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RESEARCH ARTICLE

# Abstract Volterra Relation

**A. Viselter and S. Kantorovitz**[*]

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**Abstract**

The abstract Volterra relation is the relation $VD(S) \subseteq D(S)$, $SV - VS \subseteq V^2$, satisfied by a closed operator $S$ and a bounded operator $V$ over some complex Banach space $X$. Results of several articles of the second author are extended from the case of $S$ bounded to the general case. These include results on similarity, quasi-affinity, $C^n$-classification, and growth of related semigroups.

**Keywords:** Volterra commutation relation, semigroups of operators, holomorphic semigroups, boundary group, similarity, quasi affinity, fractional powers of operators.

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## Introduction

This article deals with various settings of the abstract Volterra relation, which is the relation

$$VD(S) \subseteq D(S), \quad SV - VS \subseteq V^2,$$

satisfied by a closed operator $S$ and a bounded operator $V$ over some complex Banach space $X$.

Our first case is of an unbounded Volterra systems, which is a system $\{S_n, V_n\colon 1 \le n \le N\}$, in which for all $n$, the pair $(S_n, V_n)$ satisfies the Volterra relation, plus a few additional requirements. This setting is motivated by the classical example, in which $X = L^p([0,\infty)^N)$ (for $1 < p < \infty$ fixed), $S_n\colon f(x) \mapsto x_n f(x)$, and $V_n$ is the "weighted Volterra operator"

$$(V_n f)(x) = \int_0^{x_n} e^{-\varepsilon(x_n - t)} f(x_1, \ldots, x_{n-1}, t, x_{n+1}, \ldots, x_N)\, dt$$

for some $\varepsilon > 0$ (see Example 4.2) . For $\zeta \in \mathbb{C}^N$, we set $T_\zeta := S + \sum_{n=1}^N \zeta_n V_n$, where $S = \sum_{n=1}^N S_n$. The operators $iT_\zeta$ generate a $C_0$-group, denoted by $T_\zeta(\cdot)$. Among our results in this case, we prove that for fixed $t \in \mathbb{R}$, $T_\zeta(t)$ is an entire function of $\zeta \in \mathbb{C}^N$; we find an upper bound for the growth of $\|T_\zeta(t)\|$; we prove that for $\alpha, \zeta \in \mathbb{C}^N$ with $Re(\alpha) = Re(\zeta)$, $T_\alpha$ and $T_\zeta$ are similar, and that

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$T_\zeta$ is of class $C^{n+m_1+\cdots+m_N}$ for $S$ of class $C^n$ and $\zeta \in \mathbb{C}^N$ with $(\forall 1 \le k \le N)$ $|Re(\zeta_k)| \le m_k$.

In our second case, based on two alternative standing hypothesis, $S$ is a closed operator, $V \in B(X)$ is injective and $(S, V)$ satisfy the Volterra relation. Moreover, $-A := -V^{-1}$ either satisfies $(0, \infty) \subseteq \rho(-A)$ and $\sup_{\lambda>0} \|\lambda R(\lambda; -A)\| < \infty$, or is the generator of a uniformly bounded $C_0$-semigroup. The former is motivated by Example 4.3, in which $X = L^p(0, \infty)$ ($1 \le p < \infty$ fixed), $S$: $f(x) \mapsto -xf(x)$, and $A = \varepsilon I - \frac{d}{dx}$ for some $\varepsilon > 0$ (with a suitable domain of definition). Instead of using similarity, we use the weaker relation of quasi-affinity ($B$ is a quasi-affine transform of $A$ if $QA \subseteq BQ$ for some injective $Q \in B(X)$). We prove that for $\frac{1}{2} \le \alpha < 1$ or $0 < \alpha < 1$, respectively, and $t \ge 0$, $\widetilde{S}$ is a quasi-affine transform of $\widetilde{S} + tV^\alpha$, and $\widetilde{S} - tV^\alpha$ is a quasi-affine transform of $\widetilde{S}$, where $V^\alpha$ is Balakrishnan's fractional power and $\widetilde{S}$ is either a restriction of $S$ to some known manifold of $X$ or $S$ itself, respectively. In addition, for all $\zeta, \beta \in \mathbb{C}$ with $1 < Re(\beta)$, $S + \zeta V^\beta$ is similar to $S$. Following the first case, we also extend this one to unbounded Volterra systems.

## 1. Preliminaries

Throughout this paper, $X$ will denote a complex Banach space.

Let $S$ be a closed operator over $X$ with domain $D(S)$, and $V \in B(X)$ such that

$$VD(S) \subseteq D(S). \tag{1.1}$$

When this is the case, the Lie product operator $[S, V] := SV - VS$ satisfies $D([S, V]) = D(S)$. When (1.1) is true, we say that $S$ and $V$ satisfy the *abstract Volterra relation* if

$$[S, V] \subseteq V^2.$$

The reader of this article is assumed to be familiar with the basic theory of semigroups of operators, that can be found in the first chapters of [5]. The second chapter of this article is based on the results of [6], chapter 11 and [9], that are generalized in the spirit of [7]. The third chapter is a generalization of [8].

We begin with some preliminaries concerning regular semigroups.

**Notation 1.1.** For $\zeta \in \mathbb{C}$, we use the notations $\xi = Re(\zeta)$, $\eta = Im(\zeta)$.

**Definition 1.2.** A *regular* semigroup is a $C_0$-semigroup holomorphic in $\mathbb{C}^+$, which is bounded in the rectangle $Q := \{\zeta \in \mathbb{C}\colon\ 0 < \xi \le 1, |\eta| \le 1\}$. We set

$$\nu := \sup_{\zeta \in Q} \log \|V(\zeta)\|.$$

Then $0 \le \nu < \infty$.

**Theorem 1.3.** *Let $V(\cdot)$ be a regular semigroup. Then for each $\eta \in \mathbb{R}$, $V(\xi + i\eta)$ converges strongly as $\xi \to 0^+$ to a bounded operator, $V(i\eta)$, with the following properties:*

(1) $\{V(i\eta)\colon \eta \in \mathbb{R}\}$ *is a* $C_0$*-group.*

(2) $V(i\eta)$ *commutes with* $V(\zeta)$ *for all* $\eta \in \mathbb{R}$, $\zeta \in \mathbb{C}^+$.

(3) $V(\xi + i\eta) = V(\xi)V(i\eta)$ *for all* $\xi > 0$, $\eta \in \mathbb{R}$.

(4) $V(\cdot)$ *is of exponential type* $\leq \nu$, *that is, there exists a constant* $K > 0$ *such that* $\|V(\zeta)\| \leq Ke^{\nu|\zeta|}$ *for all* $\zeta \in \overline{\mathbb{C}^+}$.

**Proof.** Theorem 17.9.1 from [4].

**Definition 1.4.** ([4], page 235) Let $V(\cdot)$ be a regular semigroup. The *Nörlund function* of $V(\cdot)$, $\gamma(\cdot)$, is given by

$$(\forall \xi > 0) \quad \gamma(\xi) := \limsup_{|\eta| \to \infty} |\eta|^{-1} \log \|V(\xi + i\eta)\|.$$

Let $(\alpha_0, \alpha_1)$ be the largest $\alpha$-interval such that the equation

$$\gamma(\xi) = \frac{\pi}{2\alpha}$$

has a (necessarily unique) solution for all $0 \leq \alpha_0 < \alpha < \alpha_1 \leq \infty$. The symbols $\alpha_0$, $\alpha_1$ will be used in this sense in the rest of the paper.

**Lemma 1.5.** *Let $V(\cdot)$ be a regular semigroup with $\alpha_1 > 1$. Set $V = V(1)$. Then every (bounded) operator commuting with $V$ commutes with $V(\zeta)$ for all $\zeta \in \overline{\mathbb{C}^+}$.*

**Proof.** Lemma 2.3 from [9].

**Corollary 1.6.** *Let $V_i(\cdot)$, $i = 1, 2$ be two regular semigroups with $\alpha_1 > 1$. Set $V_i = V_i(1)$, $i = 1, 2$, and assume that $V_1$, $V_2$ commute. Then $V_1(\zeta)$ commutes with $V_2(\lambda)$ for all $\zeta, \lambda \in \overline{\mathbb{C}^+}$.*

We next present some of the basics of Balakrishnan's fractional powers theory.

**Definition 1.7.** ([1], (2.1)) Let $A$ be a closed operator over $X$. Suppose that $(0, \infty) \subseteq \rho(-A)$ and that there exists an $M < \infty$ such that for all $\lambda > 0$,

$$\|\lambda R(\lambda; -A)\| \leq M.$$

In this case we define, for each $\alpha \in \mathbb{C}$ with $0 < Re(\alpha) < 1$ and $x \in D(A)$,

$$J^\alpha x := \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha - 1} R(\lambda; -A) Ax \, d\lambda$$

(we take the principal value of $\lambda^\alpha$, so that $\lambda^\alpha$ is positive for positive $\alpha$).

The operator $J^\alpha$ (with domain $D(A)$) is well-defined, since it is easily seen that the integral above converges absolutely.

This definition can be extended to all $\alpha$ such that $Re(\alpha) > 0$ (see [1], (2.2), (2.3) and (2.4)), to form a family of operators, $J^\alpha$, $Re(\alpha) > 0$, such that for $0 < Re(\alpha) < 1$, $D(J^\alpha) = D(A)$, and for $n-1 \leq Re(\alpha) < n$, $n \geq 2$, $D(J^\alpha) = D(A^n)$.

By Lemma 2.1 from [1], $J^\alpha$ is closable for all $\alpha \in \mathbb{C}$ with $Re(\alpha) > 0$, which allows us to define ([1], (2.7))

$$A^\alpha := \overline{J^\alpha}.$$

If $A$ is bounded, $J^\alpha = A^\alpha$ are bounded for all $\alpha$ with $Re(\alpha) > 0$. For more basic properties of the fractional powers, we refer the reader to [1].

We recall that a $C_0$-semigroup, $T(\cdot)$, is called *uniformly bounded* if there exists an $M < \infty$ such that for all $t \geq 0$, $\|T(t)\| \leq M$.

We use Balakrishnan's fractional powers under two alternative standing hypothesis:

**Standing Hypothesis 1.8.** Let $X$ be a complex Banach space.

(1) $A$ is a closed, densely defined operator, such that $(0,\infty) \subseteq \rho(-A)$ and there exists an $M < \infty$ with

$$(\forall \lambda > 0) \quad \|\lambda R(\lambda; -A)\| \leq M.$$

(2) $-A$ is the generator of a uniformly-bounded $C_0$-semigroup.

Note that (1) means that the assumptions of Definition 1.7 are satisfied by $A$, plus the fact that $A$ is densely-defined. Moreover, by the Hille-Yoshida Theorem, (2) is more restrictive than (1).

**Theorem 1.9.** *Assume that $A$ satisfies the assumptions of Standing Hypothesis 1.8, (1). Then:*

(1) *For $0 < \alpha \leq \frac{1}{2}$, $-A^\alpha$ generates a $C_0$-semigroup holomorphic in some sector of $\mathbb{C}$, that contains the positive real axis.*

(2) *For $0 < \alpha \leq \frac{1}{2}$, $(0,\infty) \subseteq \rho(-A^\alpha)$, and for all $\mu > 0$, $x \in X$,*

$$(\mu I + A^\alpha)^{-1} x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty R(\lambda; -A) \frac{\lambda^\alpha}{\mu^2 - 2\mu\lambda^\alpha \cos \pi\alpha + \lambda^{2\alpha}} x \, d\lambda$$

*(the integrand is of class $L^1(0,\infty)$).*

*Suppose that $V \in B(X)$ is injective, such that $A = V^{-1}$. Then:*

(3) *For $\alpha \in \mathbb{C}^+$, $V^\alpha$ is injective, and $A^\alpha = (V^\alpha)^{-1}$.*

(4) *Let* $\alpha \in \mathbb{C}^+$. *Assume that* $T \in B(X)$ *is such that* $T$, $R(\mu; A)$ *commute for some* $\mu \in \rho(A)$. *Then* $TA^\alpha \subseteq A^\alpha T$.

*If* $A$ *satisfies the assumptions of Standing Hypothesis 1.8, (2), then (1) and (2) above are true for all* $0 < \alpha < 1$, *and:*

(5) $J^\alpha$ *is closed; in particular,* $A^\alpha = J^\alpha$.

**Proof.** Lemma 3.5, Theorem 5.1, Lemma 6.3 and the beginning of its proof in [1]; the discussion after the proof of Theorem 5.1.2 and Corollaries 5.1.12, (ii) and 5.2.2 in [2]; and the discussion on page 260 and Theorem 1, page 263 in [11].

The following lemma will be used repeatedly in the 3rd chapter.

**Lemma 1.10.** *Let* $V, C \in B(X)$ *be commuting operators, and let* $S$ *be a closed operator with domain* $D(S)$. *Suppose that* $VD(S) \subseteq D(S)$ *and that* $[S, V] \subseteq C$. *Then for all* $\lambda \in \rho(V)$,

$$R(\lambda; V)D(S) \subseteq D(S)$$

*and*

$$[S, R(\lambda; V)] \subseteq CR(\lambda; V)^2.$$

**Proof.** This is the case (a) $\Rightarrow$ (b) in Lemma 12.1 from [6].

## 2. Unbounded Volterra systems

### 2.1. General theory

**Definition 2.1.** An Unbounded Volterra System over a Banach space $X$ is a system $\{S_n, V_n\colon\ n = 1, \dots, N\}$, such that:

(1) For each $n = 1, \dots, N$, $iS_n$ is the generator of the $C_0$-group $S_n(t)$.

(2) For each $1 \le k, l \le N$, $t \in \mathbb{R}$, $S_k(t)S_l(t) = S_l(t)S_k(t)$.

(3) Denote by $iS$ the generator of the $C_0$-group $S(t) = S_1(t) \cdots S_N(t)$; then $S = \sum_{n=1}^N S_n$ (defined on $D(S) = \bigcap_{n=1}^N D(S_n)$ ).

(4) For each $1 \le n \le N$, $V_n \in B(X)$.

(5) For each $1 \le k, l \le N$, $[V_k, V_l] = 0$, $V_k D(S_l) \subseteq D(S_l)$ and $[S_l, V_k] \subseteq \delta_{lk} V_k^2$.

**Comment 2.2.** The fact that $S(t)$ as defined above is indeed a $C_0$-group is a result of 1,2. In this case, it is easy to prove that $\sum_{n=1}^N S_n \subseteq S$. We require equality.

**Notation 2.3.** For each $\zeta = (\zeta_1, \ldots, \zeta_N) \in \mathbb{C}^N$, $0 \le j \le N$, we set $T_{\zeta,j} := S + \sum_{k=1}^{j} \zeta_k V_k$. Let $T_{\zeta,j}(\cdot)$ be the $C_0$-group generated by $iT_{\zeta,j}$ (see Theorem 1.38 from [5]). We also set $T_\zeta := T_{\zeta,N}$, $T_\zeta(\cdot) := T_{\zeta,N}(\cdot)$.

**Lemma 2.4.** *Let $S(t)$ be a $C_0$-group, whose generator is $iS$, and let $\{V_n\colon 1 \le n \le N\} \subseteq B(X)$ be such that for each $1 \le k,l \le N$, $V_k D(S) \subseteq D(S)$, $[S, V_k] \subseteq V_k^2$, and $[V_k, V_l] = 0$. Then for each $1 \le n \le N$, $\rho(V_n)$ contains $i\mathbb{R}\backslash\{0\}$, and if $A \cup B = \{1, 2, \ldots, N\}$ and $A \cap B = \emptyset$,*

$$(\forall K \in \mathbb{Z}^N, t \in \mathbb{R}) \quad T_K(t) = \prod_{n \in A} (I - itV_n)^{-K_n} S(t) \prod_{n \in B} (I + itV_n)^{K_n}$$

*and*

$$\|T_K(t)\| \le \|S(t)\| \prod_{n=1}^{N} (1 + |t|\|V_n\|)^{|K_n|}.$$

**Proof.** We will prove by induction on $j$ ($j \le n$) that if $A \cup B = \{1, \ldots, j\}$ and $A \cap B = \emptyset$, then for each $K \in \mathbb{Z}^N$ (equivalently, $K \in \mathbb{Z}^j$),

$$T_{K,j}(t) = \prod_{n \in A} (I - itV_n)^{-K_n} S(t) \prod_{n \in B} (I + itV_n)^{K_n}.$$

For $j = 0$, the proof is trivial, since $T_{\zeta,0} = S$, and so $T_{\zeta,0}(\cdot) = S(\cdot)$. We assume that the claim above is true for some $1 \le j < N$, and prove its validity for $j+1$: let $K = (K_1, \ldots, K_{j+1})$ be in $\mathbb{Z}^{j+1}$, and set $S' := T_{K,j}$. Observe that $S'$ actually doesn't depend on $K_{j+1}$, but only on $K_1, \ldots, K_j$. According to the properties of $S, V_1, \ldots, V_N$, the pair $(S', V_{j+1})$ satisfies the assumptions of Theorem 1.3 from [9] (used here with $A = -I$ in its notations); therefore, $\rho(V_{j+1})$ contains $i\mathbb{R}\backslash\{0\}$, and since $S' + K_{j+1} V_{j+1} = T_{K,j+1}$, $iT_{K,j+1}$ generates the $C_0$-group $T_{K,j+1}(t)$, and we have $T_{K,j+1}(t) = T_{K,j}(t)(I + itV_{j+1})^{K_{j+1}} = (I - itV_{j+1})^{-K_{j+1}} T_{K,j}(t)$. If $A, B$ are as above, and if we assume (without loss of generality) that $j+1 \in B$, then by the induction assumption (when setting $A' = A, B' = B\backslash\{j+1\}$) and the first equality in the equation above, we have

$$\begin{aligned} T_{K,j+1}(t) &= \prod_{n \in A'} (I - itV_n)^{-K_n} S(t) \prod_{n \in B'} (I + itV_n)^{K_n} (I + itV_{j+1})^{K_n} \\ &= \prod_{n \in A} (I - itV_n)^{-K_n} S(t) \prod_{n \in B} (I + itV_n)^{K_n}. \end{aligned}$$

This concludes the proof of the first statement of the Lemma.

To prove the second, set $A := \{1 \le n \le N\colon K_n < 0\}$, $B := \{1 \le n \le N\colon K_n \ge 0\}$. By the first statement, for all $K \in \mathbb{Z}^N$, $\|T_K(t)\| = \|\prod_{n \in A} (I - itV_n)^{|K_n|} S(t) \prod_{n \in B} (I + itV_n)^{|K_n|}\| \le \|S(t)\| \prod_{n=1}^{N} (1 + |t|\|V_n\|)^{|K_n|}$, and the proof is complete. ∎

**Lemma 2.5.** *Under the assumptions of an Unbounded Volterra System, for all* $\zeta \in \mathbb{C}^N$, $K \in \mathbb{Z}^N$ *and* $t \in \mathbb{R}$,

$$T_{\zeta+K}(t) = T_\zeta(t) \prod_{n=1}^{N} (I + itV_n)^{K_n} = \prod_{n=1}^{N} (I - itV_n)^{-K_n} T_\zeta(t).$$

**Proof.** Let $\zeta \in \mathbb{C}^N$ be given. By using Lemma 2.4 with $T_\zeta$ instead of $S$, and noting that for each $K \in \mathbb{Z}^N$, $T_\zeta + \sum_{n=1}^N K_n V_n = T_{\zeta+K}$, we have the wanted equality (by first setting $A = \emptyset, B = \{1, 2, \ldots, N\}$, then the opposite). ∎

**Definition 2.6.** ([3], page 224) Let $G$ be an open set in $\mathbb{C}^N$. A function $f$ defined on $G$ with values in $X$ is said to be *analytic* on $G$ if $f$ is continuous, and the first partial derivatives $\partial f/\partial z_i$, $i = 1, 2, \ldots, N$, exist at every point of $G$.

**Lemma 2.7.** *Under the assumptions of an Unbounded Volterra System, for all* $t \in \mathbb{R}$ *fixed,* $T_\zeta(t)$ *is an entire function of* $\zeta \in \mathbb{C}^N$*; moreover, for* $t$ *in a neighborhood of* $0$,

$$(\forall \zeta \in \mathbb{C}^N) \quad T_\zeta(t) = S(t) \prod_{n=1}^{N} (I + itV_n)^{\zeta_n} = \prod_{n=1}^{N} (I - itV_n)^{-\zeta_n} S(t).$$

**Proof.** Let $M, \omega \geq 0$ be such that $\|S(t)\| \leq M e^{\omega|t|}$ for each $t \in \mathbb{R}$.

We will prove inductively that for each $j$, $0 \leq j \leq N$, $T_{\zeta,j}(t)$ is an entire function of the variable $(\zeta_1, \ldots, \zeta_j) \in \mathbb{C}^j$ for every $t \in \mathbb{R}$ fixed, that $\|T_{\zeta,j}(t)\| \leq M \cdot exp(\omega|t| + M|t| \sum_{k=1}^j |\zeta_k| \|V_k\|)$ for each $t \in \mathbb{R}$, $\zeta \in \mathbb{C}^N$ (or $\zeta = (\zeta_1, \ldots, \zeta_j) \in \mathbb{C}^j$, equivalently), and that there exists a neighborhood of zero such that for each $t$ in the neighborhood and $\zeta \in \mathbb{C}^N$,

$$T_{\zeta,j}(t) = S(t) \prod_{k=1}^{j} (I + itV_k)^{\zeta_k} = \prod_{k=1}^{j} (I - itV_k)^{-\zeta_k} S(t).$$

In particular, those exponentials make sense.

For $j = 0$, since $T_{\zeta,0}(t) = S(t)$, the claims are obvious.

We assume that the claims above are true for some $j$, $0 \leq j < N$, and prove their validity for $j+1$. By Definition 2.1, the pair $(T_{\zeta,j}, V_{j+1})$ satisfies the assumptions of Corollary 11.6 from [6] for every fixed $\zeta \in \mathbb{C}^N$. Applying that Corollary (and its proof's inner results), one gets the equality $T_{\zeta,j+1} = \sum_{m=0}^\infty U_m(t)(i\zeta_{j+1})^m$, when for each $m \in \mathbb{N}, t \in \mathbb{R}, x \in X$,

$$U_0(t) := T_{\zeta,j}(t) \ , \ U_m(t)x := \int_0^t T_{\zeta,j}(t-s) V_{j+1} U_{m-1}(s) x \, ds$$

(actually, the functions $\{U_m\}_{m=0}^{\infty}$ from $\mathbb{R}$ to $B(X)$ depend on $(\zeta_1, \ldots, \zeta_j)$, and the precise notation should be $U_{(\zeta_1,\ldots,\zeta_j),m}(t)$).

By the induction assumption, since $T_{\zeta,j}$ is an entire function of $(\zeta_1, \ldots, \zeta_j) \in \mathbb{C}^j$ for every $t \in \mathbb{R}$ fixed, each of the functions $U_m(t)$ $(m = 0, 1, \ldots)$ is an entire function of $(\zeta_1, \ldots, \zeta_j) \in \mathbb{C}^j$ for every $t \in \mathbb{R}$ fixed. Moreover,

$$\|U_m(t)\| \leq M(M\|V_{j+1}\||t|)^m \cdot e^{\omega|t|+M|t|\sum_{k=1}^{j}|\zeta_k|\|V_k\|}/m!$$

(here, $\omega + M\sum_{k=1}^{j}|\zeta_k|\|V_k\|$ replaces $\omega$ in the Corollary). Therefore, for every $t \in \mathbb{R}$ fixed, $\limsup_{m\to\infty}\|U_m(t)\|^{1/m} = 0$ uniformly for $(\zeta_1, \ldots, \zeta_j)$ in every bounded subset of $\mathbb{C}^j$ (since $\sqrt[m]{m!} \to \infty$ as $m \to \infty$), and so the series $\sum_{m=0}^{\infty} U_m(t)(i\zeta_{j+1})^m$ converges uniformly for $(\zeta_1, \ldots, \zeta_j)$ in every bounded subset of $\mathbb{C}^{j+1}$. Thus, $T_{\zeta,j+1}(t)$ is an entire function of the variable $(\zeta_1, \ldots, \zeta_j, \zeta_{j+1})$ for every $t \in \mathbb{R}$ fixed. Moreover, by the estimate of $\|U_m(t)\|$ above (and the power-series representation $exp(a) = \sum_{m=0}^{\infty} a^m/m!$),

$$\|T_{\zeta,j+1}(t)\| \leq M \cdot e^{\omega|t|+M|t|\sum_{k=1}^{j+1}|\zeta_k|\|V_k\|}.$$

By the induction hypothesis and Corollary 11.6 from [6], there exists a neighborhood of zero such that for all $t$ in that neighborhood and $\zeta \in \mathbb{C}^N$,

$$\begin{aligned} T_{\zeta,j+1}(t) &= T_{\zeta,j}(t)(I + itV_{j+1})^{\zeta_{j+1}} = S(t)\prod_{k=1}^{j+1}(I + itV_k)^{\zeta_k} \\ &= (I - itV_{j+1})^{-\zeta_{j+1}}T_{\zeta,j}(t) = \prod_{k=1}^{j+1}(I - itV_k)^{-\zeta_k}S(t), \end{aligned}$$

as wanted. ∎

**Definition 2.8.** An Unbounded Volterra System $\{S_k, V_k \colon k = 1, \ldots N\}$ over a Banach space $X$ is said to be *regular* if for each $1 \leq n \leq N$, $V_n(\cdot)$ is a regular $C_0$-semigroup with $\alpha_1 > 1$, such that $V_n = V_n(1)$.

In a regular system, the boundary groups $V_n(i\eta)$ exist, and there are positive finite constants $\nu_n$ and $K_n$ such that $\|V_n(i\eta)\| \leq K_n \exp(\nu_n|\eta|)$ for all $n = 1, \ldots, N$ and $\eta \in \mathbb{R}$. It follows that there exist positive finite constants $\nu, K$ such that $\|V_n(i\eta)\| \leq Ke^{\nu|\eta|}$ for all $n, \eta$.

**Theorem 2.9.** *Let $\{S_k, V_k \colon k = 1, \ldots N\}$ be an unbounded regular Volterra system, let $\alpha, \zeta \in \mathbb{C}^N$ be such that $Re(\alpha) = Re(\zeta)$, and denote $i\eta = \alpha - \zeta$. For $\theta \in \mathbb{R}^N$, we define $Q_\theta = \prod_{n=1}^{N} V_n(i\theta_n)$. Then*

$$T_\alpha = Q_\eta^{-1} T_\zeta Q_\eta$$

*(that is, $T_\alpha$ and $T_\zeta$ are similar when $Re(\alpha) = Re(\zeta)$).*

**Proof.** Define, for each $\theta \in \mathbb{R}^N$, $0 \le j \le N$, $Q_{\theta,j} := \prod_{n=1}^{j} V_n(i\theta_n)$ ($Q_{\theta,0} = I$). We will prove inductively that $T_{\alpha,j} = Q_{\eta,j}^{-1} T_{\zeta,j} Q_{\eta,j}$ for all $0 \le j \le N$. For j=0, the claim is obvious. We assume that $T_{\alpha,j} = Q_{\eta,j}^{-1} T_{\zeta,j} Q_{\eta,j}$ for some $0 \le j < N$, and prove it for $j+1$. The properties of an Unbounded Volterra System imply that the pair $(T_{\alpha,j}, V_{j+1})$ satisfies the assumptions of Corollary 11.13 from [6]. Since $Re(\alpha) = Re(\zeta)$ implies that $Re(\alpha_{j+1}) = Re(\zeta_{j+1})$, we get that

$$T_{\alpha,j+1} = T_{\alpha,j} + \alpha_{j+1} V_{j+1} = V_{j+1}^{-1}(i\eta_{j+1})(T_{\alpha,j} + \zeta_{j+1} V_{j+1}) V_{j+1}(i\eta_{j+1}).$$

But by the induction assumption, $T_{\alpha,j} = Q_{\eta,j}^{-1} T_{\zeta,j} Q_{\eta,j}$. Moreover, $\zeta_{j+1} V_{j+1}$ commutes with $Q_{\eta,j}$, $Q_{\eta,j}^{-1}$ (by Definition 2.1, (5) and Lemma 1.5), and so $\zeta_{j+1} V_{j+1} = Q_{\eta,j}^{-1}(\zeta_{j+1} V_{j+1}) Q_{\eta,j}$, and in conclusion,

$$\begin{aligned} T_{\alpha,j+1} &= V_{j+1}^{-1}(i\eta_{j+1}) Q_{\eta,j}^{-1}(T_{\zeta,j} + \zeta_{j+1} V_{j+1}) Q_{\eta,j} V_{j+1}(i\eta_{j+1}) \\ &= Q_{\eta,j+1}^{-1} T_{\zeta,j+1} Q_{\eta,j+1}, \end{aligned}$$

as wanted. Finally, the last step of the induction (j=N) gives us

$$T_\alpha = T_{\alpha,N} = Q_{\eta,N}^{-1} T_{\zeta,N} Q_{\eta,N} = Q_\eta^{-1} T_\zeta Q_\eta. \qquad \blacksquare$$

By Lemma 11.14 from [6], we thus have:

**Corollary 2.10.** *Let $\zeta, \alpha \in \mathbb{C}^N$ be such that $Re(\zeta) = Re(\alpha)$, and denote $i\eta = \alpha - \zeta$. Then*

$$(\forall t \in \mathbb{R}) \quad T_\alpha(t) = Q_{-\eta} T_\zeta(t) Q_\eta.$$

The following theorem plays a fundamental role in the proof of Theorem 2.15.

**Theorem 2.11.** (the "Three Lines Theorem") *Let $f$ be a function of a complex variable $z$ with values in $X$ (we set $x = Re(z)$, $y = Im(z)$). Suppose that $f$ is defined, analytic and bounded in the strip $x_0 \le x \le x_1$, $-\infty < y < \infty$. For $x_0 \le x \le x_1$, define*

$$M(x) := \sup_{-\infty < y < \infty} \|f(x+iy)\|.$$

*Then $\log M(x)$ is a* convex *function (of the real variable $x$).*

**Proof.** Theorem 3 from [3], chapter VI.10.1, page 520.

**Notation 2.12.** For $\zeta \in \mathbb{C}^N$, we shall set $\xi = Re(\zeta)$, $\eta = Im(\zeta)$, and use the $\ell^1$ norm $|\zeta| = \sum_{n=1}^{N} |\zeta_n|$.

The following lemma illustrates our usage of the "3-Lines Theorem" in the proof of Theorem 2.15.

**Lemma 2.13.** *Let $m \in \mathbb{Z}^N$. Define $\pi := \{\zeta \in \mathbb{C}^N\colon (\forall 1 \le k \le N) \quad m_k \le \xi_k \le m_k + 1\}$, and let $\phi\colon \pi \to X$ be an analytic* bounded *function. For all $\xi$, $m_k \le \xi_k \le m_k + 1$ ($\forall k$), set $M(\xi) := \sup_{\eta \in \mathbb{R}^N} \|\phi(\xi + i\eta)\|$. Assume that there exist constants $\nu \in \mathbb{R}$, $a, b_1, \ldots, b_N \ge 0$ such that*

$$M(\delta) \le a \cdot e^{\nu \sum_{k=1}^N \delta_k^2} \prod_{k=1}^N b_k^{|\delta_k|}$$

*for all $\delta = (\delta_1, \ldots, \delta_N) \in \mathbb{R}^N$ with $\delta_k \in \{m_k, m_k + 1\}$, $k = 1, 2, \ldots, N$. Then for all $\zeta \in \pi$,*

$$M(\xi) \le a \cdot e^{\nu \sum_{k=1}^N (\xi_k^2 + \frac{1}{4})} \prod_{k=1}^N b_k^{|\xi_k|}.$$

**Proof.** Let $\xi_1 \in \mathbb{R}$ be such that $m_1 \le \xi_1 \le m_1 + 1$, and let $0 \le t \le 1$ be such that $\xi_1 = tm_1 + (1-t)(m_1+1)$. Notice that when this is the case, then $|\xi_1| = t|m_1| + (1-t)|m_1+1|$ (since $m_1 \in \mathbb{Z}$). We apply the "3-Lines Theorem" for $\phi(\cdot, m_2, \ldots, m_N)$, $x = m_1$, $y = m_1 + 1$, and get

$$\begin{aligned} M(\xi_1, m_2, \ldots, m_N) &\le M(m_1, \ldots, m_N)^t \cdot M(m_1+1, m_2, \ldots, m_N)^{1-t} \\ &\le a^{t+(1-t)} \cdot e^{\nu t \sum_{k=1}^N m_k^2} \cdot e^{\nu(1-t)[(m_1+1)^2 + \sum_{k=2}^N m_k^2]} \\ &\quad \cdot \prod_{k=1}^N b_k^{t|m_k|} b_1^{(1-t)|m_1+1|} \cdot \prod_{k=2}^N b_k^{(1-t)|m_k|} \\ &= a \cdot e^{\nu[tm_1^2 + (1-t)(m_1+1)^2 + \sum_{k=2}^N m_k^2]} \cdot b_1^{|\xi_1|} \cdot \prod_{k=2}^N b_k^{|m_k|}. \end{aligned}$$

But

$$\begin{aligned} tm_1^2 + (1-t)(m_1+1)^2 &= tm_1^2 + (1-t)(m_1^2 + 2m_1 + 1) \\ &= tm_1^2 + m_1^2 + 2m_1 + 1 - tm_1^2 - 2tm_1 - t \\ &= m_1^2 + 2m_1 + 1 - 2tm_1 - t \\ &= (m_1 + 1 - t)^2 - t^2 + t \\ &\le (m_1 + 1 - t)^2 + \frac{1}{4} = \xi_1^2 + \frac{1}{4} \end{aligned}$$

(since $-t^2 + t \le 1/4$ for all $t \in \mathbb{R}$). Thus,

$$M(\xi_1, m_2, \ldots, m_N) \le a \cdot e^{\nu(\xi_1^2 + \frac{1}{4} + \sum_{k=2}^N m_k^2)} b_1^{|\xi_1|} \prod_{k=2}^N b_k^{|m_k|}.$$

Let $\xi_2 \in \mathbb{R}$ be such that $m_2 \le \xi_2 \le m_2 + 1$, and let $0 \le t \le 1$ be such that $\xi_2 = tm_2 + (1-t)(m_2+1)$. We use again the "3-Lines Theorem" for

$\phi(\xi_1, \cdot, m_3, \ldots, m_N)$, $x = m_2$, $y = m_2 + 1$, and continue inductively. The last step would give us the lemma's statement. ■

**Remark 2.14.** In the proof above we didn't use the fact that $\phi$ is an analytic function on $\pi$, but only that it is analytic in each one of the variables $\zeta_1, \ldots, \zeta_N$ separately. The *continuity* of $\phi$ was not used.

**Theorem 2.15.** *Let $\{S_k, V_k\colon\ k=1, \ldots N\}$ be an Unbounded Regular Volterra System. There exists a constant $H > 0$, independent of $t, \zeta$, such that*

$$(\forall t \in \mathbb{R}, \zeta \in \mathbb{C}^N) \quad \|T_\zeta(t)\| \le H \cdot \|S(t)\| \prod_{n=1}^{N} (1 + |t|\|V_n\|)^{|\xi_n|} \cdot e^{2\nu|\eta|}.$$

**Proof.** Fix $t \in \mathbb{R}$, and define $\phi_t\colon \mathbb{C}^N \to B(X)$ by $\phi_t(\zeta) := e^{\nu \sum_{k=1}^{N} \zeta_k^2} T_\zeta(t)$ for each $\zeta \in \mathbb{C}^N$ (recall that $\nu$ is such that there exists a constant $K$, independent of $n, \eta$, with $\|V_n(i\eta)\| \le K e^{\nu|\eta|}$ for all $n, \eta$). For convenience, throughout the following proof we will use "$\sum$" instead of "$\sum_{k=1}^{N}$".

By Theorem 2.9 with $\zeta, \xi$ instead of $\alpha, \zeta$ respectively,

$$\begin{aligned}(\forall \zeta \in \mathbb{C}^N) \quad \|\phi_t(\zeta)\| &\le e^{\nu \sum (\xi_k^2 - \eta_k^2)} \|Q_{-\eta}\| \|T_\xi(t)\| \|Q_\eta\| \\ &\le K^{2N} e^{\nu \sum (\xi_k^2 - \eta_k^2 + 2|\eta_k|)} \|T_\xi(t)\| \\ &\le K^{2N} e^{\nu \sum (\xi_k^2 + 1)} \|T_\xi(t)\|. \qquad (2.1)\end{aligned}$$

By Lemma 2.7, $\phi_t$ is an *entire* function over $\mathbb{C}^N$. In particular, $\phi_t$ is *bounded* in every "poly-strip" of the form $\{\zeta \in \mathbb{C}^N\colon\ (\forall 1 \le k \le N) \quad m_k - 1 \le \xi_k \le m_k\}$, when $m_k \in \mathbb{Z}$ for all $1 \le k \le N$ (the entireness of $T_\zeta(t)$ implies that $T_\xi(t)$ is bounded on the compact set $\{\xi\colon\ (\forall k) \mu_k - 1 \le \xi_k \le m_k\}$ in $\mathbb{R}^N$; the rest follows directly from (2.1)).

Moreover, by Lemma 2.4,

$$\|\phi_t(m_1 + i\eta_1, \ldots, m_N + i\eta_N)\| \le K^{2N} e^{\nu \sum (m_k^2 + 1)} \|S(t)\| \prod_{k=1}^{N} (1 + |t|\|V_k\|)^{|m_k|}.$$

By Lemma 2.13 (with $a = K^{2N} e^{\nu N} \|S(t)\|$, $b_k = 1 + |t|\|V_k\|$), for $\zeta$ in every poly-strip of the above form, hence for every $\zeta \in \mathbb{C}^N$,

$$\|\phi_t(\zeta)\| \le K^{2N} e^{\nu \sum (\xi_k^2 + \frac{5}{4})} \|S(t)\| \prod_{k=1}^{N} (1 + |t|\|V_k\|)^{|\xi_k|}. \qquad (2.2)$$

Thus, for all $\xi \in \mathbb{R}^N$,

$$\|T_\xi(t)\| = e^{-\nu \sum \xi_k^2} \|\phi_t(\xi)\| \le K^{2N} e^{\frac{5}{4}\nu N} \|S(t)\| \prod_{k=1}^{N} (1 + |t|\|V_k\|)^{|\xi_k|}.$$

The theorem now follows from this and Theorem 2.9 (by taking $H := K^{4N}e^{\frac{5}{4}\nu N}$). ■

### 2.2. $C^n$ classification

The following theorem is based on an extension of the term $C^n$-classification for *unbounded* operators, taken from [6], Section 11.19.

**Theorem 2.16.** *If $S$ is of class $C^n$, then $T_\zeta$ is of class $C^{n+m_1+m_2+\dots m_N}$ for all $\zeta \in \pi_{m_1,\dots,m_N} := \{z \in \mathbb{C}^N\colon\ (\forall 1 \le k \le N)\ |Re(z_k)| \le m_k\}$.*

**Proof.** Fix a $\zeta \in \pi_{m_1,\dots,m_N}$. We prove inductively that for each $0 \le j \le N$, $T_{\zeta,j}$ is of class $C^{n+m_1+\dots+m_j}$. For $j = 0$, the claim is obvious (since $T_{\zeta,0} = S$ is of class $C^n$). Assume that the claim is true for some $0 \le j < N$. The pair $(T_{\zeta,j}, V_{j+1})$ satisfies the assumptions of Standing Hypothesis 11.10, [6]; thus, by Theorem 11.20 from [6], since $T_{\zeta,j}$ is of class $C^{n+m_1+\dots+m_j}$ and $|Re(\zeta_{j+1})| \le m_{j+1}$, $T_{\zeta,j+1} = T_{\zeta,j} + \zeta_{j+1}V_{j+1}$ is of class $C^{n+m_1+\dots+m_j+m_{j+1}}$, as wanted. The last step of the induction ($j = N$) gives the wanted result. ■

## 3. Quasi-affinity and the unbounded Volterra relation

Let $S, V$ satisfy the assumptions of Standing Hypothesis 11.10, [6]. By the discussion preceding Theorem 12.6 on page 102 from [6], restricted to the case $\zeta_1 = \zeta \in \mathbb{C},\ \zeta_2 = \zeta_3 = \ldots = 0,\ \alpha_1 = \alpha,\ Re(\alpha) \ge 1,\ \alpha \ne 1$, we conclude that $S + \zeta V(\alpha)$ is similar to $S$. The case $\alpha = 1$ is exactly the case of Corollary 11.13 from [6]. Hence, $S + \zeta V$ is similar to $S$ if $Re(\zeta) = 0$. These results pose the question of similarity of the perturbations $S + \zeta V(\alpha)$ and $S$ when $0 \le Re(\alpha) < 1$. This question is partially answered in [8], for $S$ bounded. In the setting of [8], it is not assumed that $V$ can be embedded in a regular semigroup as before. Instead, Balakrishnan's fractional powers theory is used. In this chapter, the results of this article are generalized to the case of $S$ unbounded, and to the case of unbounded Volterra systems.

**Definition 3.1.** [8] A quasi-affinity between the Banach spaces $X$,$Y$ is a bounded linear operator $Q\colon X \to Y$ which is injective and has a dense range. A quasi-affine operator over a Banach space $X$ is an injective operator $Q\colon X \to X$.

Note that a quasi-affine operator $Q$ over a Banach space $X$ is a quasi-affinity between $X$ and $\overline{range\ Q}$.

**Definition 3.2.** Let $A$, $B$ be unbounded operators. We say that $B$ is a quasi-affine transform of $A$ if $B \supseteq QAQ^{-1}$ for some quasi-affine operator, $Q$, over $X$.

(Equivalently, $QA \subseteq BQ$, or, if $A$, $B$ are bounded, $QA = BQ$).

We begin with the following extension of Theorem 3.3.2 in [4] to an infinite interval.

**Lemma 3.3.** *Let $X$ be a Banach space, and let $f\colon (0,\infty) \to X$ be a continuous function. Let $S$ be a closed operator over $X$ such that $Im(f) \subseteq D(S)$, $S \circ f$ is also continuous, and $f, S \circ f \in L^1(0,\infty)$. Then $\int_0^\infty f(\lambda)\,d\lambda \in D(S)$, and*

$$S\int_0^\infty f(\lambda)\,d\lambda = \int_0^\infty (S \circ f)(\lambda)\,d\lambda.$$

**Proof.** Fix $0 < a < b < \infty$. Over the (finite) interval $[a,b]$, the continuous functions $f$, $S \circ f$ are Riemann-integrable. Let $\{I_n\}$ be Riemann sums for the integral $\int_a^b f(\lambda)\,d\lambda$ (converging to it). Each $I_n$ is a linear combination of a *finite* number of values of $f$ in $[a,b]$. Therefore, for each $n \in \mathbb{N}$, $I_n \in D(S)$, and by the linearity of $S$ and the continuity of $S \circ f$, $\{SI_n\}$ converges to $\int_a^b (S \circ f)(\lambda)\,d\lambda$. Since $S$ is closed, we have $\int_a^b f(\lambda)\,d\lambda \in D(S)$, and

$$S\int_a^b f(\lambda)\,d\lambda = \int_a^b (S \circ f)(\lambda)\,d\lambda.$$

But $f$, $S \circ f \in L^1(0,\infty)$, and therefore $\int_a^b f(\lambda)\,d\lambda \to \int_0^\infty f(\lambda)\,d\lambda$ and $S\int_a^b f(\lambda)\,d\lambda = \int_a^b (S \circ f)(\lambda)\,d\lambda \to \int_0^\infty (S \circ f)(\lambda)\,d\lambda$ as $a \to 0^+, b \to \infty$. Again, since $S$ is closed, $\int_0^\infty f(\lambda)\,d\lambda \in D(S)$ and

$$S\int_0^\infty f(\lambda)\,d\lambda = \int_0^\infty (S \circ f)(\lambda)\,d\lambda. \qquad \blacksquare$$

### 3.1. Quasi affinity

We shall assume that the following hypothesis is true throughout the following subsection.

**Standing Hypothesis 3.4.** Let $X$ be a complex Banach space, and let $S$, $V$ be such that:

(1) $S$ is a closed operator over $X$, $V \in B(X)$ is injective, $VD(S) \subseteq D(S)$ and $[S,V] \subseteq V^2$.

(2) $A := V^{-1}$ is densely defined, $(0,\infty) \subseteq \rho(-A)$ and there exists a constant $M < \infty$ such that

$$(\forall \lambda > 0) \quad \|\lambda R(\lambda; -A)\| \le M.$$

**Remark 3.5.** A consequence of (2) is that for each $\lambda > 0$, $\lambda \in \rho(-V)$ and $\|\lambda R(\lambda; -V)\| \le M + 1$.

**Lemma 3.6.** *Let $\alpha$ be such that $0 < Re(\alpha) < 1$ and $x \in D(S)$. Then $V^\alpha x \in D(S)$, and*

$$SV^\alpha x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1} SR(\lambda; -V)Vx\,d\lambda.$$

**Proof.** We use Lemma 3.3. By Standing Hypothesis 3.4, (1), $Vx \in D(S)$, thus, by Lemma 1.10 (with $C = V^2$) and Remark 3.5, $R(\lambda; -V)Vx \in D(S)$ for all $\lambda > 0$. The function $\lambda^{\alpha-1}R(\lambda; -V)Vx$ is surely a continuous function of $\lambda > 0$. Moreover, by the second statement of Lemma 1.10,

$$g(\lambda) := \lambda^{\alpha-1}SR(\lambda; -V)Vx = \lambda^{\alpha-1}[R(\lambda; -V)SVx + V^2R(\lambda; -V)^2Vx]$$

for all $\lambda > 0$. Thus, by Remark 3.5,

$$\|g(\lambda)\| \leq \lambda^{\alpha-2}(M+1)\|SVx\| + \lambda^{\alpha-3}(M+1)^2\|V^3x\|$$

(we used the commutativity of $V$, $R(\lambda; -V)$ for all $\lambda \in \rho(-V)$). Thus, $g \in L^1(1, \infty)$. On the other hand, by Lemma 1.10,

$$\begin{aligned} g(\lambda) &= \lambda^{\alpha-1}S[I - \lambda R(\lambda; -V)]x \\ &= \lambda^{\alpha-1}[Sx - \lambda R(\lambda; -V)Sx - \lambda V^2R(\lambda; -V)^2x] \\ &= \lambda^{\alpha-1}[Sx - \lambda R(\lambda; -V)Sx - \lambda(I - \lambda R(\lambda; -V))^2x], \end{aligned}$$

and so

$$\|g(\lambda)\| \leq \lambda^{\alpha-1}[\|Sx\| + (M+1)\|Sx\| + \lambda(1+1+M)^2\|x\|].$$

Hence, $g \in L^1(0, 1)$. In conclusion, $g \in L^1(0, \infty)$. Therefore, by Lemma 3.3, $V^\alpha x \in D(S)$, and

$$SV^\alpha x = \frac{\sin \pi\alpha}{\pi}\int_0^\infty \lambda^{\alpha-1}SR(\lambda; -V)Vx\, d\lambda. \qquad \blacksquare$$

**Lemma 3.7.** *Let $\alpha$ be such that $0 < Re(\alpha) < 1$. Then $D([S, V^\alpha]) = D(S)$, and*

$$[S, V^\alpha] \subseteq \alpha V^{\alpha+1}.$$

**Proof.** Since $[S, V^\alpha] = SV^\alpha - V^\alpha S$, it is clear that $D([S, V^\alpha]) \subseteq D(S)$. By Lemma 3.6, the opposite is also true, and we have $D([S, V^\alpha]) = D(S)$. Moreover, since $R(\lambda; -V)V = I - \lambda R(\lambda; -V)$, for $x \in D(S)$,

$$\begin{aligned} \frac{\pi}{\sin \pi\alpha}[S, V^\alpha]x &= \int_0^\infty \lambda^{\alpha-1}[SR(\lambda; -V)Vx - R(\lambda; -V)VSx]\, d\lambda \\ &= \int_0^\infty \lambda^{\alpha-1}[Sx - \lambda SR(\lambda; -V)x - Sx + \lambda R(\lambda; -V)Sx]\, d\lambda \\ &= \int_0^\infty \lambda^\alpha[R(\lambda; -V), S]x\, d\lambda. \end{aligned}$$

Therefore, by Lemma 1.10,

$$\frac{\pi}{\sin \pi\alpha}[S, V^\alpha]x = \int_0^\infty \lambda^\alpha R(\lambda; -V)^2V^2x\, d\lambda.$$

Since $\frac{d}{d\lambda}R(\lambda;-V) = -R(\lambda;-V)^2$, integrating by parts gives

$$\frac{\pi}{\sin \pi\alpha}[S, V^\alpha]x = [-\lambda^\alpha R(\lambda;-V)V^2 x]_{\lambda=0}^\infty + \int_0^\infty \alpha\lambda^{\alpha-1}R(\lambda;-V)V^2 x. \quad (3.1)$$

Regarding the first factor in (3.1),

$$\|\lambda^\alpha R(\lambda;-V)V^2 x\| = \|\lambda^\alpha(I - \lambda R(\lambda;-V))Vx\| \leq \lambda^\alpha(1+1+M)\|Vx\| \to 0$$

as $\lambda \to 0^+$, and

$$\|\lambda^\alpha R(\lambda;-V)V^2 x\| \leq \lambda^{\alpha-1}(M+1)\|V^2 x\| \to 0$$

as $\lambda \to \infty$. Hence, by the boundedness of $V$, (3.1) becomes

$$\begin{aligned} [S, V^\alpha]x &= \frac{\sin \pi\alpha}{\pi}\alpha \int_0^\infty \lambda^{\alpha-1}R(\lambda;-V)V^2 x \, d\lambda \\ &= \frac{\sin \pi\alpha}{\pi}\alpha V \int_0^\infty \lambda^{\alpha-1}R(\lambda;-V)Vx \, d\lambda = \alpha V V^\alpha x = \alpha V^{\alpha+1}x, \end{aligned}$$

that is, $[S, V^\alpha] \subseteq \alpha V^{\alpha+1}$, as wanted. ■

**Corollary 3.8.** *Let $\alpha$ be such that $Re(\alpha) > 0$. Then $D([S, V^\alpha]) = D(S)$, and*

$$[S, V^\alpha] \subseteq \alpha V^{\alpha+1}.$$

In other words, the conclusion of Lemma 3.7 is true for all $\alpha \in \mathbb{C}^+$.

**Proof.** First, we assume that $Re(\alpha) = 1$. Obviously, $D([S, V^\alpha]) \subseteq D(S)$. Let $x \in D(S)$ and let $(\alpha_n)_{\mathbb{N}}$ be a sequence in $\{z \in \mathbb{C}\colon\ 0 < Re(z) < 1\}$ such that $\lim_{n\to\infty} \alpha_n = \alpha$. Since $V$ is bounded, the function $\mathbb{C}^+ \ni \alpha \mapsto V^\alpha y$ is analytic for all $y \in X$ (see Lemma 2.2 in [1]), thus continuous, and so $\lim_{n\to\infty} V^{\alpha_n}x = V^\alpha x$. For each $n \in \mathbb{N}$, one can apply Lemma 3.7 to get that $V^{\alpha_n}x \in D(S)$, and

$$SV^{\alpha_n}x = V^{\alpha_n}Sx + \alpha_n V^{\alpha_n+1}x \to V^\alpha Sx + \alpha V^{\alpha+1}x$$

as $n \to \infty$. Since $S$ is closed, we conclude that $V^\alpha x \in D(S)$ and $SV^\alpha x = V^\alpha Sx + \alpha V^{\alpha+1}x$, as wanted.

We now turn to the remaining case. Assume $Re(\alpha) > 1$. Let $\beta, n$ be such that $n \in \mathbb{N}$, $0 < Re(\beta) \leq 1$ and $\alpha = n + \beta$ ($n = [Re(\alpha)]$ if $Re(\alpha) \notin \mathbb{N}$, otherwise $n = Re(\alpha) - 1$). As before, $D([S, V^\alpha]) \subseteq D(S)$. Let $x \in D(S)$. Then $V^\alpha x = V^{n+\beta}x = V^n V^\beta x \in D(S)$ by the first part of Lemma 3.6. Moreover, by the identity $[S, V^n] \subseteq nV^{n+1}$ (easily proven by induction) and Lemma 3.7,

$$\begin{aligned} SV^\alpha x &= SV^n V^\beta x = V^n SV^\beta x + nV^{n+1}V^\beta x \\ &= V^n V^\beta Sx + \beta V^n V^{\beta+1}x + nV^{n+\beta+1}x = V^\alpha Sx + \alpha V^{\alpha+1}x, \end{aligned}$$

as wanted. ■

**Theorem 3.9.** *For all $\beta \in \mathbb{C}$ with $Re(\beta) > 1$ and for all $\zeta \in \mathbb{C}$, $S + \zeta V^\beta$ is similar to $S$. More specifically,*

$$S + \zeta V^\beta = e^{(-\zeta/(\beta-1))V^{\beta-1}} S e^{(\zeta/(\beta-1))V^{\beta-1}}.$$

**Proof.** By Corollary 3.8,

$$[S, V^{\beta-1}/(\beta-1)] \subseteq V^\beta.$$

The theorem's statement now follows from Corollary 12.2, [6] with $S, V^{\beta-1}/(\beta-1), V^\beta$ replacing $S, V, C$ respectively. ■

**Corollary 3.10.** *Let $\alpha$ be such that $0 < Re(\alpha) < 1$. Then*

$$[S, A^\alpha] \subseteq -\alpha V^{1-\alpha}.$$

**Proof.** We use the identity $A^\alpha = (V^\alpha)^{-1}$ (c.f. Theorem 1.9, (3)). Let $x \in D([S, A^\alpha])$, that is, $x \in D(S)$, $x \in D(A^\alpha)$, $Sx \in D(A^\alpha)$ and $A^\alpha x \in D(S)$. Set $y = A^\alpha x$ (equivalently, $x = V^\alpha y$). Then $y, V^\alpha y \in D(S)$, $SV^\alpha y \in D(A^\alpha)$, and by Lemma 3.7,

$$\begin{aligned} [S, A^\alpha]x &= SA^\alpha x - A^\alpha Sx = Sy - A^\alpha SV^\alpha y = A^\alpha(V^\alpha Sy - SV^\alpha y) \\ &= -A^\alpha[S, V^\alpha]x = -\alpha A^\alpha V^{\alpha+1} y = -\alpha A^\alpha V^\alpha V y \\ &= -\alpha V y = -\alpha V^{1-\alpha} V^\alpha A^\alpha x = -\alpha V^{1-\alpha} x. \end{aligned}$$ ■

**Notation 3.11.** For $\frac{1}{2} \le \alpha < 1$, set $A_\alpha := -A^{1-\alpha}/(1-\alpha)$, and denote by $T_\alpha(\cdot)$ the $C_0$-semigroup generated by $A_\alpha$ (see Theorem 1.9, (1)).

**Notation 3.12.** For any two linear operators $A$, $S$ over $X$, we use the notation $d_A S := [A, S]$.

By Corollary 3.10, $d_{A_\alpha} S \equiv [A_\alpha, S] \subseteq V^\alpha$.

**Lemma 3.13.** *Let $x \in D([S, A_\alpha])$. Then for all $\frac{1}{2} \le \alpha < 1$, $\mu > 0$, $R(\mu; -A_\alpha)x \in D(S)$ and $SR(\mu; -A_\alpha)x \in D(A_\alpha)$.*

**Proof.** Since $R(\mu; -A_\alpha) = (1-\alpha)R((1-\alpha)\mu; -A^{1-\alpha})$, it is sufficient to prove the above when $R(\mu; -A_\alpha)$ is replaced by $R(\mu; -A^\alpha)$ and $0 < \alpha \le \frac{1}{2}$. Fix $0 < \alpha \le \frac{1}{2}$, $\mu > 0$ and $x \in D(S)$. By Theorem 1.9, (2), $(0, \infty) \subseteq \rho(-A^\alpha)$, and for all $\mu > 0$, $x \in X$,

$$R(\mu; -A^\alpha)x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \frac{\lambda^\alpha}{\mu^2 - 2\mu\lambda^\alpha \cos \pi\alpha + \lambda^{2\alpha}} R(\lambda; -A)x \, d\lambda.$$

By the identify

$$(\forall \lambda > 0) \quad R(\lambda; -A) = \lambda^{-1} V R(\lambda^{-1}; -V), \tag{3.2}$$

Standing Hypothesis 3.4, (1) and Lemma 1.10, for all $x \in D(S)$, $R(\lambda;-A)x \in D(S)$ and

$$\begin{aligned}
SR(\lambda;-A)x &= \lambda^{-1}SVR(\lambda^{-1};-V)x \\
&= \lambda^{-1}[VS+V^2]R(\lambda^{-1};-V)x \\
&= \lambda^{-1}[VR(\lambda^{-1};-V)Sx - VV^2R(\lambda^{-1};-V)^2x + V^2R(\lambda^{-1};-V)x] \\
&= R(\lambda;-A)Sx + \lambda^{-1}V^2R(\lambda^{-1};-V)[I - VR(\lambda^{-1};-V)]x \\
&= R(\lambda;-A)Sx + \lambda^{-1}V^2R(\lambda^{-1};-V)[I - I + \lambda^{-1}R(\lambda^{-1};-V)]x \\
&= R(\lambda;-A)Sx + [\lambda^{-1}VR(\lambda^{-1};-V)]^2x \\
&= R(\lambda;-A)Sx + R(\lambda;-A)^2x. \qquad (3.3)
\end{aligned}$$

(note that $[S,-V] \subseteq -V^2$, hence $[S,R(\lambda;-V)] \subseteq -V^2R(\lambda;-V)^2$).

Denote the scalar factor of $R(\lambda;-A)$ in the integral above by $f(\lambda)$, and set $g(\lambda) = f(\lambda)SR(\lambda;-A)x$. Let $C_1$ be such that for $\lambda > 0$ *large enough*, $|f(\lambda)| \leq C_1\lambda^{-\alpha}$. Thus, for such $\lambda$, by (3.3),

$$\|g(\lambda)\| \leq C_1\lambda^{-\alpha-1}M[\|Sx\| + \lambda^{-1}M\|x\|].$$

Therefore $g(\lambda) \in L^1(1,\infty)$. There exists also a constant $C_2$ such that for $\lambda > 0$ *small enough*, $|f(\lambda)| \leq C_2\lambda^{\alpha}$. By (3.2) and Remark 3.5, $\|R(\lambda;-A)\| \leq (M+1)\|V\|$ for all $\lambda > 0$; Hence, for such $\lambda$, by (3.3),

$$\|g(\lambda)\| \leq C_2\lambda^{\alpha}(M+1)\|V\|[\|Sx\| + (M+1)\|V\|\|x\|],$$

and we have $g(\lambda) \in L^1(0,1)$. In conclusion, $g(\lambda) \in L^1(0,\infty)$, and by Lemma 3.3, $R(\mu;-A^{\alpha})x \in D(S)$, and

$$SR(\mu;-A^{\alpha})x = \frac{\sin\pi\alpha}{\pi}\int_0^{\infty}\frac{\lambda^{\alpha}}{\mu^2 - 2\mu\lambda^{\alpha}\cos\pi\alpha + \lambda^{2\alpha}}SR(\lambda;-A)x\,d\lambda.$$

Now, assume that $x \in D([S,A^{\alpha}])$. We want to show that $SR(\mu;-A^{\alpha})x \in D(A^{\alpha})$. Again, Lemma 3.3 is used. By (3.3), $g(\lambda)$ is a continuous function of $\lambda > 0$, and $SR(\lambda;-A)x \in R(\lambda;-A)X = D(A) \subseteq D(A^{\alpha})$. Moreover, by Theorem 1.9, (4) and Corollary 3.10,

$$\begin{aligned}
A^{\alpha}SR(\lambda;-A)x &= A^{\alpha}R(\lambda;-A)Sx + A^{\alpha}R(\lambda;-A)^2x \\
&= R(\lambda;-A)A^{\alpha}Sx + R(\lambda;-A)^2A^{\alpha}x \\
&= R(\lambda;-A)SA^{\alpha}x + \alpha R(\lambda;-A)V^{1-\alpha}x + R(\lambda;-A)^2A^{\alpha}x.
\end{aligned}$$

Set $h(\lambda) = f(\lambda)A^{\alpha}SR(\lambda;-A)x$. Then for $\lambda > 0$ large enough,

$$\|h(\lambda)\| \leq C_1\lambda^{-\alpha-1}M[\|SA^{\alpha}x\| + \alpha\|V^{1-\alpha}x\| + \lambda^{-1}M\|A^{\alpha}x\|],$$

thus $h(\lambda) \in L^1(1,\infty)$, and for $\lambda > 0$ small enough,

$$\|h(\lambda)\| \leq C_2\lambda^\alpha(M+1)\|V\|[\|SA^\alpha x\| + \alpha\|V^{1-\alpha}x\| + (M+1)\|V\|\|A^\alpha x\|],$$

hence $h(\lambda) \in L^1(0,1)$. In conclusion, $h(\lambda) \in L^1(0,\infty)$, and by Lemma 3.3, $SR(\mu;-A^\alpha)x \in D(A^\alpha)$. ∎

**Notation 3.14.** For $\frac{1}{2} \leq \alpha < 1$ fixed, set

$$\mathcal{D} = \{x \in D(d_{A_\alpha}S)\colon\ A_\alpha x \in D(d_{A_\alpha}S)\}.$$

**Lemma 3.15.** *For all $\frac{1}{2} \leq \alpha < 1$, $t \geq 0$, $\overline{S_{|\mathcal{D}}}$ is a quasi-affine transform of $\overline{S_{|\mathcal{D}}} + tV^\alpha$ and $\overline{S_{|\mathcal{D}}} - tV^\alpha$ is a quasi-affine transform of $\overline{S_{|\mathcal{D}}}$. More specifically, $T_\alpha(t)$ is a quasi-affinity, and*

$$T_\alpha(t)(\overline{S_{|\mathcal{D}}} + tV^\alpha) \subseteq \overline{S_{|\mathcal{D}}}T_\alpha(t),$$

$$T_\alpha(t)\overline{S_{|\mathcal{D}}} \subseteq (\overline{S_{|\mathcal{D}}} - tV^\alpha)T_\alpha(t).$$

**Proof.** The idea is to use Lemma 12.12 from [6]. First, by Theorem 1.9, (1) and the discussion preceding Remark 2.4 in [8], $T_\alpha(t)$ is 1-1 (meaning, quasi-affine) for all $t \geq 0$, and by Corollary 3.10,

$$d_{A_\alpha}S \equiv [A_\alpha, S] \subseteq V^\alpha.$$

**Claim 1.** $\mathcal{D} = D(d^2_{A_\alpha}S)$.

The inclusion '$\supseteq$' is surely true. To prove the opposite, we notice that it is enough to prove that if $x \in D(d_{A_\alpha}S)$ then $(d_{A_\alpha}S)x \in D(A_\alpha)$. But by Corollary 3.10, $(d_{A_\alpha}S)x = V^\alpha x \in D(A_\alpha)$ by Theorem 1.9, (4).
By Theorem 1.9, (4), for all $x \in D(d^2_{A_\alpha}S)$, $(d^2_{A_\alpha}S)x = A_\alpha V^\alpha x - V^\alpha A_\alpha x = 0$.

**Claim 2.** *$\mathcal{D}$ is $R(\lambda;-A_\alpha)$-invariant for all $\lambda > 0$.*

To prove the claim, let $x \in \mathcal{D}$, $\lambda > 0$ be given. Surely $R(\lambda;-A_\alpha)x \in D(A_\alpha)$. By Lemma 3.13, $R(\lambda;-A_\alpha)x \in D(S)$ and $SR(\lambda;-A_\alpha)x \in D(A_\alpha)$. Moreover, $A_\alpha R(\lambda;-A_\alpha)x = x - \lambda R(\lambda;-A_\alpha)x \in D(S)$, and we have $R(\lambda;-A_\alpha)x \in D(d_{A_\alpha}S)$. Hence, $A_\alpha R(\lambda;-A_\alpha)x = x - \lambda R(\lambda;-A_\alpha)x \in D(d_{A_\alpha}S)$ as well, and in conclusion, $R(\lambda;-A_\alpha)x \in \mathcal{D}$.

By Lemma 12.12, [6] with $-A_\alpha, S, V^\alpha$ replacing $A, S, C$ respectively, we have

$$T_\alpha(t)(\overline{S_{|\mathcal{D}} + t(V^\alpha)_{|\mathcal{D}}}) \subseteq S_{|\mathcal{D}}T_\alpha(t).$$

But $V^\alpha$ is bounded, and so $\overline{S_{|\mathcal{D}} + t(V^\alpha)_{|\mathcal{D}}} = \overline{S_{|\mathcal{D}}} + tV^\alpha$. Hence,

$$T_\alpha(t)(\overline{S_{|\mathcal{D}}} + tV^\alpha) \subseteq \overline{S_{|\mathcal{D}}}T_\alpha(t).$$

For the second equality in the statement of the theorem, define $S_\alpha = S - tV^\alpha$. Since $V$, $V^\alpha$ commute (this is trivial from the definition of $V^\alpha$ and the boundedness of $V$), $[S_\alpha, V] \subseteq V^2$, meaning- Standing Hypothesis 3.4 is satisfied by the pair $(S_\alpha, V)$, and so, if we set

$$\mathcal{D}' := \{x \in D(d_{A_\alpha} S_\alpha):\ A_\alpha x \in D(d_{A_\alpha} S_\alpha)\},$$

then what we have proved so far gives

$$\begin{aligned} T_\alpha(t)\overline{S_{|\mathcal{D}'}} &= T_\alpha(t)(\overline{S_{|\mathcal{D}'} - t(V^\alpha)_{|\mathcal{D}'}} + tV^\alpha) \subseteq (\overline{S_{|\mathcal{D}'} - t(V^\alpha)_{|\mathcal{D}'}})T_\alpha(t) \\ &= (\overline{S_{|\mathcal{D}'}} - tV^\alpha)T_\alpha(t), \end{aligned}$$

and the desired conclusion follows from the fact that $\mathcal{D} \subseteq \mathcal{D}'$, which is easily verified (note that $x \in D(A_\alpha) \Rightarrow$ there exists a $y \in X$ such that $x = V^{1-\alpha}y$, hence $V^\alpha x = V^\alpha V^{1-\alpha} y = V^{1-\alpha}(V^\alpha y) \in D(A_\alpha)$; see Theorem 1.9, (3)). ∎

As a result, we have the following

**Theorem 3.16.** *Suppose that $\mathcal{D}$ is a core for $S$. Then for all $\frac{1}{2} \leq \alpha < 1$ and $t \geq 0$, $S$ is a quasi-affine transform of $S + tV^\alpha$ and $S - tV^\alpha$ is a quasi-affine transform of $S$. More specifically, $T_\alpha(t)$ is a quasi-affinity, and*

$$T_a(t)(S + tV^\alpha) \subseteq ST_\alpha(t),$$

$$T_\alpha(t)S \subseteq (S - tV^\alpha)T_a(t).$$

### 3.2. Limits of similarities

We shall assume that the following hypothesis is true throughout the following subsection.

**Standing Hypothesis 3.17.** Let $X$ be a complex Banach space, and let $S$, $V$ be such that:

(1) $S$ is a closed operator over $X$, $V \in B(X)$ is injective, $VD(S) \subseteq D(S)$, and $[S, V] \subseteq V^2$.

(2) $-A := -V^{-1}$ generates a uniformly bounded $C_0$-semigroup, $T(\cdot)$ ($\|T(t)\| \leq M < \infty$ for all $t > 0$).

By the Hille-Yoshida Theorem, in this case we have $(0, \infty) \subseteq \rho(-A)$ and

$$\|[\lambda R(\lambda; -A)]^n\| \leq M$$

for all $\lambda > 0$, $n \in \mathbb{N}$. Therefore, Standing Hypothesis 3.17 is more restrictive than Standing Hypothesis 3.4.

**Notation 3.18.** For $\varepsilon > 0$, we set $A_\varepsilon := R(\varepsilon; -V)$.

**Lemma 3.19.** *For all $x \in D(S)$, $\alpha \in \mathbb{C}$ with $0 < Re(\alpha) < 1$, and $\varepsilon > 0$, we have $A_\varepsilon^{1-\alpha}x \in D(S)$ and*

$$SA_\varepsilon^{1-\alpha}x = \frac{\sin \pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}SR(\lambda; -A_\varepsilon)A_\varepsilon x \, d\lambda.$$

**Proof.** Fix $x \in D(S)$, $0 < Re(\alpha) < 1$, $\varepsilon > 0$. By (19) from [8], for all $\lambda > 0$,

$$R(\lambda; -A_\varepsilon)A_\varepsilon x = \lambda^{-1}R\left(\frac{1+\lambda\varepsilon}{\lambda}; -V\right)x \in D(S) \tag{3.4}$$

(cf. Lemma 1.10). For all $\lambda > 0$, set $f(\lambda) := \lambda^{-\alpha}SR(\lambda; -A_\varepsilon)A_\varepsilon x$. By (3.4) and Lemma 1.10, for all $\lambda > 0$,

$$\begin{aligned} f(\lambda) =& \lambda^{-\alpha-1}\left[R\left(\frac{1+\lambda\varepsilon}{\lambda}; -V\right)Sx - V^2R\left(\frac{1+\lambda\varepsilon}{\lambda}; -V\right)^2 x\right] \\ =& \lambda^{-\alpha-1}\left[R\left(\frac{1+\lambda\varepsilon}{\lambda}; -V\right)Sx - \left(I - \frac{1+\lambda\varepsilon}{\lambda}R\left(\frac{1+\lambda\varepsilon}{\lambda}; -V\right)\right)^2 x\right]. \end{aligned} \tag{3.5}$$

By the first equality in (3.5),

$$\begin{aligned} \|f(\lambda)\| &\le \lambda^{-\alpha-1}\left[\frac{\lambda}{1+\lambda\varepsilon}(1+M)\|Sx\| + \|V\|^2\frac{\lambda^2}{(1+\lambda\varepsilon)^2}(1+M)^2\|x\|\right] \\ &\le \lambda^{-\alpha}[(1+M)\|Sx\| + \|V\|^2\lambda(1+M)^2\|x\|], \end{aligned}$$

and so $f \in L^1(0,1)$. Moreover, by the second equality in (3.5),

$$\|f(\lambda)\| \le \lambda^{-\alpha-1}\left[\frac{\lambda}{1+\lambda\varepsilon}\|Sx\| + (1+1+M)^2\|x\|\right].$$

For all $\lambda > 0$, $\frac{\lambda}{1+\lambda\varepsilon} \le \frac{1}{\varepsilon}$, and so

$$\|f(\lambda)\| \le \lambda^{-\alpha-1}\left[\frac{1}{\varepsilon}\|Sx\| + (2+M)^2\|x\|\right].$$

Thus $f \in L^1(1,\infty)$. In conclusion, $f \in L^1(0,\infty)$. Hence, by Lemma 3.3, $A_\varepsilon^{1-\alpha}x \in D(S)$ and

$$SA_\varepsilon^{1-\alpha}x = \frac{\sin \pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}SR(\lambda; -A_\varepsilon)A_\varepsilon x \, d\lambda. \qquad \blacksquare$$

**Lemma 3.20.** *For all $\alpha \in \mathbb{C}$ with $0 < Re(\alpha) < 1$, $\varepsilon > 0$,*

$$[A_\varepsilon^{1-\alpha}, S] \subseteq (1-\alpha)A_\varepsilon^{2-\alpha}V^2.$$

**Proof.** By Lemma 3.19, $D([A_\varepsilon^{1-\alpha}, S]) = D(S)$. Fix $x \in D(S)$. By (3.4),

$$A_\varepsilon^{1-\alpha} x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha-1} R\left(\frac{1+\lambda\varepsilon}{\lambda}; -V\right) x\, d\lambda.$$

By Lemma 1.10, $[S, R(\mu; -V)] \subseteq -R(\mu; -V)^2 V^2$. Thus, by (3.4) and Lemma 3.19,

$$\begin{aligned} [A_\varepsilon^{1-\alpha}, S]x &= \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-1-\alpha} \left[R\left(\frac{1+\lambda\varepsilon}{\lambda}; -V\right), S\right] x\, d\lambda \\ &= \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-1-\alpha} R\left(\frac{1+\lambda\varepsilon}{\lambda}; -V\right)^2 V^2 x\, d\lambda \\ &= \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{-\alpha+1} R(\lambda; -A_\varepsilon)^2 A_\varepsilon^2 V^2 x\, d\lambda \\ &= \frac{\sin \pi\alpha}{\pi} \int_0^\infty -\lambda^{-\alpha+1} \frac{d}{d\lambda} R(\lambda; -A_\varepsilon) A_\varepsilon^2 V^2 x\, d\lambda. \end{aligned}$$

We integrate by parts: by Lemma 3.1 from [8],

$$\|\lambda^{-\alpha+1} R(\lambda; -A_\varepsilon) A_\varepsilon^2 V^2 x\| \le \lambda^{-\alpha} M \|A_\varepsilon^2 V^2 x\| \to 0$$

as $\lambda \to \infty$, and

$$\begin{aligned} \|\lambda^{-\alpha+1} R(\lambda; -A_\varepsilon) A_\varepsilon^2 V^2 x\| &= \|\lambda^{-\alpha+1} [I - \lambda R(\lambda; -A_\varepsilon)] A_\varepsilon V^2 x\| \\ &\le \lambda^{1-\alpha} (1+M) \|A_\varepsilon V^2 x\| \to 0 \end{aligned}$$

as $\lambda \to 0^+$. Hence,

$$\begin{aligned} \frac{\pi}{\sin \pi\alpha} [A_\varepsilon^{1-\alpha}, S]x &= [-\lambda^{-\alpha+1} R(\lambda; -A_\varepsilon) A_\varepsilon^2 V^2 x]_{\lambda=0}^\infty \\ &\quad + (1-\alpha) \int_0^\infty \lambda^{-\alpha} R(\lambda; -A_\varepsilon) A_\varepsilon^2 V^2 x\, d\lambda \\ &= \frac{\pi}{\sin \pi\alpha} (1-\alpha) A_\varepsilon^{1-\alpha} A_\varepsilon V^2 x = \frac{\pi}{\sin \pi\alpha} (1-\alpha) A_\varepsilon^{2-\alpha} V^2 x. \end{aligned}$$

Thus $[A_\varepsilon^{1-\alpha}, S] \subseteq (1-\alpha) A_\varepsilon^{2-\alpha} V^2$. ∎

Since $A_\varepsilon^{1-\alpha}$, $A_\varepsilon^{2-\alpha} V^2$ commute, Corollary 12.2 from [6] gives:

**Corollary 3.21.** *For each $\alpha \in \mathbb{C}$ with $0 < Re(\alpha) < 1$, $\varepsilon > 0$ and $\zeta \in \mathbb{C}$, $S + \zeta A_\varepsilon^{2-\alpha} V^2$ is similar to $S$. Moreover,*

$$S + \zeta A_\varepsilon^{2-\alpha} V^2 = e^{(\zeta/(1-\alpha)) A_\varepsilon^{1-\alpha}} S e^{-(\zeta/(1-\alpha)) A_\varepsilon^{1-\alpha}}.$$

**Notation 3.22.** For $0 < \alpha < 1$, set $A_\alpha := -A^{1-\alpha}/(1-\alpha)$ (do not confuse with $A_\varepsilon$!), and denote by $T_\alpha(\cdot)$ the $C_0$-semigroup generated by $A_\alpha$ (see Theorem 1.9).

By Theorem 3.4 from [8], we get the following

**Theorem 3.23.** *For all $0 < \alpha < 1$, $t \geq 0$, $S$ is a quasi-affine transform of $S + tV^\alpha$ and $S - tV^\alpha$ is a quasi-affine transform of $S$. Moreover,*

$$T_\alpha(t)(S + tV^\alpha) \subseteq ST_\alpha(t)$$

*and*

$$T_\alpha(t)S \subseteq (S - tV^\alpha)T_\alpha(t).$$

**Proof.** Applying $\exp[-\zeta A_\varepsilon^{1-\alpha}/(1-\alpha)]$ to both sides of the result of Corollary 3.21 with $0 < \alpha < 1$ and $t > 0$, and applying Theorem 3.4 from [8], we obtain $T_\alpha(t)(S + tV^\alpha) \subseteq ST_\alpha(t)$. For the second result, set $S_\alpha := S - tV^\alpha$. Since $(S_\alpha, V)$ satisfies the assumptions of Standing Hypothesis 3.17, we can apply the first result to it; the second result follows. ∎

### 3.3. Quasi affinity in unbounded Volterra systems

We generalize Theorems 3.9 and 3.23 to the case of Unbounded Volterra Systems. The generalization of Theorem 3.9 will assume the following.

**Standing Hypothesis 3.24.** Let $X$ be a complex Banach space, $N \in \mathbb{N}$ be given, and let the pairs $(S_1, V_1), (S_2, V_2), ..., (S_N, V_N)$ satisfy the assumption of Standing Hypothesis 3.4, such that for all $0 \leq k, l \leq N$, $k \neq l$, we have $V_l D(S_k) \subseteq D(S_k)$, $[S_k, V_l] \subseteq 0$ and $[V_l, V_k] = 0$. We set $S := \sum_{n=1}^N S_n$.

**Theorem 3.25.** *Let $\beta \in \mathbb{C}^N$ be such that $1 < Re(\beta_n)$ for all $1 \leq n \leq N$, and let $\zeta = (\zeta_1, \ldots, \zeta_N) \in \mathbb{C}^N$ be given. Then $S + \sum_{n=1}^N \zeta_n V_n^{\beta_n}$ is similar to $S$. More specifically,*

$$S + \sum_{n=1}^N \zeta_n V_n^{\beta_n} = e^{-Q} S e^Q$$

*where*

$$Q := \sum_{n=1}^N \frac{\zeta_n V_n^{\beta_n - 1}}{\beta_n - 1}.$$

**Proof.** For $0 \leq j \leq N$, set $S_j := S + \sum_{n=1}^j \zeta_n V_n^{\beta_n}$, $Q_j := \sum_{n=1}^j \frac{\zeta_n V_n^{\beta_n - 1}}{\beta_n - 1}$. We will prove inductively that for all such $j$, $S_j = e^{-Q_j} S e^{Q_j}$. The case $j = 0$ is trivial. Assume the result for some $j$, $0 \leq j < N$. The pair $(S_j, V_{j+1})$ satisfies the assumptions of Standing Hypothesis 3.4. This is true since by the induction hypothesis, $D(S_j) = D(S)$, and since $V_{j+1}$ commutes with $V_n^\gamma$ for all $1 \leq n \leq j$, $Re(\gamma) > 0$ (because $V_{j+1}, V_n$ commute). Therefore, the results of Theorem 3.9 can be used. We thus have

$$\begin{aligned} S_{j+1} &= S_j + \zeta_{j+1} V_{j+1}^{\beta_{j+1}} = e^{-(\zeta_{j+1}/(\beta_{j+1}-1))V_{j+1}^{\beta_{j+1}}} S_j e^{(\zeta_{j+1}/(\beta_{j+1}-1))V_{j+1}^{\beta_{j+1}}} \\ &= e^{-Q_{j+1}} S e^{Q_{j+1}}. \end{aligned}$$

The theorem now follows by taking $j = N$. ∎

**Standing Hypothesis 3.26.** We assume the same as in Standing Hypothesis 3.24, only that the pairs $(S_1, V_1), \ldots, (S_N, V_N)$ satisfy the assumptions of Standing Hypothesis 3.17 (instead of 3.4).

**Notation 3.27.** For all $1 \leq n \leq N$, we set $A_n := V_n^{-1}$, and denote the $C_0$-semigroup whose generator is $-A_n$ by $T_n(\cdot)$. For $0 < Re(\alpha) < 1$, denote by $T_{\alpha,n}(\cdot)$ the $C_0$-semigroup generated by $-A_n^{1-\alpha}/(1-\alpha)$.

Under this hypothesis, we get the following generalization of Theorem 3.23:

**Theorem 3.28.** *Let $\alpha, t \in \mathbb{R}^N$ be such that $0 < \alpha_n < 1$, $t_n > 0$ for all $n = 1, \ldots, N$. Then $S$ is a quasi-affine transform of $S + \sum_{n=1}^N t_n V_n^{\alpha_n}$ and $S - \sum_{n=1}^N t_n V_n^{\alpha_n}$ is a quasi-affine transform of $S$. More specifically,*

$$Q\left(S + \sum_{n=1}^N t_n V_n^{\alpha_n}\right) \subseteq SQ$$

*and*

$$QS \subseteq \left(S - \sum_{n=1}^N t_n V_n^{\alpha_n}\right) Q$$

*for*

$$Q := \prod_{n=1}^N T_{\alpha_n,n}(t_n).$$

**Proof.** For each $1 \leq n \leq N$, $T_{\alpha_n,n}(t_n)$ is 1-1, and therefore $Q$ is a quasi-affinity. For $0 \leq j \leq N$, define $Q_j := \prod_{n=1}^j T_{\alpha_n,n}(t_n)$. We will prove inductively that for all such $j$,

$$Q_j\left(S + \sum_{n=1}^j t_n V_n^{\alpha_n}\right) \subseteq SQ_j$$

and

$$Q_j S \subseteq \left(S - \sum_{n=1}^j t_n V_n^{\alpha_n}\right) Q_j.$$

For $j=0$, $IS \subseteq SI$ trivially (an empty sum is regarded as $0$, an empty product as $I$). We assume the result for $j$, $0 \leq j < N$, and prove it for $j+1$. The pair $(S+\sum_{n=1}^{j} t_n V_n^{\alpha_n}, V_{j+1})$ satisfies the assumptions of Standing Hypothesis 3.17, and so, by Theorem 3.23,

$$\begin{aligned}
&T_{\alpha_{j+1},j+1}(t_{j+1})\left(S+\sum_{n=1}^{j+1} t_n V_n^{\alpha_n}\right)\\
&\quad = T_{\alpha_{j+1},j+1}(t_{j+1})\left(S+\sum_{n=1}^{j} t_n V_n^{\alpha_n} + t_{j+1}V_{j+1}^{\alpha_{j+1}}\right)\\
&\quad \subseteq \left(S+\sum_{n=1}^{j} t_n V_n^{\alpha_n}\right) T_{\alpha_{j+1},j+1}(t_{j+1}).
\end{aligned}$$

Hence, by the induction's assumption, we have

$$\begin{aligned}
Q_{j+1}\left(S+\sum_{n=1}^{j+1} t_n V_n^{\alpha_n}\right) &= Q_j T_{\alpha_{j+1},j+1}(t_{j+1})\left(S+\sum_{n=1}^{j+1} t_n V_n^{\alpha_n}\right)\\
&\subseteq Q_j\left(S+\sum_{n=1}^{j} t_n V_n^{\alpha_n}\right) T_{\alpha_{j+1},j+1}(t_{j+1})\\
&\subseteq SQ_j T_{\alpha_{j+1},j+1}(t_{j+1}) = SQ_{j+1}.
\end{aligned}$$

In the same manner,

$$Q_{j+1}S \subseteq \left(S-\sum_{n=1}^{j+1} t_n V_n^{\alpha_n}\right) Q_{j+1}.$$

The theorem now follows by taking $j=N$. ∎

As for $Q$ above, the order of the multiplication doesn't really matter, as seen in the following lemma.

**Lemma 3.29.** *For all $1 \leq k,l \leq N$, $t_1,t_2>0$, $0<\alpha,\beta<1$, $T_{\alpha,1}(t_1)$ and $T_{\beta,2}(t_2)$ commute (cf. Notation 3.27).*

**Proof.** *Step I*: For all $\lambda \in \rho(-A_k)$, $\mu \in \rho(-A_l)$, $R(\lambda;-A_k)$ and $R(\mu;-A_l)$ commute.

To prove this, write $R(\lambda;-A_k) = \lambda^{-1}V_k R(\lambda^{-1};-V_k)$ for $\lambda \neq 0$ and $R(0;-A_k) = -V_k$. Doing the same for $l$, the step follows immediately from the commutativity of $V_k, V_l$ (see Standing Hypothesis 3.24).

*Step II*: $T_k(t_1)$ commutes with $T_l(t_2)$.

This follows from the first step and the formula

$$T_i(t) = s - \lim_{n\to\infty} [\frac{n}{t} R(\frac{n}{t}; -A_i)]^n$$

for $i = k, l$ and $t > 0$ (cf. Theorem 1.36 from [5]).

*Step III*: $T_{\alpha,1}(t_1)$ and $T_{\beta,2}(t_2)$ commute.

This follows from the second step, formula (2) in page 260, [11], and the discussion that follows. ∎

Of course, the results of Lemma 3.29 are still true for $\frac{1}{2} \le \alpha, \beta < 1$ if we assume only Standing Hypothesis 3.24.

## 4. Examples

**Theorem 4.1.** *Fix an $N \in \mathbb{N}$. For all $\varepsilon > 0$, $\zeta \in \mathbb{C}^+$, $1 \le n \le N$ and a locally integrable function on $[0,\infty)^N$, $f$, we define, for all $x = (x_1, \ldots, x_N) \in [0,\infty)^N$,*

$$(J^{\zeta}_{\varepsilon,n} f)(x) := \Gamma(\zeta)^{-1} \int_0^{x_n} e^{-\varepsilon(x_n - t)} (x_n - t)^{\zeta - 1} f(x_1, \ldots, x_{n-1}, t, x_{n+1}, \ldots, x_N)\, dt.$$

*Then for all $1 < p < \infty$, we have the following results:*

(1) *$J^{\zeta}_{\varepsilon,n}$ is a (well-defined) bounded operator over $L^p([0,\infty)^N)$.*

(2) *$(J^{\zeta}_{\varepsilon,n})_{\zeta\in\mathbb{C}^+}$ is a regular $C_0$-semigroup over $L^p([0,\infty)^N)$.*

(3) *Fix $\pi/2 < \nu < \pi$. Then the boundary group $(J^{i\eta}_{\varepsilon,n})_{\eta\in\mathbb{R}}$ satisfies*

$$(\forall \eta \in \mathbb{R}) \quad \|J^{i\eta}_{\varepsilon,n}\| \le C_{\nu,p,n} \cdot e^{\nu|\eta|},$$

*where $C_{\nu,p,n}$ is a constant, independent of $\varepsilon, \eta$.*

**Proof.** The theorem follows from Theorem 8.3 in [6] by a simple usage of the Fubini-Tonelli Theorems.

We are now ready to present the following example to the results of Chapter 2.

**Example 4.2.** Fix $N \in \mathbb{N}$, $1 < p < \infty$, $\varepsilon > 0$. Denote $X := L^p([0,\infty)^N)$. For each $1 \le n \le N$, define the $C_0$-group $S_n(\cdot)$ by

$$(\forall t \in \mathbb{R}, f \in X, x \in [0,\infty)^N) \quad (S_n(t)f)(x) := e^{itx_n} f(x).$$

Then for all $n$, $S_n(\cdot)$ is surely a $C_0$-group over $X$, whose generator is $iS_n$, where

$$D(S_n) = \{f \in X\colon\ x_n f(x) \in X\},$$

and for all $f \in D(S_n)$, $(S_n f)(x) = x_n f(x)$. For each $1 \leq n \leq N$, define $V_n := J_{\varepsilon,n} = J^1_{\varepsilon,n}$. We shall prove that $\{S_n, V_n\colon 1 \leq n \leq N\}$ satisfies the assumption of Definition 2.1: (1), (2) and (4) follow immediately. We verify (3) as follows: first, note that

$$(S(t)f)(x) = e^{it(x_1+\cdots+x_N)} f(x)$$

(for all $f \in X, x \in [0,\infty)^N$). Denote the generator of $S(\cdot)$ by $iS$. The same argument as above gives that $D(S) = \{f \in X\colon (x_1 + \cdots + x_N)f(x) \in X\}$, and that if $f \in D(S)$, then $(Sf)(x) = (x_1 + \cdots + x_N)f(x)$ for all $x$. Surely, $\bigcap_{n=1}^N D(S_n) \subseteq D(S)$. The reversed inclusion follows since for each $f \in X$, $x \in [0,\infty)^N$, $1 \leq n \leq N$,

$$|x_n f(x)| = x_n |f(x)| \leq (x_1 + \cdots + x_N)|f(x)| = |(x_1 + \cdots + x_N)f(x)|,$$

thus $x_n f(x) \in X$, and so $f \in D(S_n)$.

We verify next Condition (5) in Definition 2.1. Let $1 \leq k, l \leq N$ and $f \in D(S_k)$ be given. Then for each $x \in [0,\infty)^N$,

$$x_k (J_{\varepsilon,l} f)(x) = \int_0^{x_l} e^{-\varepsilon(x_l - t)} x_k f(x_1, \ldots, x_{l-1}, t, x_{l+1}, \ldots, x_N)\, dt.$$

Now, if $k \neq l$, then (by the definition of $J_{\varepsilon,l}$) $x_k (J_{\varepsilon,l} f)(x) = (J_{\varepsilon,l} S_k f)(x) \in X$. Hence, $V_l D(S_k) \subseteq D(S_k)$, and $[S_k, V_l] \subseteq 0$. Else, if $k = l$, we have

$$\begin{aligned} x_k (J_{\varepsilon,k} f)(x) &= \int_0^{x_k} e^{-\varepsilon(x_k - t)} t f(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_N)\, dt \\ &\quad + \int_0^{x_k} e^{-\varepsilon(x_k - t)} (x_k - t) f(x_1, \ldots, x_{k-1}, t, x_{k+1}, \ldots, x_N)\, dt \\ &= (J_{\varepsilon,k} S_k f)(x) + (J^2_{\varepsilon,k} f)(x). \end{aligned}$$

Thus $V_k D(S_k) \subseteq D(S_k)$ and $[S_k, V_k] \subseteq V_k^2$. Finally, the commutativity of $V_k, V_l$ follows immediately from Fubini's Theorem.

Our next step will be showing that $\{S_n, V_n\colon 1 \leq n \leq N\}$ is a *regular* unbounded Volterra system. Fix $1 \leq n \leq N$. By Theorem 4.1, (2), $(J^\zeta_{\varepsilon,n})_{\zeta \in \mathbb{C}^+}$ is a regular $C_0$-semigroup. Thus, it is sufficient to show that $\alpha_1 > 1$. Let $\gamma_n(\cdot)$ be the Nörlund function of $V_n$. Fix $\pi/2 < \nu < \pi$. Then by Definition 1.4 and Theorem 4.1, for all $\xi > 0$,

$$\begin{aligned} \gamma_n(\xi) &= \limsup_{|\eta| \to \infty} |\eta|^{-1} \log \|V_n(\xi + i\eta)\| \\ &\leq \limsup_{|\eta| \to \infty} |\eta|^{-1} (\log \|V_n(\xi)\| + \log \|V_n(i\eta)\|) \\ &\leq \limsup_{|\eta| \to \infty} |\eta|^{-1} \log(C_{\nu,p,n} e^{\nu|\eta|}) = \nu. \end{aligned}$$

Therefore, if the equation $\gamma_n(\xi) = \frac{\pi}{2\alpha}$ has a $\xi$-solution, then $\frac{\pi}{2\alpha} \le \nu$; by letting $\nu \to \frac{\pi}{2}^+$, we obtain $\alpha \ge 1$. Hence, $1 \le \alpha_0 < \alpha_1$, and the proof is complete.

For $\zeta \in \mathbb{C}^N$, let $T_\zeta(\cdot)$ be the $C_0$-group generated by $i(S+\sum_{n=1}^N \zeta_n J_{\varepsilon,n})$. Then by the results of the $2^{nd}$ chapter, if $Re(\alpha) = Re(\zeta)$ then $S+\sum_{n=1}^N \alpha_n J_{\varepsilon,n}$ is similar to $S + \sum_{n=1}^N \zeta_n J_{\varepsilon,n}$ (Theorem 2.9). Moreover,

$$\|S(t)\| = 1 \tag{4.1}$$

for all $t \in \mathbb{R}$, and so there exists a constant $H > 0$ with

$$\|T_{\xi+i\eta}(t)\| \le H \prod_{n=1}^N (1 + |t|\|J_{\varepsilon,n}\|)^{|\xi_n|} e^{2\nu|\eta|}$$

for all $\xi, \eta, t \in \mathbb{R}$ (where $\nu$ is as in the paragraph following Definition 2.8).

In order to use Theorem 2.16, we first show that $S$ is of class $C$ (that is, $C^0$), using the notations and definitions of [6], Section 11.19. By (4.1), $S(\cdot)$ is temperate. Now, fix a $\varphi \in C_C^\infty$. Let $\psi \in \mathcal{S}$ such that $\varphi = \widetilde{\psi}$. We need to show the existence of a constant $B$, independent of $\varphi$, with $\|\tau(\varphi)\| \le B\|\varphi\|_U$ ($\|\cdot\|_U$ is the supremum norm). To do that, let $f \in L^p([0,\infty)^N)$ be given. Then for all $x \in [0,\infty)^N$,

$$\begin{aligned}(\tau(\varphi)f)(x) &= \int_{\mathbb{R}} (S(t)f)(x) \cdot \psi(t)\,dt = \int_{\mathbb{R}} e^{it(x_1+\cdots+x_N)} f(x) \cdot \psi(t)\,dt \\ &= \varphi(x_1 + \cdots + x_N) f(x).\end{aligned}$$

Thus $\|\tau(\varphi)f\|_p \le \|\varphi\|_U \|f\|_p$, hence $\|\tau(\varphi)\| \le \|\varphi\|_U$.

Therefore, by Theorem 2.16, if $|\zeta_k| \le m_k$ ($m_k \in \mathbb{N}$) for all $1 \le k \le N$, then $S + \sum_{n=1}^N \zeta_n J_{\varepsilon,n}$ is of class $C^{m_1+\cdots+m_N}$.

We now bring an example to the results of Chapter 3.

**Example 4.3.** Fix $1 \le p < \infty$, and set $X = L^p(0,\infty)$. Define $A$ to be the derivation operator, $A = \frac{d}{dx}$, over the domain

$$D(A) = \{f \in X\colon\ f \text{ is absolutely continuous, and } f' \in X\}.$$

It is a well-known fact that $\sigma(A) = \{\lambda\colon\ Re(\lambda) \le 0\}$, and that for $\lambda$ satisfying $Re(\lambda) > 0$, we have

$$(\forall f \in X, x \in [0,\infty)) \quad (R(\lambda;A)f)(x) = \int_0^\infty e^{-\lambda t} f(x+t)\,dt.$$

Moreover, $A$ is the generator of the contractive $C_0$-semigroup $(T(t)f)(x) = f(x+t)$, defined over $X$ (c.f. [3], pages 629-630). This motivates us to define, for $\varepsilon > 0$ fixed, $A(\varepsilon) := \varepsilon I - A$. Also define $S$ by

$$(Sf)(x) = -xf(x)$$

$(\forall x \in [0,\infty))$ for each $f \in X$ that satisfies $xf(x) \in X$.

By the Hille-Yoshida Theorem, $A$ (and so $A(\varepsilon)$) is densely-defined, and for all $\lambda > 0$,

$$\|\lambda R(\lambda; A)\| \leq 1.$$

Let $\lambda > 0$ be given. Then $\lambda I + A(\varepsilon) = (\lambda + \varepsilon)I - A$, thus $\lambda \in \rho(-A(\varepsilon))$, and

$$\|\lambda R(\lambda; -A(\varepsilon))\| = \|\lambda R(\lambda + \varepsilon; A)\| \leq \frac{\lambda}{\lambda + \varepsilon} \leq 1.$$

Meaning, the second requirement in Standing Hypothesis 3.4 is satisfied (with $A(\varepsilon)$ replacing $A$). We now prove the first one. Set $V(\varepsilon) = A(\varepsilon)^{-1}$. For all $f \in X$, by Fubini's Theorem,

$$\begin{aligned}(V(\varepsilon)^2 f)(x) &= \int_0^\infty e^{-\varepsilon t} \left( \int_0^\infty e^{-\varepsilon s} f(x + t + s)\, ds \right) dt \\ &= \iint\limits_{[0,\infty)\times[0,\infty)} e^{-\varepsilon(t+s)} f(x + t + s)\, ds\, dt.\end{aligned}$$

We use the change of variables $u := t + s$, $v := t - s$ to obtain that

$$(V(\varepsilon)^2 f)(x) = \int_0^\infty e^{-\varepsilon u} u f(x + u)\, du.$$

If we assume now that $f \in D(S)$, then

$$\begin{aligned}-x(V(\varepsilon)f)(x) &= -x \int_0^\infty e^{-\varepsilon t} f(x + t)\, dt = \int_0^\infty e^{-\varepsilon t} [-(x + t)f(x + t)]\, dt \\ &\quad + \int_0^\infty e^{-\varepsilon t} t f(x + t)\, dt = (V(\varepsilon)Sf)(x) + (V(\varepsilon)^2 f)(x) \in X.\end{aligned}$$

This proves that $V(\varepsilon)D(S) \subseteq D(S)$. In conclusion, $S$ is a closed operator over $X$, $V(\varepsilon) \in B(X)$ is injective, $V(\varepsilon)D(S) \subseteq D(S)$ and $[S, V(\varepsilon)] \subseteq V(\varepsilon)^2$. The requirements of Standing Hypothesis 3.4 are hereby satisfied by the pair $(S, V(\varepsilon))$, and so the results of Theorem 3.9 and Lemma 3.15 are true. That is, for all $\beta$ with $Re(\beta) > 1$ and for all $\zeta \in \mathbb{C}$,

$$S + \zeta V(\varepsilon)^\beta = e^{(-\zeta/(\beta-1))V(\varepsilon)^{\beta-1}} S e^{(\zeta/(\beta-1))V(\varepsilon)^{\beta-1}}$$

and for all $\frac{1}{2} \leq \alpha < 1$, if we denote by $T_{\varepsilon,\alpha}(\cdot)$ the $C_0$-semigroup generated by $A_{\varepsilon,\alpha} := -A(\varepsilon)^{1-\alpha}/(1-\alpha)$ and $\mathcal{D} := \{x \in D(d_{A_{\varepsilon,\alpha}} S)\colon A_{\varepsilon,\alpha} x \in D(d_{A_{\varepsilon,\alpha}} S)\}$, then for all $t \geq 0$, $T_{\varepsilon,\alpha}(t)$ is quasi-affine, and

$$T_{\varepsilon,\alpha}(t)(\overline{S_{|\mathcal{D}}} + tV(\varepsilon)^\alpha) \subseteq \overline{S_{|\mathcal{D}}} T_{\varepsilon,\alpha}(t),$$

$$T_{\varepsilon,\alpha}(t)\overline{S_{|\mathcal{D}}} \subseteq (\overline{S_{|\mathcal{D}}} - tV(\varepsilon)^\alpha) T_{\varepsilon,\alpha}(t).$$

This example can easily by extended to the case of an unbounded Volterra system, satisfying Standing Hypothesis 3.24, where $X = L^p([0,\infty)^N)$ ($N \in \mathbb{N}$ and $1 \le p < \infty$ are fixed), and the operators $S_n$, $A_n$ are defined by

$$(S_n f)(x) = -x_n f(x), \quad A_n = \frac{\partial}{\partial x_n} \qquad n = 1, 2, \ldots, N$$

(with the suitable domains of definition).

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Department of Mathematics
Bar Ilan University
52900 Ramat-Gan
Israel
viselta@macs.biu.ac.il

Department of Mathematics
Bar Ilan University
52900 Ramat-Gan
Israel
kantor@macs.biu.ac.il

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