T_a(0))$.

Theorem 2.3. *Let $(T_a(t))_{a\in\mathcal{A},t\geq 0}$ be a local symmetric pre-semigroup of operators. Then there exists a selfadjoint operator H over $\mathcal{H}$ such that for every $a \in \mathcal{A}, x \in \mathcal{D}, t \in [0, \varepsilon_a(x))$,*

$$T_a(t)x = e^{tH} T_a(0)x. \tag{2.1}$$

Moreover, write $\mathcal{M} := \operatorname{span}\{T_a(t)x : a \in \mathcal{A}, x \in \mathcal{D}, 0 < t < \varepsilon_a(x)\}$. Then H is essentially selfadjoint *over $\mathcal{M} \oplus \mathcal{M}^\perp$, and H is* unique *if we require that $\mathcal{M}^\perp \subseteq \ker H$.*

Note that the trivial case, in which $\mathcal{A}$ is a singleton $\mathcal{A} = \{a_0\}$, $T_{a_0}(0) \subseteq I$ and $\mathcal{D}$ is dense, yields Fröhlich's Theorem ([2, Theorem I.1]). It is easily seen that the assumptions of Fröhlich's Theorem imply Postulates (P1) and (P2).

Proof. Existence. Fix $a \in \mathcal{A}$, $x \in \mathcal{D}$. We first prove that $t \mapsto T_a(t)x$ is continuous on $(0, \varepsilon_a(x))$. If $T_a(t_0)x = 0$ for some $t_0 \in (0, \varepsilon_a(x))$, then Postulate (P2) implies that $T_a(t)x = 0$ for *all* $t \in (0, \varepsilon_a(x))$. Else, define

$$(\forall t \in (0, \varepsilon_a(x))) \quad g(t) := \log \|T_a(t)x\| .$$

If $s, u \in (0, \varepsilon_a(x))$, then by Postulate (P2) and the Cauchy-Schwarz inequality,

$$(T_a(\frac{s+u}{2})x, T_a(\frac{s+u}{2})x) = (T_a(s)x, T_a(u)x) \le \|T_a(s)x\| \, \|T_a(u)x\| . \tag{2.2}$$

Hence $g(\frac{s+u}{2}) \le \frac{1}{2}g(s) + \frac{1}{2}g(u)$, i.e., g is convex on $(0, \varepsilon_a(x))$. By Postulate (P1), g is continuous (c.f. [18, 9]). If now $s, u \in (0, \varepsilon_a(x))$, then (2.2) yields that

$$\begin{aligned} \|T_a(s)x - T_a(u)x\|^2 &= \|T_a(s)x\|^2 + \|T_a(u)x\|^2 - 2\mathtt{Re}(T_a(s)x, T_a(u)x) \\ &= \|T_a(s)x\|^2 + \|T_a(u)x\|^2 - 2\left\|T_a(\frac{s+u}{2})x\right\|^2 . \end{aligned}$$

Therefore $t \mapsto T_a(t)x$ is continuous on $(0, \varepsilon_a(x))$.

For every $n \in \mathbb{N}$, let $\delta_n \in C^\infty(\mathbb{R})$ be such that

$$\operatorname{supp} \delta_n \subseteq [0, \frac{1}{n}], \quad \delta_n \ge 0 \quad \text{and} \quad \int_{\mathbb{R}} \delta_n(s)ds = 1. \tag{2.3}$$

Given $a \in \mathcal{A}$, $x \in \mathcal{D}$, $0 \le t < \varepsilon_a(x)$, let $n_0 = n_0(a, x, t)$ be minimal such that $t + \frac{1}{n_0} < \varepsilon_a(x)$. Then for every $n \ge n_0$, $k \in \mathbb{Z}^+$, by Postulate (P1),

$$\Psi(a, x, t, n, k) := \int_{\mathbb{R}} \delta_n^{(k)}(s) T_a(t+s)x ds$$

is well-defined. Moreover, (2.3) implies that $\Psi(a, x, t, n, 0) \to T_a(t)x$ as $n \to \infty$. Denote

$$\Omega := \left\{(a, x, t, n, k) : a \in \mathcal{A}, x \in \mathcal{D}, 0 < t < \varepsilon_a(x), n \ge n_0(a, x, t), k \in \mathbb{Z}^+\right\}$$

and $\mathcal{D}_1 := \operatorname{span}\{\Psi(a, x, t, n, k)\}_\Omega$. Fix a, x, t, n, k as above. We wish to prove that the derivative $\lim_{h \to 0} \frac{1}{h}[\Psi(a, x, t+h, n, k) - \Psi(a, x, t, n, k)]$ defines a (well-defined) *symmetric* operator over $\mathcal{D}_1$. Let $\varepsilon' = \varepsilon'_a(x, t) :=$

$\frac{1}{2}\min(\varepsilon_a(x)-(t+\frac{1}{n_0}),t)>0$. Then for $h\in(-\varepsilon',\varepsilon')$, $\Psi(a,x,t+h,n,k)$ is well defined. Moreover,

$$\begin{aligned}\Psi(a,x,t+h,n,k)-\Psi(a,x,t,n,k)=&\\ &=\int_{\mathbb{R}}\delta_n^{(k)}(s)(T_a(t+s+h)x-T_a(t+s)x)ds\\ &=\int_{\mathbb{R}}(\delta_n^{(k)}(s-h)-\delta_n^{(k)}(s))T_a(t+s)xds.\end{aligned}$$

The function $v\mapsto T_a(v)x$ is continuous on the closed interval $[\frac{1}{2}t,\frac{1}{2}(\varepsilon_a(x)+(t+\frac{1}{n_0}))]$, hence it is bounded there by some $M<\infty$, and we obtain

$$\begin{aligned}\left\|\frac{1}{h}[\Psi(a,x,t+h,n,k)-\Psi(a,x,t,n,k)]+\int_{\mathbb{R}}\delta_n^{(k+1)}(s)T_a(t+s)xds\right\|\leq&\\ M\int_{\mathbb{R}}\left|\frac{1}{h}[\delta_n^{(k)}(s-h)-\delta_n^{(k)}(s)]+\delta_n^{(k+1)}(s)\right|ds&\end{aligned}$$

for $h\in(-\varepsilon',\varepsilon')$. The right side tends to 0 as $h\to 0$, and we can thus define

$$\begin{aligned}H_0\Psi(a,x,t,n,k)&:=\lim_{h\to 0}\frac{1}{h}[\Psi(a,x,t+h,n,k)-\Psi(a,x,t,n,k)]\\ &=-\int_{\mathbb{R}}\delta_n^{(k+1)}(s)T_a(t+s)xds\in\mathcal{D}_1.\end{aligned}\tag{2.4}$$

To demonstrate why this defines a well-defined linear operator $H_0:\mathcal{D}_1\to\mathcal{D}_1$, fix $a_i\in\mathcal{A}$, $x_i\in\mathcal{D}$, $0<t_i<\varepsilon_{a_i}(x_i)$, $n_i\geq n_0(a_i,x_i,t_i)$ and $k_i\in\mathbb{Z}^+$ (i=1,2). Postulate (P2) furnishes the existence of $r>0$ such that for $h\in(-r,r)$, if $0\leq s_1^i<s_2^i\ldots<s_{\ell_i}^i\leq 1/n_i$ and $c_1^i,\ldots,c_{\ell_i}^i\in\mathbb{R}$ (i=1,2), then

$$\begin{aligned}(\sum_{j=1}^{\ell_1}c_j^1T_{a_1}(t_1+s_j^1+h)x_1,\sum_{k=1}^{\ell_2}c_k^2T_{a_2}(t_2+s_k^2)x_2)=&\\ (\sum_{j=1}^{\ell_1}c_j^1T_{a_1}(t_1+s_j^1)x_1,\sum_{k=1}^{\ell_2}c_k^2T_{a_2}(t_2+s_k^2+h)x_2).&\end{aligned}$$

By the definition of the Riemann integral, we thus have, for $h\in(-r,r)$,

$$\begin{aligned}(\Psi(a_1,x_1,t_1+h,n_1,k_1),\Psi(a_2,x_2,t_2,n_2,k_2))=&\\ (\Psi(a_1,x_1,t_1,n_1,k_1),\Psi(a_2,x_2,t_2+h,n_2,k_2)).&\end{aligned}\tag{2.5}$$

Fix $\Psi \in \mathcal{D}_1$. Suppose that $(a_i, x_i, t_i, n_i, k_i), (a'_j, x'_j, t'_j, n'_j, k'_j) \in \Omega$ and $c_i, c'_j \in \mathbb{C}$ $(1 \leq i \leq k,\ 1 \leq j \leq l)$ are such that $\sum_{i=1}^{k} c_i \Psi(a_i, x_i, t_i, n_i, k_i) = \sum_{j=1}^{l} c'_j \Psi(a'_j, x'_j, t'_j, n'_j, k'_j) = \Psi$. If $(b_\iota, y_\iota, \tau_\iota, \eta_\iota, \kappa_\iota) \in \Omega$, $\gamma_\iota \in \mathbb{C}$ $(1 \leq \iota \leq \varrho)$ and $|h|$ is small enough, (2.5) yields

$$\begin{aligned}
&(\underbrace{\sum_{i=1}^{k} c_i \Psi(a_i, x_i, t_i + h, n_i, k_i)}_{\phi(h)}, \sum_{\iota=1}^{\varrho} \gamma_\iota \Psi(b_\iota, y_\iota, \tau_\iota, \eta_\iota, \kappa_\iota)) = \\
&= (\sum_{i=1}^{k} c_i \Psi(a_i, x_i, t_i, n_i, k_i), \sum_{\iota=1}^{\varrho} \gamma_\iota \Psi(b_\iota, y_\iota, \tau_\iota + h, \eta_\iota, \kappa_\iota)) \\
&= (\sum_{j=1}^{l} c'_j \Psi(a'_j, x'_j, t'_j, n'_j, k'_j), \sum_{\iota=1}^{\varrho} \gamma_\iota \Psi(b_\iota, y_\iota, \tau_\iota + h, \eta_\iota, \kappa_\iota)) \\
&= (\underbrace{\sum_{j=1}^{l} c'_j \Psi(a'_j, x'_j, t'_j + h, n'_j, k'_j)}_{\varphi(h)}, \sum_{\iota=1}^{\varrho} \gamma_\iota \Psi(b_\iota, y_\iota, \tau_\iota, \eta_\iota, \kappa_\iota)).
\end{aligned} \tag{2.6}$$

Taking derivatives with respect to t in both ends of (2.6) (i.e., applying H_0 on the left side of the inner products) implies that $(\lim_{h\to 0} \frac{1}{h}[\phi(h) - \phi(0)], \Phi) = (\lim_{h\to 0} \frac{1}{h}[\varphi(h) - \varphi(0)], \Phi)$ for all $\Phi \in \mathcal{D}_1$. But since the derivatives themselves belong to $\mathcal{D}_1$ (c.f. (2.4)), this indicates that $H_0 : \mathcal{D}_1 \to \mathcal{D}_1$ is a well-defined operator. Moreover, by (2.6), H_0 is *symmetric* over $\mathcal{D}_1$.

Set $\mathcal{H}_1 := \overline{\mathcal{D}_1}$. Then H_0 is a symmetric densely-defined operator over the Hilbert space $\mathcal{H}_1$, which we would like to extend to a selfadjoint operator. We do not know, in advance, whether it has equal deficiency indices. We thus use the method of [2], and consider the operator $\tilde{H}_0 : \mathcal{D}_1 \times \mathcal{D}_1 \to \mathcal{D}_1 \times \mathcal{D}_1$, defined by $\tilde{H}_0(x, y) = (H_0 x, -H_0 y)$ for all $x, y \in \mathcal{D}_1$. This is a symmetric densely-defined operator over the Hilbert space $\tilde{\mathcal{H}}_1 = \mathcal{H}_1 \times \mathcal{H}_1$, which has equal deficiency indices (c.f., for example, [6, Lemma 2.44]). Hence, it has a selfadjoint extension, $\tilde{H}$ (over $\tilde{\mathcal{H}}_1$).

Let $E(\cdot)$ be the spectral measure of $\tilde{H}$. Denote, for $m \in \mathbb{N}$, $E_m := E([-m, m])$, $\tilde{H}_m := \tilde{H} E_m$. Fix $a \in \mathcal{A}$, $x \in \mathcal{D}$, n such that $\frac{1}{n} < \varepsilon_a(x)$ and $k \in \mathbb{Z}^+$. For every $0 < t < \epsilon := \varepsilon_a(x) - \frac{1}{n}$, define $\Psi_m(t) := E_m\binom{\Psi(a,x,t,n,k)}{0}$

for all $m \in \mathbb{N}$ (note that the set of (a, x, t, n, k) that can be chosen this way forms exactly Ω). Then by the definition of H_0, $\tilde{H}_0$ and the boundedness of E_m,

$$\Psi'_m(t) = E_m\left(\begin{smallmatrix} H_0\Psi(a,x,t,n,k) \\ 0 \end{smallmatrix} \right) = E_m\tilde{H}_0\left(\begin{smallmatrix} \Psi(a,x,t,n,k) \\ 0 \end{smallmatrix} \right) = \tilde{H}_m\Psi_m(t).$$

Since $\tilde{H}_m$ is a *bounded* operator (over $\tilde{\mathcal{H}}_1$), then for all $0 < t' \le t < \epsilon$,

$$\begin{aligned} \Psi_m(t) &= e^{(t-t')\tilde{H}_m}\Psi_m(t') = e^{(t-t')\tilde{H}_m}E_m\Psi_m(t') \\ &= e^{(t-t')\tilde{H}}E_m\Psi_m(t') = e^{(t-t')\tilde{H}}\Psi_m(t'). \end{aligned}$$

Now, since $s - \lim_{n\to\infty} E_m = I$ (in $\tilde{\mathcal{H}}_1$), and since $e^{(t-t')\tilde{H}}$ is selfadjoint, hence closed, $\left(\begin{smallmatrix} \Psi(a,x,t',n,k) \\ 0 \end{smallmatrix} \right) \in D(e^{(t-t')\tilde{H}})$, and

$$\left(\begin{smallmatrix} \Psi(a,x,t,n,k) \\ 0 \end{smallmatrix} \right) = e^{(t-t')\tilde{H}}\left(\begin{smallmatrix} \Psi(a,x,t',n,k) \\ 0 \end{smallmatrix} \right) \tag{2.7}$$

for all $(a, x, t, n, k) \in \Omega$, $0 < t' \le t$. From (2.7) we conclude that for every $\Psi \in \mathcal{D}_1$, $e^{\tau\tilde{H}}\left(\begin{smallmatrix} \Psi \\ 0 \end{smallmatrix} \right) \in \mathcal{H}_1 \times \{0\}$ for all $0 \le \tau < \rho$ ($\rho > 0$ is small enough) . Since $\tilde{H}$ is selfadjoint, we can analytically expand $e^{\tau\tilde{H}}\left(\begin{smallmatrix} \Psi \\ 0 \end{smallmatrix} \right)$ to $e^{z\tilde{H}}\left(\begin{smallmatrix} \Psi \\ 0 \end{smallmatrix} \right)$ for all z in the strip $\{z : 0 < \mathtt{Re}(z) < \rho\}$. By the analyticity in that strip and the continuity in $\{z : 0 \le \mathtt{Re}(z) < \rho\}$, we have $e^{is\tilde{H}}\left(\begin{smallmatrix} \Psi \\ 0 \end{smallmatrix} \right) \in \mathcal{H}_1 \times \{0\}$ for all $s \in \mathbb{R}$, $\Psi \in \mathcal{D}_1$. Since $\mathcal{D}_1$ is dense in $\mathcal{H}_1$, $e^{is\tilde{H}}(\mathcal{H}_1 \times \{0\}) \subseteq \mathcal{H}_1 \times \{0\}$, and so the orthogonal projection $P := \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix} \right)$ commutes with $e^{is\tilde{H}}$ for all $s \in \mathbb{R}$ (as selfadjoint operators over $\tilde{\mathcal{H}}_1$), equivalently- with $\tilde{H}$ (c.f. Proposition 1.2). Hence, $H := P\tilde{H}_{|D(\tilde{H})\cap(\mathcal{H}_1\times\{0\})}$ is a selfadjoint operator over $\mathcal{H}_1$.

Let $a \in \mathcal{A}$, $x \in \mathcal{D}$ and $0 < t < \varepsilon_a(x)$ be given. For all n, k such that $(a, x, t, n, k) \in \Omega$ and $0 < t' < t$,

$$\Psi(a, x, t, n, k) = e^{(t-t')H}\Psi(a, x, t', n, k)$$

by (2.7). Take $k = 0$ and let $n \to \infty$, and one has (as $e^{(t-t')H}$ is closed)

$$T_a(t)x = e^{(t-t')H}T_a(t')x, \tag{2.8}$$

equivalently- $e^{(t'-t)H}T_a(t)x = T_a(t')x$. Let now $t' \to 0^+$. The right side tends to $T_a(0)x$ (c.f. Postulate (P1)). The Spectral Theorem and Lebesgue's

Convergence Theorems hence yield that $T_a(t)x \in D(e^{-tH})$ and $e^{-tH}T_a(t)x = T_a(0)x$, that is,

$$T_a(t)x = e^{tH}T_a(0)x. \tag{2.9}$$

Finally, replace H by the operator $H \oplus 0_{\mathcal{H}_1^\perp}$. Then the (new) H is a selfadjoint operator over $\mathcal{H}$, and since $T_a(0)x \in \mathcal{H}_1$, (2.9) is still true.

H is essentially selfadjoint over $\mathcal{M} \oplus \mathcal{M}^\perp$. By our construction,

$$\overline{\mathcal{M}} = \mathcal{H}_1 \;\Rightarrow\; \mathcal{M}^\perp = \mathcal{H}_1^\perp \subseteq \ker H. \tag{2.10}$$

Let $a \in \mathcal{A}$, $x \in \mathcal{D}$ and $0 < t < \varepsilon_a(x)$ be given. By (2.1) (or (2.8)) and the Spectral Theorem, $T_a(t)x \in D(e^{sH}) \cap D(e^{-sH}) \subseteq D(H)$ for every $0 < s < \varepsilon_a(x,t) := \min(\varepsilon_a(x) - t, t)$. Hence $\mathcal{M} \subseteq D(H)$. Denote $H_1 := H_{|\mathcal{M}\oplus\mathcal{M}^\perp}$. H_1 is symmetric as a restriction of the selfadjoint operator H. We assert that $\ker(H_1^* - iI) = \{0\}$. Suppose that $H_1^* y = iy$. Fix a, x, t as above. Then for $0 < s < \varepsilon_a(x,t)$,

$$\frac{d}{ds}(e^{sH}T_a(t)x, y) = (He^{sH}T_a(t)x, y) = (H_1e^{sH}T_a(t)x, y) = -i(e^{sH}T_a(t)x, y),$$

and by the continuity of $(e^{sH}T_a(t)x, y)$ in s we infer that

$$(\forall 0 \le s < \varepsilon_a(x,t)) \quad (e^{sH}T_a(t)x, y) = e^{-is}(T_a(t)x, y).$$

By analytic continuation, we conclude that

$$(\forall r \in \mathbb{R}) \quad (e^{irH}T_a(t)x, y) = e^{r}(T_a(t)x, y).$$

But the left side is bounded in r, thus necessarily $(T_a(t)x, y) = 0$. So $y \in \mathcal{M}^\perp$, and by (2.10) and the definition of H_1, $0 = Hy = H_1y = H_1^*y = iy$. The proof that $\ker(H_1^* + iI) = \{0\}$ is identical. H_1 is therefore essentially selfadjoint.

Uniqueness. Suppose that H' is another selfadjoint operator that satisfies both (2.1) (or (2.8)) and the right side of (2.10). Let $a \in \mathcal{A}$, $x \in \mathcal{D}$ and $0 < t < \varepsilon_a(x)$ be given. Then for all $0 \le h < \varepsilon_a(x) - t$ and $y \in \mathcal{H}$,

$$(e^{hH}T_a(t)x, y) = (T_a(t+h)x, y) = (e^{hH'}T_a(t)x, y).$$

By analytic continuation,

$$(\forall s \in \mathbb{R}) \quad (e^{isH}T_a(t)x, y) = (e^{isH'}T_a(t)x, y).$$

Therefore $e^{isH}T_a(t)x = e^{isH'}T_a(t)x$ for all $a \in \mathcal{A}$, $x \in \mathcal{D}$, $0 < t < \varepsilon_a(x)$ and $s \in \mathbb{R}$. By (2.10) and the boundedness of the operators, $e^{isH} = e^{isH'}$ for all $s \in \mathbb{R}$, which implies that $H = H'$. $\square$

Remark 2.4. The requirement in Postulate (P1) of continuity in 0^+ of the functions $T_a(\cdot)x$ is superfluous if, instead of (2.1), one is satisfied with (2.8). Indeed, this requirement is not used anywhere else in the course of the proof.

Remark 2.5. Our proof of Theorem 2.3 contains that of [2, Theorem I.1]. It is also possible to use the latter theorem directly, together with our new ideas and several more, to prove our theorem. However, it is required in [2] that the Hilbert space $\mathcal{H}$ be separable. This assumption, nevertheless, is not used during the proof of the theorem, so that $\mathcal{H}$ may actually be an arbitrary complex Hilbert space. Likewise, no request for separability is made by us in our results, but for the sake of completeness we have included full details.

A common (and perhaps the most important) example of a local symmetric pre-semigroup of operators is when $\mathcal{A}$ is a subsemigroup of $(\mathbb{R}^+, +)$, and $T_a(t) = T(a + t)$.

Definition 2.6. A family of linear operators $T(v)$, $v \geq a_0$ for some $a_0 \in \mathbb{R}^+$, is called a *local symmetric semigroup of operators* (over $\mathbb{R}$) if there exist a nonempty subsemigroup $\mathcal{A}$ of the semigroup $(\mathbb{R}^+, +)$ and a dense subspace $\mathcal{D}$ of $\mathcal{H}$, such that:

(S1) Domain: for every $a \in \mathcal{A}$, $x \in \mathcal{D}$, there exists $\varepsilon_a(x) > 0$ such that $x \in D(T(a + t))$ for every $0 \leq t < \varepsilon_a(x)$; $t \mapsto \|T(a + t)x\|$ is either measurable over some nonempty subinterval, or bounded on a measurable subset of positive measure, of $(0, \varepsilon_a(x))$; and $\lim_{t\to 0^+} T(a + t)x = T(a)x$.

(S2) Semigroup symmetry: for every $a, b \in \mathcal{A}$ and $x, y \in \mathcal{D}$, if $t, s, u \geq 0$ satisfy $t + u < \varepsilon_a(x)$ and $s + u < \varepsilon_b(y)$, then the following equality holds:

$$(T(a + t + u)x, T(b + s)y) = (T(a + t)x, T(b + s + u)y).$$

(S3) Semigroup law: for every $a, b \in \mathcal{A}$ and $x \in \mathcal{D}$, $\varepsilon_{a+b}(x) \geq \varepsilon_a(x)$ and if $0 \leq t < \varepsilon_a(x)$ then $T(a + t)x \in D(T(b))$ and

$$T(b)T(a + t)x = T(a + b + t)x.$$

(S4) Symmetry: for every $a \in \mathcal{A}$, $T(a)$ is symmetric and $T(a)_{|\mathcal{D}}$ is essentially selfadjoint. We denote $H(a) := \overline{(T(a)_{|\mathcal{D}})} = \overline{T(a)}$.

For the rest of this section, $T(\cdot)$ will denote a local symmetric semigroup of operators (over $\mathbb{R}$). Postulates (S1), (S2) correspond, of course, to (P1), (P2) of Definition 2.1 for $T_a(t) := T(a+t)$. We therefore have the following corollary of Theorem 2.3.

Corollary 2.7. *$H(ma) = H(a)^m$, and the operator H discussed in Theorem 2.3 commutes with $H(a)$, for every $a \in \mathcal{A}$, $m \in \mathbb{N}$.*

Proof. Fix $a \in \mathcal{A}$, $m \in \mathbb{N}$. For every $x \in \mathcal{D}$, $x \in D(T(a)^m)$ and $T(a)^m x = T(ma)x$ by Postulate (S3). Hence, $H(a)^m x = T(a)^m x = T(ma)x$ (c.f. Postulate (S4)), which implies that $H(ma) = \overline{(T(ma)_{|\mathcal{D}})} \subseteq \overline{H(a)^m} = H(a)^m$. Since both $H(ma)$, $H(a)^m$ are selfadjoint, we infer that $H(ma) = H(a)^m$.

Let $a \in \mathcal{A}$, $x, y \in \mathcal{D}$ be given. Then for every $0 \leq t < \varepsilon_a(x)$,

$$\begin{aligned}(e^{tH}T(3a)x, y) &= (T(3a + t)x, y) = (T(2a)T(a + t)x, y)\\ &= (T(a + t)x, T(2a)y) = (T(a)x, T(2a + t)y)\\ &= (x, T(a)T(2a + t)y) = (x, T(3a + t)y)\\ &= (x, e^{tH}T(3a)y)\end{aligned}$$

by (2.1) and Postulates (S2)-(S4). Hence, by analytic extension,

$$(e^{isH}T(3a)x, y) = (x, e^{-isH}T(3a)y) = (e^{isH}x, T(3a)y)$$

for all $s \in \mathbb{R}$. Thus, for every $x \in \mathcal{D}$ fixed, $(e^{isH}x, T(3a)y)$ is a continuous function of y in $\mathcal{D}$. By the definition of the Hilbert adjoint, $e^{isH}x \in D(T(3a)^*) = D(H(3a))$, and $H(3a)e^{isH}x = e^{isH}T(3a)x$. In conclusion, $e^{isH}T(3a)_{|\mathcal{D}} \subseteq H(3a)e^{isH}$, and so $e^{isH}\overline{(T(3a)_{|\mathcal{D}})} = e^{isH}H(3a) \subseteq H(3a)e^{isH}$ for every $s \in \mathbb{R}$. Hence, $H(3a)$ and e^{isH} commute (as selfadjoint operators) for all $s \in \mathbb{R}$, which implies (c.f. Proposition 1.2) that $H(3a) = H(a)^3$ commutes with H. As a result, since $H(a)$ is selfadjoint and 3 is odd, so that $\lambda \mapsto \lambda^{1/3}$ is a well-defined continuous real function, $\left(H(a)^3\right)^{1/3} = H(a)$ commute with H. $\square$

The main advantage of our following final result is that it supplies a spectral representation that depends on a *single*, selfadjoint operator.

Theorem 2.8. *(1) Suppose that $\mathcal{A} \nsubseteq \{0\}$ and $\lim_{n\to\infty} \varepsilon_{na}(x) = \infty$ for all $0 \neq a \in \mathcal{A}$ and $x \in \mathcal{D}$. Then there exists a unique positive selfadjoint operator A such that*

$$T(a+t)x = A^{a+t}x \tag{2.11}$$

for all $0 \neq a \in \mathcal{A}$, $x \in \mathcal{D}$ and $0 \leq t < \varepsilon_a(x)$. This operator also satisfies $H(a) = A^a$ for all $0 \neq a \in \mathcal{A}$.

(2) If $0 \in \mathcal{A}$, there exists a unique positive selfadjoint operator $A_\circ$ such that for all $x \in \mathcal{D}$ and $0 < t < \varepsilon_0(x)$,

$$T(t)x = A_\circ^t x. \tag{2.12}$$

(3) If all of the above is true, then A and $A_\circ$ are related by the equation $A = A_\circ F$, where $F \leq H(0)$ is an orthogonal projection that commutes with $A_\circ$, which is unique with respect to these properties.

Once again, the trivial case $\mathcal{A} = \{0\}$, $T(0) \subseteq I$ reduces Theorem 2.8 to the case of Fröhlich's Theorem ([2, Theorem I.1]), as the hypotheses of the latter imply those of the former.

Proof. Existence. Let $E_a(\cdot)$ denote the resolution of the identity of $H(a)$ for all $a \in \mathcal{A}$. Fix $0 \neq a, b \in \mathcal{A}$, and let $x \in \mathcal{D}$ be given. Since $\lim_{n\to\infty} \varepsilon_{na}(x) = \infty$ and $b > 0$, there exists some $n_0 = n_0(a, b, x)$ such that $\varepsilon_{na}(x) > b$ for all $n \geq n_0$. As a result, for all $n \geq n_0$,

$$H(a)^n H(b)x = H(na + b)x = e^{bH} H(na)x = e^{bH} H(a)^n x$$

(c.f. (2.1), Postulate (S3) and the last Corollary). Therefore, Corollaries 1.4 and 2.7 (which will be used repeatedly in the present proof) yield

$$E_a(\mathbb{R}\backslash\{0\})H(b)x = e^{bH} E_a(\mathbb{R}\backslash\{0\})x,$$

which is true for all $x \in \mathcal{D}$; that is to say, $E_a(\mathbb{R}\backslash\{0\})H(b)_{|\mathcal{D}} \subseteq e^{bH} E_a(\mathbb{R}\backslash\{0\})$. As e^{bH} is closed, we obtain

$$E_a(\mathbb{R}\backslash\{0\})H(b) = E_a(\mathbb{R}\backslash\{0\})\overline{(H(b)_{|\mathcal{D}})} \subseteq e^{bH} E_a(\mathbb{R}\backslash\{0\}). \tag{2.13}$$

Since $e^{bH} E_a(\mathbb{R}\backslash\{0\})$ is symmetric, so is $E_a(\mathbb{R}\backslash\{0\})H(b)$. Thus, by Lemma 1.5, $H(b)$ commutes with $E_a(\mathbb{R}\backslash\{0\})$. Hence,

$$\overline{E_a(\mathbb{R}\backslash\{0\})H(b)} = H(b)E_a(\mathbb{R}\backslash\{0\}) = e^{bH} E_a(\mathbb{R}\backslash\{0\}) \tag{2.14}$$

(equality holds since a selfadjoint operator is maximal symmetric).

From (2.13) and the fact that e^{bH} is injective we infer that $\ker H(b) \subseteq \ker E_a(\mathbb{R}\backslash\{0\}) = \ker H(a)$. By symmetry,

$$\operatorname{Im} E_a(\{0\}) = \ker H(a) = \ker H(b) = \operatorname{Im} E_b(\{0\}), \tag{2.15}$$

and so (2.14) becomes

$$(\forall 0 \neq a, b \in \mathcal{A}) \quad H(b) = e^{bH} E_a(\mathbb{R}\backslash\{0\}) = A^b$$

for $A := e^H E_a(\mathbb{R}\backslash\{0\})$ (which is *independent* of the choice of a by virtue of (2.15)). Therefore

$$e^{tH} H(a) = A^{a+t}$$

for all $0 \neq a \in \mathcal{A}$, $t \geq 0$, whence (2.11) easily follows.

If $0 \in \mathcal{A}$, then by Corollary 2.7, $H(0)^2 = H(0)$. Thus $H(0)$, since it is selfadjoint, is a (necessarily bounded) orthogonal projection. Moreover, for $a \in \mathcal{A}$, $H(0)H(a)_{|\mathcal{D}} \subseteq H(a)_{|\mathcal{D}}$, and we infer (as in (2.13)) that $H(0)H(a) \subseteq H(a)$. By Lemma 1.5, $H(0)$ commutes with $H(a)$, hence $H(a)H(0) = H(a)$. Consequently, $E_a(\mathbb{R}\backslash\{0\}) \leq H(0)$ (both orthogonal projections). To complete the proof of existence, take $A_\circ := e^H H(0)$ and $F := E_a(\mathbb{R}\backslash\{0\})$.

Uniqueness. The uniqueness of A follows immediately from the equality $H(a) = A^a$, that holds for each $0 \neq a \in \mathcal{A}$.

Suppose that (2.12) is also true for $A'_\circ$. Let $G(\cdot)$ and $G'(\cdot)$ be the spectral measures of $A_\circ$ and $A'_\circ$, respectively. Fix $x \in \mathcal{D}$, $y \in \mathcal{H}$. For all $0 < t < \varepsilon_0(x)$, (2.12) implies that

$$\int_{\mathbb{R}^+} \lambda^t (G(d\lambda)x, y) = \int_{\mathbb{R}^+} \lambda^t (G'(d\lambda)x, y).$$

Therefore, on account of the Uniqueness Theorem for the bilateral Laplace transform (c.f. [21, Ch. VI, Theorem 6a]), the measures $(G(\cdot)x, y)$ and $(G'(\cdot)x, y)$, when restricted to the σ-algebra of Borel subsets of $(0, \infty)$, are equal. Since $(G(\mathbb{R}^+)x, y) = (x, y) = (G'(\mathbb{R}^+)x, y)$, those measures are equal as Borel measures over $\mathbb{R}^+$. This is true for all x, y as above, thus $G(\cdot) = G'(\cdot)$, so that $A_\circ = A'_\circ$. The definition of $A_\circ$ yields that $\ker A_\circ = \ker H(0)$. Consequently, if F' is an orthogonal projection that commutes with $A_\circ$, and also $F' \leq H(0)$, $A = A_\circ F'$, then $\ker F' = \ker A$, so necessarily $F' = F$. $\square$

Remark 2.9. One of the consequences of Theorem 2.8 is that $H(a), H(b)$ commute for all $a, b \in \mathcal{A}$. This is *not* a trivial byproduct of Definition 2.6.

3. Application

As an application, we have the following technique of establishing the existence of a representing measure for a positive semi-definite functional over some types of function algebras. In [20], the author defined a special function algebra $\mathcal{R}$, and proved a representation theorem for positive semi-definite functionals over $\mathcal{R}$. It was then shown that such a theorem may be used to derive a fractional moments theorem (c.f. [20, Theorems 2.1, 2.6]). However, in these results, it was required that the functions $\mathbb{R}^+ \ni t \mapsto t^\alpha$ belong to $\mathcal{R}$ for all α in a dense subset of $\mathbb{R}^+$ (c.f. [20, Remark 2.7]). In the following discussion we demonstrate how Theorem 2.8 may be employed to prove a representation theorem for a wider class of function algebras.

Assume that $\mathcal{R}$ is an algebra of complex functions, which contains the constant functions, and such that $\overline{r} \in \mathcal{R}$ for all $r \in \mathcal{R}$ (such an algebra is said to be selfadjoint). Let $\Lambda : \mathcal{R} \to \mathbb{C}$ be a positive semi-definite linear functional over $\mathcal{R}$, i.e., $\Lambda(|r|^2) \geq 0$ for all $r \in \mathcal{R}$. Then Λ induces a semi-inner product over $\mathcal{R}$ in the standard manner, by defining $(r, q) := \Lambda(r\overline{q})$ for $r, q \in \mathcal{R}$. Thus, if $\mathcal{N}$ denotes the ideal $\left\{r \in \mathcal{R} : \Lambda(|r|^2) = 0\right\}$, then $\mathcal{R}/\mathcal{N}$ is an inner-product space. Hence, its completion $\mathcal{H}$ is a complex Hilbert space.

Let $\mathcal{A}$ be a subsemigroup of $\mathbb{R}^+$. Suppose that $\mathcal{R}$ is an algebra of complex functions over $\mathbb{R}^+$, that satisfies the following (we use t as the independent variable):

(1) $t^a, (t^{2a} + 1)^{-1} \in \mathcal{R}$ for all $a \in \mathcal{A}$.
(2) There exists an ideal $\mathcal{Q}$ in $\mathcal{R}$ such that $\mathcal{N} \subseteq \mathcal{Q}$, and for every $r \in \mathcal{Q}$, $a \in \mathcal{A}$, there exists $\varepsilon_a(r) > 0$, for which $t^\alpha r \in \mathcal{Q}$ if $a \leq \alpha < a + \varepsilon_a(r)$ (we take $\varepsilon_a(r)$ to be the maximal with this property).
(3) If $a \in \mathcal{A}$ and $r_0 \in \mathcal{N}$, then $t^\alpha r_0 \in \mathcal{N}$ for $a \leq \alpha < a + \sup_{r \in \mathcal{Q}} \varepsilon_a(r)$.

Given $r \in \mathcal{Q}$ and $a \in \mathcal{A}$, (2) and (3) imply that $\varepsilon_a(r + r_0) = \varepsilon_a(r)$ for all $r_0 \in \mathcal{N}$. Hence we may set $\varepsilon_a(r + \mathcal{N}) := \varepsilon_a(r)$. For $0 \leq s < \varepsilon_a(r + \mathcal{N})$, define

$$T(a + s)(r + \mathcal{N}) := t^{a+s} r + \mathcal{N}.$$

This definition is legal by virtue of (3). For $r + \mathcal{N} \in (\mathcal{Q}/\mathcal{N})^\perp$, set $\varepsilon_a(r + \mathcal{N}) := \infty$ and $T(a + s)(r + \mathcal{N}) := 0$ for every $s \geq 0$.

We prove that $T(\cdot)$ satisfies the requirements of Definition 2.6 with $\mathcal{D} := \mathcal{Q}/\mathcal{N} \oplus (\mathcal{Q}/\mathcal{N})^{\perp}$. From the definitions of the inner-product and $T(\cdot)$ it follows that Postulates (S2), (S3) hold, and that $T(a)$ (whose domain is exactly $\mathcal{D}$) is symmetric for all $a \in \mathcal{A}$. Moreover, if $r \in \mathcal{Q}$, $a \in \mathcal{A}$, then since $\mathcal{Q}$ is an ideal in $\mathcal{R}$, $(t^a \pm i)^{-1} r = (t^a \mp i)(t^{2a}+1)^{-1} r \in \mathcal{Q}$ (by virtue of (1)). Consequently, $r + \mathcal{N} \in \operatorname{Im}(T(a) \pm iI)$. Since obviously $(\mathcal{Q}/\mathcal{N})^{\perp} \subseteq \operatorname{Im}(T(a) \pm iI)$, we infer that $\operatorname{Im}(T(a) \pm iI)$ are dense in $\mathcal{H}$, hence $T(a)$ is essentially selfadjoint. (This technique of establishing the essentially selfadjointness of $T(a)$ it taken from [19, page 1269]). In order to make Postulate (S1) true, we add the following requirement:

(4) For fixed $r \in \mathcal{Q}$ and $a \in \mathcal{A}$, the function $s \mapsto \Lambda(t^{2a+2s} |r|^2)$ is either measurable over some nonempty subinterval, or bounded on a measurable subset of positive measure, of $(0, \varepsilon_a(r))$; and $\lim_{s \to 0^+} \Lambda((t^{a+s} - t^a)^2 |r|^2) = 0$.

Theorem 2.8 is now applicable, and yields the following result.

Theorem 3.1. *Let $\mathcal{R}$ be a selfadjoint algebra of complex functions over $\mathbb{R}^+$, and let Λ be a positive semi-definite functional over $\mathcal{R}$. If (1)-(4) are satisfied by $\mathcal{R}$ and Λ, then there exists a unique positive selfadjoint operator A over the Hilbert space $\mathcal{H}$ associated with Λ, whose kernel contains $(\mathcal{Q}/\mathcal{N})^{\perp}$, and such that*

$$A^{\alpha}(r + \mathcal{N}) = t^{\alpha} r + \mathcal{N}$$

for every $r \in \mathcal{Q}$, $a \in \mathcal{A}$ and $a \leq \alpha < a + \varepsilon_a(r)$.

This means, for example, that given $r \in \mathcal{R}$, there exists a positive Borel measure μ_r over $\mathbb{R}^+$ such that

$$\Lambda(t^{\alpha} |r|^2) = \int_{\mathbb{R}^+} \lambda^{\alpha} d\mu_r$$

for every $a \in \mathcal{A}$, $a \leq \alpha < a + \varepsilon_a(r)$.

A similar theorem, for algebras $\mathcal{R}$ consisting of functions of k positive real variables $t_1, \ldots, t_k$, is easily obtained. Postulates (1)-(4) should be substituted by the corresponding ones, in which t is replaced by t_i for $i = 1, \ldots, k$.

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References

[1] Devinatz, A., *A note on semi-groups of unbounded self-adjoint operators*, Proc. Amer. Math. Soc. 5, no. 1 (1954), 101-102.

[2] Fröhlich, J., *Unbounded, symmetric semigroups on a separable Hilbert space are essentially selfadjoint*, Adv. in Appl. Math. 1, no. 3 (1980), 237-256.

[3] Fuglede, B., *A commutativity theorem for normal operators*, Proc. Natl. Acad. Sci. USA 36, no. 1 (1950), 35-40.

[4] Hille, E., *On semi-groups of transformations in Hilbert space*, Proc. Natl. Acad. Sci. USA 24, no. 3 (1938), 159-161.

[5] Hille, E., *Notes on linear transformations. II. Analyticity of semi-groups*, Ann. of Math. (2) 40, no. 1 (1939), 1-47.

[6] Kantorovitz, S., *Semigroups of Operators and Spectral Theory*, Pitman Research Notes in Math., no. 330, Longman (1995).

[7] Kantorovitz, S. and Hughes, R. J., *Spectral representation of local semigroups*, Math. Ann. 259, no. 4 (1982), 455-470.

[8] Kantorovitz, S. and Hughes, R. J., *Spectral analysis of certain operator functions*, J. Operator Theory 22, no. 2 (1989), 243-265.

[9] Kemperman, J. H. B., *On the regularity of generalized convex functions*, Trans. Amer. Math. Soc. 135 (1969), 69-93.

[10] Klein, A. and Landau, L. J., *Construction of a unique selfadjoint generator for a symmetric local semigroup*, J. Funct. Anal. 44, no. 2 (1981), 121-137.

[11] Sz.-Nagy, B. v., *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 5, part 5, Springer, Berlin, 1942.

[12] Nussbaum, A. E., *Spectral representation of certain one-parametric families of symmetric operators in Hilbert space*, Trans. Amer. Math. Soc. 152, no. 2 (1970), 419-429.

[13] Nussbaum, A. E., *Multi-parameter local semi-groups of hermetian operators*, J. Funct. Anal. 48, no. 2 (1982), 213-223.

[14] Reed, M. and Simon, B., *Methods of Modern Mathematical Physics I: Functional Analysis*, Academic Press, New York-London, 1972.

[15] Ricker, W., *Spectral representation of local semigroups associated with Klein-Landau systems*, Math. Proc. Cambridge Philos. Soc. 95, no. 1 (1984), 93-100.

[16] Ricker, W., *Spectral representation of local semigroups in locally convex spaces*, Czechoslovak Math. J. 35, no. 2 (1985), 248-259.

[17] Shucker, D. S., *Extensions and generalizations of a theorem of Widder and of the theory of symmetric local semigroups*, J. Funct. Anal. 58, no. 3 (1984), 291-309.

[18] Sierpiński, W., *Sur les fonctions convexes mesurables*, Fundam. Math. 1 (1920), 125-128.

[19] Vasilescu, F.-H., *Hamburger and Stieltjes moment problems in several variables*, Trans. Amer. Math. Soc. 354, no. 3 (2002), 1265-1278.

[20] Viselter, A., *Generalized Widder Theorem via fractional moments*, J. Funct. Anal. 256, no. 2 (2009), 594-602.

[21] Widder, D. V., *The Laplace Transform*, Princeton University Press, Princeton, 1941.

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