PROBLEM SESSION: 14TH BRAUER GROUP MEETING

NOTES BY URIYA FIRST

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1. Linkage of Fields (Adam Chapman)

We restrict the discussion to quaternion algebras for simplicity, but it can be similarly extended to other objects such as symbol algebras and Pfister forms.

Let F be a field. A set of quaternion algebras $Q_1, \dots, Q_n$ over F is called *linked* if they share a common maximal subfield (up to isomorphism). The field F is called *n-linked* if every n quaterion algebras over F are linked.

For example, it is well-known that F is 2-linked if and only if the symbol length of ${}_2\operatorname{Br} F$ is 1 (i.e., every 2-torsion Brauer class is represented by a quaternion algebra). If $\sqrt{-1} \in F$ and there is a quaternion division algebra over F, then this moreover implies that $u(F)$ (the u-invariant of F) is 4 or 8; see [9], [4]. When F is nonreal, the field F is 3-linked if and only if $u(F) \le 4$; see [1], [5].

Question 1.1. *Let $r \ge 2$. Is there an r-linked field that is not $(r+1)$-linked? In particular, is there a 4-linked field that is not 5-linked?*

The answer is known to be positive for $r = 2, 3$. For example:

- $\mathbb{C}((x))((y))((z))$ is 2-linked and not 3-linked.
- $\mathbb{C}(x, y)$ is 3-linked and not 4-linked [6].
- $\mathbb{F}_2^{\mathrm{alg}}(x, y)$ is 3-linked and not 4-linked.

Also, $\mathbb{Q}[i]$ is n-linked for all $2 \le n \in \mathbb{N}$.

Notice that the above examples are in characteristics 2 and 0. Can we find examples in characteristic $p \neq 0, 2$? One possible approach to constructing examples in any prescribed characteristic is the following: Start with a field E admitting

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"many" quaternion division algebras. Extend E to a compositum of all function fields of anisotropic Albert forms and all anisotropic 3-fold Pfister forms. Repeat until one gets a new field E'. This field is 3-linked.

Question 1.2. *Is E' 4-linked?*

2. Parametric Dimension (Danny Neftin)

Let F be a field, e.g. $\mathbb{Q}$, and let G be a linear algebraic group over F. We would like to have a rational parametrization of G-torsors of F. The extent to which this is possible is measured by the *parametric dimension*.

The (generalized or rational) parametric dimension of G over F, denoted $\mathrm{pd}_F(G)$, is defined as the minimal number $t \in \mathbb{N} \cup \{0\}$ (or ∞) such that there exist finitely many G-torsors $T_1, \dots, T_r$ over $K := F(x_1, \dots, x_t)$ such that every G-torsor over F is a specialization of at least one of the T_i. It was defined and studied for finite constant groups in [8, Section 5], [14] and is also related to the *arithmetic dimension* of O'Neil [16]. There is a variation on the definition where one allows each T_i to be defined over an F-field K_i for transcendence degree $\le t$; denote it by $\mathrm{pd}'_F(G)$. It is clear that $\mathrm{pd}'_F(G) \le \mathrm{ed}_F(G)$, where the right hand side is the essential dimension of G.

It is interesting to compare the behaviour of parametric dimension and essential dimension over number fields. It is also interesting to study the relation between $\mathrm{pd}'_K G$ and $\mathrm{pd}_K G$.

A particularly interesting example [14, Example A.1], [8, Section 5] is the group $G = C_2^s = C_2 \times \cdots \times C_2$ (s times), where C_2 is the (constant) algebraic group with 2 elements. The G-torsors over F classify G-Galois extensions of F, which are all of the form $F[\sqrt{\alpha_1}, \dots, \sqrt{\alpha_s}]$ for some $\alpha_1, \dots, \alpha_s \in F^\times$ (if $\mathrm{char}\, F \neq 2$).[1] Consequently $\mathrm{pd}_{\mathbb{Q}} C_2^s \le s$, because $\mathbb{Q}(\sqrt{x_1}, \dots, \sqrt{x_s})/\mathbb{Q}(x_1, \dots, x_s)$ specializes to any C_2^s-Galois extension of $\mathbb{Q}$. For comparison, $\mathrm{ed}_{\mathbb{Q}} C_2^s = s$. However, $\mathrm{pd}_{\mathbb{Q}} C_2^5 \le 4$.

To see this, we can use the Hasse–Minkowski Theorem, from which it follows that the set of values represented by a quadratic form over $\mathbb{Q}$ of dimension 4 is $\mathbb{Q}_+$, $\mathbb{Q}_-$, $\mathbb{Q} - \{0\}$, or $\mathbb{Q}$. Consider the C_2^5-Galois extension

$$\mathbb{Q}(\sqrt{x_1}, \dots, \sqrt{x_4}, \sqrt{x_1 + \cdots + x_4})/\mathbb{Q}(x_1, \dots, x_4). \tag{1}$$

In order to specialize it to $\mathbb{Q}[\sqrt{\alpha_1}, \dots, \sqrt{\alpha_5}]/\mathbb{Q}$ with $\alpha_1, \dots, \alpha_5 > 0$, we can choose $\beta_1, \dots, \beta_4 \in \mathbb{Q}$ and specialize $x_i \mapsto \alpha_i \beta_i^2$ for $i = 1, \dots, 4$, so that $x_1 + \cdots + x_4$ is specialized to $\alpha_1\beta_1^2 + \cdots + \alpha_4\beta_4^2$. By the Hasse-Minkowski Theorem, we can choose the β_i such that $\alpha_1\beta_1^2 + \cdots + \alpha_4\beta_4^2 = \alpha_5$. We conclude that (1) specializes to $\mathbb{Q}[\sqrt{\alpha_1}, \dots, \sqrt{\alpha_5}]/\mathbb{Q}$. Other choices of signs can be handled similarly, by considering $\mathbb{Q}(\sqrt{x_1}, \dots, \sqrt{x_4}, \sqrt{x_1 + x_2 + x_3 - x_4})/\mathbb{Q}(x_1, \dots, x_4)$. (This illustrates why we might need more than one torsor to specialize from.)

Question 2.1. *What can we say about $\mathrm{pd}_{\mathbb{Q}} C_2^s$ as $s \to \infty$? In particular, does $\mathrm{pd}_{\mathbb{Q}} C_2^s$ grow to ∞? is it bounded from above?*

The results of [14] imply that if a certain local-to-global principle holds, then $\mathrm{pd}_{\mathbb{Q}} G \le 2$ for every finite (constant) group G.

Question 2.2. *Is there a connected linear algebraic group G over $\mathbb{Q}$ with $\mathrm{pd}_{\mathbb{Q}} G \ge 3$?*

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[1] The notation $F[\sqrt{\alpha_1}, \dots, \sqrt{\alpha_s}]$ stands for the F-algebra $F[T_1, \dots, T_s]/(T_1^2 - \alpha_1, \dots, T_s^2 - \alpha_s)$ with its evident C_2^s-action, which is always a C_2^s-Galois algebra over F, but not necessarily a field. All C_2^s-Galois algebras over F are of this form (if $\mathrm{char}\, F \neq 2$).

3. Divisibility of Symbols (Eliyahu Matzri)

Let F be a field, let $x_1, \ldots, x_4, y_1, y_2 \in F^\times$, and let p be a prime number. Consider the symbol $\alpha := \{x_1, \ldots, x_4\}$ in $K_4^M F/p$ (Milnor's 4-th K-theory modulo p), and let K be the fraction field of the Severi–Brauer variety of the symbol algebra $(y_1, y_2)_{p,F}$.

Question 3.1. *Suppose that $\alpha_K = 0$. Does it imply that there are $t_1, t_2 \in F$ such that $\alpha = \{y_1, y_2, t_1, t_2\}$?*

The question extends naturally to symbols in $K_n^M F$ for all $n \geq 2$. The case $n = 3$ is known and follows from an exact sequence of Merkurjev and Karpenko. In the case $n = 4$, it is known that $\{y_1, y_2\}$ divides in $\{x_1, \ldots, x_4\}$ in the ring $K_*^M F/p$, but it is not known if $\{x_1, \ldots, x_4\}$ can be expressed as the product of $\{y_1, y_2\}$ and another symbol.

One can also consider a generalization of the question to higher symbols $\{x_1, \ldots, x_s\}$, $\{y_1, \ldots, y_r\}$ ($x_1, \ldots, x_s, y_1, \ldots, y_r \in F^\times$ and $s \geq r$) by replacing the Severi–Brauer variety of $(y_1, y_2)_{p,F}$ with a norm variety for $\{y_1, \ldots, y_r\}$.

4. Isotropicity of Pfister Forms over Purely Inseparable Extensions (Detlev Hoffmann)

Question 4.1. *Is there a field F of characteristic 2, a 3-fold Pfister quadratic form q, and a purely inseparable extension K/F such that q_K is isotropic, but for every $\alpha, \beta \in K$, the form $q_{F(\alpha,\beta)}$ is anisotropic?*

To put the question in context, Hoffmann [12] showed that a $(n+1)$-fold Pfister quadratic form over F which becomes isotropic over K is already isotropic over an F-subfield of K generated by $2^n - 1$ elements. In particular, for q as in the question, there exist $\alpha, \beta, \gamma \in K$ such that $q_{F(\alpha,\beta,\gamma)}$ is isotropic, so anistoropicity over all F-subfields of K generated by 2 elements is the best we can hope for. Also, if q is a 2-fold Pfister quadratic form such that q_K is isotropic, then there is $\alpha \in K$ such that $q_{F(\alpha)}$ is isotropic. Hoffmann constructs in [op. cit.] a purely inseparable field extension K/F and a 3-fold Pfister quadratic form over F such that q_K is isotropic, but $q_{F(\alpha)}$ is anisotropic for every $\alpha \in K - F$.

More generally, the question is motivated by the notion of q-mininal fields, i.e., fields extensions K of F such that q_K is isotropic, but q_L is anisotropic for every intermediate extension $F \subseteq L \subsetneq K$. Hoffmann's example shows that there are q-minimal fields K with many subfields lying between F and K.

5. Pythagoras Number of Function Fields (Marco Zaninelli)

Let K be a hereditarily Pythagorean field, i.e., every sum of squares in a finite real extension of K is a square. Suppose also that $\operatorname{char} K = 0$. Let C be a conic over K, say $aX^2 + bY^2 = Z^2$ in $\mathbb{P}^2$, and let α is the Brauer class of the symbol algebra $(a, b)_{2,K}$ associated to C. Then there is an exact sequence

$$0 \to \langle \alpha \rangle \to \operatorname{Br} K \to \operatorname{Br} C \to 0$$

(use Lichtenbaum's Theorem [10, Theorem 5.4.11] and (2) below). Here, and $\operatorname{Br} C$ coincides with the unramified Brauer group of $K(C)$, denoted $\operatorname{Br}_{\mathrm{nr}} K(C)$.

Suppose now that the conic C is replaced with an elliptic curve X. The Leray spectral sequence [7, Proposition 5.4.2] gives a 5-term exact sequence

$$0 \to \operatorname{Pic} X \to (\operatorname{Pic} \overline{X})^{\operatorname{Gal}(\overline{K}/K)} \to \operatorname{Br} K \to \operatorname{Br}_1 X \to \mathrm{H}^1(K, \operatorname{Pic} \overline{X}). \tag{2}$$

Here, $\overline{K}$ is an algebraic closure of K, $\overline{X} = X_{\overline{K}}$ and $\operatorname{Br}_1 X = \ker(\operatorname{Br} X \to \operatorname{Br} \overline{X})$.

We are interested in finding the Pythagoras number of $F := K(X)$. Observe that if $f \in F^\times$ is a sum of squares in F, then f is a sum of *two* squares if and only if $(-1, f)_{2,F} = 0$ in $\mathrm{Br}\, F$. It can be further shown that $(-1, f)_{2,F}$ lives in $\mathrm{Br}_1 X$ [17]. We would like to know if it vanishes or not.

Turning to (2), we look at the image of $(-1, f)_{2,K}$ in $\mathrm{H}^1(K, \mathrm{Pic}\, \overline{X})$. We know that $\mathrm{Pic}\, \overline{X} \cong \mathbb{Z} \times X(\overline{K})$, so $\mathrm{H}^1(K, \mathrm{Pic}\, \overline{X}) \cong \mathrm{H}^1(K, X(\overline{K}))$. In fact, since $(-1, f)_{2,K}$ is 2-torsion in $\mathrm{Br}_1 X$, the image of $(-1, f)_{2,K}$ lives in ${}_2\mathrm{H}^1(K, X(\overline{K}))$.

Question 5.1. *Describe ${}_2\mathrm{H}^1(K, X(\overline{K}))$. Under what assumptions does the image of $(-1, f)_{2,K}$ in ${}_2\mathrm{H}^1(K, X(\overline{K}))$ vanish?*

An answer would help determining the Pythagoras number of $F = K(X)$. The simpler case where X is a conic was solved in [17].

We now give some examples of hereditarily Pythagorean fields K to which this problem can be specialized. Every such K has a henselian (possibly non-discrete) valuation ν such that its residue field $k(\nu)$ is hereditarily Pythagorean with 1 or 2 orderings. For example $\mathbb{R}(\!(t)\!)$ is such a field; ν is the t-adic valuation and residue field $\mathbb{R}$ has one ordering.

The hereditarily Pythagorean fields with one ordering were classified. An example with two orderings can be constructed as follows: Let K_1, K_2 be the real closures of $\mathbb{Q}(\sqrt{2})$ relative to its two orderings, and let $K = K_1 \cap K_2$, where the intersection is taken in some algebraic closure of $\mathbb{Q}(\sqrt{2})$. Then K is hereditarily Pythagorean with 2 orderings. What is the answer to question 5.1 for this K? Note that for this K, $\mathrm{Gal}(\overline{K}/K)$ is generated (as a profinite group) by 2 involutions.

6. Galois-Theoretic Nature of the Brauer Group (Ido Efrat)

Let F be a field and let p be a prime number. Denote by $G_F(p)$ the maximal pro-p quotient of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$.

Question 6.1. *Let F_1 and F_2 be fields of characteristic $\neq p$ such that F_1 and F_2 contain a primitive p-th root of unity and $G_{F_1}(p) \cong G_{F_2}(p)$. Is it the case that $(\mathrm{Br}\, F_1)_p \cong (\mathrm{Br}\, F_2)_p$?*

Here $(-)_p$ denotes the p-primary part of the group at hand.

7. Additive Commutators in Division Algebras (Boris Kunyavskii)

Theorem 7.1 (Amitsur–Rowen)**.** *Let D be a central simple algebra over a field F, let $A = \mathrm{M}_n(D)$, and let $[,] : A \times A \to A$ be the additive commutator map $[x, y] = xy - yx$. If $n \geq 2$, then every $a \in A$ with $\mathrm{Trd}_{A/F}(a) = 0$ can be written as $a = [x, y]$ for some $x, y \in A$.*

The case $n = 1$ is still open and is a very difficult problem; it is related to the cyclicity of p-algebras.

Question 7.2. *Suppose that D is a division algebra that is infinite dimensional over its center F. Let $a \in D$ be a sum of additive commutators in D. Are there $x, y \in D$ such that $a = [x, y]$?*

The reason there is hope for the infinite dimensional case is a vague analogy with the behaviour of the multiplicative commutator $(x, y) \mapsto xyx^{-1}y^{-1}$. Namely, most Chevalley groups over a field that have 'many' commutators tend to have very few of them over rings. However, the situation reverses if we go over to Chevalley groups of infinite rank, even over a ring — again, they have lots of commutators. This phenomenon was observed in various contexts by de la Harpe–Skandalis, Dennis–Vaserstein, Gupta–Holubowski and others; see [13] for more details and precise references.

8. Commutators in Matrix Groups (Mathieu Florence)

Let C be a smooth projective curve of genus g over $\mathbb{C}$. (Other fields of characteristic 0 are also interesting.) Then $\pi_1^{\mathrm{et}}(C)$ is the profinite completion of

$$\langle x_1, y_1, \dots, x_g, y_g \rangle \,/\, \langle [x_1, y_1] \cdots [x_g, y_g] \rangle$$

(here $[x, y] = xyx^{-1}y^{-1}$). Consider a (continuous) representation

$$\rho : \pi_1^{\mathrm{et}}(C) \to \mathrm{GL}_n(\mathbb{F}_p),$$

where p is a prime number. For example, if $g = 1$, then the datum of ρ is equivalent to choosing $X, Y \in \mathrm{GL}_n(\mathbb{F}_p)$ (the images of the generators x_1, y_1) such that $XY = YX$.

Question 8.1. *Can we lift ρ to a representation $\rho_\infty : \pi_1^{\mathrm{et}}(C) \to \mathrm{GL}_n(\mathbb{Z}_p)$?*

The problem is related to conjectures of De Jong about local systems on curves. By Greenberg's Theorem [11], there is $r \in \mathbb{N}$ (depending on g, n) such that if there exists a lift of ρ to $\rho_r : \pi_1^{\mathrm{et}}(C) \to \mathrm{GL}_n(\mathbb{Z}/p^r\mathbb{Z})$, then ρ_∞ exists.

M. Florence (unpublished) showed that ρ_∞ exists when $g = 1$. More general lifting results in Galois theory suggest that ρ_2 exists for all g. It is expected that answer to the question is "yes" for all g, and there should be direct clean proof.

9. Symbol Length in the Relative Brauer Group (Bill Jacob)

Let F be a field, say, of characteristic 0. Let p be a prime number such that F contains a primitive p-th root of unity. Let $\alpha, \beta \in F^\times$, let $E = F[\sqrt[p]{\alpha}, \sqrt[p]{\beta}]$, and consider

$${}_p\mathrm{Br}(E/F) := \ker({}_p\mathrm{Br}\,F \to {}_p\mathrm{Br}\,E) \subseteq {}_p\mathrm{Br}\,F$$

(the subscript p means taking the p-torsion part). Let $s\ell(E/F)$ denote the maximum possible (p-)symbol length of elements in ${}_p\mathrm{Br}(E/F)$.

It is not difficult to show that $s\ell(E/F) \leq 3$ if $p = 2$. B. Jacob and N. Schley (unpublished; inspired by the work of A. Laghribi) recently showed that $s\ell(E/F) \leq 3$ if $p = 3$. Moreover, any $x \in {}_p\mathrm{Br}(E/F)$ can be written as a sum of symbols of the form $(\alpha, ?)_{3,F} + (\beta, ?)_{3,F} + (?, ?)_{3,F}$. This is an improvement on Matzri's upper bound of 31 3-symbols [15].

Question 9.1. *Is $s\ell(E/F) \leq 3$ for all p?*

10. Galois Cohomology of Algebraic Groups via Root Datum (Mathieu Florence)

Let F be a field, let $\Gamma = \mathrm{Gal}(F^{\mathrm{sep}}/F)$, and let $\chi : \Gamma \to \prod_{p \neq \mathrm{char}\,F} \mathbb{Z}_p^\times$ denote the cyclotomic character of F.[2]

Let G be a reductive algebraic group over F. It is well-known that G is determined up to isomorphism by the root datum of $G_{F^{\mathrm{sep}}}$, denoted $R(G)$ for simplicity, and the action of Γ on $R(G)$. Consequently, $R(G)$ with its Γ-action determines $\mathrm{H}^1(F, G)$.

Question 10.1. *Can we describe $\mathrm{H}^1(F, G)$ directly from $R(G)$ with its Γ-action, possibly with reference to χ, but without referring to the field F?*

When G is a torus, it is well-known that it is possible to describe $\mathrm{H}^1(F, G)$ by means of the cocharacter lattice $X_*(G) := \mathrm{Hom}_{F^{\mathrm{sep}}\text{-grp}}(\mathbb{G}_m, G)$ (with its Γ-action) and the cyclotomic character.

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[2] To define χ, note that Γ acts on the set of n-th roots of unity in F. This induces a group homomorphism $\Gamma \to (\mathbb{Z}/n\mathbb{Z})^\times$. Taking the inverse limit over n coprime to $\mathrm{char}\,F$ gives χ.

Borovoi and Timashev [2] described $\mathrm{H}^1(\mathbb{R}, G)$ in terms of the root datum when G is a semisimple $\mathbb{R}$-group. This description seems to apply over any real-closed field, and is therefore an example where $\mathrm{H}^1(F, G)$ depends only on $R(G)$ (with its Γ-action) when $\Gamma \cong C_2$. See also the followup paper [3].

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