Elements of the sets enumerated by super-Catalan numbers

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1 Introduction

As we know several people tried to get many structures for fine numbers (see [31, Sequence A000957]), while others on Catalan numbers (see [31, Sequence A000108]). Stanley [34,35] gave more than **130** Catalan structures while Deutsch and Shapiro [11] also discovered many settings for the Fine numbers. The structures for Fine numbers and Catalan numbers are intimately related from the list of Fine number occurences in [11], which motivated us to find out more and more super-Catalan structures by the tight link between Catalan numbers and super-Catalan numbers, whose first several terms are $1, 1, 3, 11, 45, 197, \cdots$ (see [31, Sequence A001003]).

The purpose of this paper is to give a unified presentation of many new super-Catalan structures. We start the project with the idea of giving a restricted bi-color to the existed Catalan structures, and have included a selection of results in [18]. In the remainder of this section, we present a brief account of background for our investigation.

The sequence of super-Catalan number was introcuced by Friedrich Wilhelm Karl Ernst Schröder in his paper [29] during his discusses on four "bracketing problems" and the term "Schröder number" seems to have been first used by Rogers [28]. Ernst Schöder gave the n-th super-Catalan number $s(n)$ is the total number of bracketings of a string of n letters, but he did not mention any other combinatorial interpretations. While in 1994, David Hough discovered that the super-Catalan number were apparently known to Hipparchus in the second century B.C. (at least $s(9) = 103049$). The connection between bracketings and plane trees was known to Cayley [2]. The bijection between plane trees and polygon dissections appears in Etherington [15]. Currently, many good combinatorial structures enumerated by super-Catalan numbers are obtained by the references, from which we know the super-Catalan number not only counts the dissections of a convex polygon and plane trees, but also partitions (see [17]), various lattice paths (see [38]), permutations avoiding given patterns (see [8]), and so on. And in [36], Stanley narrated how the super-Catalan numbers are even more classical than has been believed before. He also recalled the three-term linear recurrence (see [6, 7, p.75])

$$3(2n-1)s(n-1) = (n+1)s(n) + (n-2)s(n-2) \qquad (n \geq 2),$$

with $s(0) = s(1) = 1$, of super-Catalan number. To give a combinatorial proof of this formula, Foata and Zeilberger [16] introduced a new combinatorial interpretation, *well-weighted binary plane trees*, for super-Catalan numbers.

Stanley studied 6 super-Catalan structures (see [34, Exercise 6.39(b),(d),(e),(h),(i) and (p)]). We can also get several structures from On-Line Encyclopedia of Integer Sequences [31, A001003]. Besides, the hilly poor noncrossing partition and (2, 3)-Motzkin path given by Yan and Yang [38], matchings on $[2n]$ avoiding both patterns 12312 and 121323 studied by Chen, Mansour and Yan [8] and small percolations referred in Fan, Mansour and Pang [17] are all combinatorial structures enumerated by super-Catalan numbers which we have known.

Recall that the number of Schröder paths of semilength n without peak at level one is the super-Catalan number s_n. By decompositon method, we get the generating function $s(x)$ for such Schröder paths satisfies

$$s(x) = 1 - xs(x) + 2xs^2(x),$$

which gives us the recursive relation $s_n = -s_{n-1} + 2\sum_{k=0}^{n} s_{k-1}s_{n-k}$ for $n \geq 1$.

As we know, a refinement of Catalan number gives $C_n = \sum_{k=0}^{n} N_{n,k}$, where $N_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ is the Narayana number with $1 \leqslant k \leqslant n$ and $N_{n,0} = 0$. In this paper, we derive the super-Catalan structures by giving bi-color to the statistic with Narayana distribution, i.e., according to the equation $s_n = \sum_{k=1}^{n} 2^{k-1}N_{n,k}$. Furthermore, we can get the equation $S_n = 2s_n = \sum_{k=1}^{n} 2^k N_{n,k}$ for the n-th large Schröder number(see [31, Sequence A006318]) by bi-color the first parameter again for the super-Catalan structures. Generally, if we define the Narayana polynomial by $N(q) = \sum_{k=0}^{n} N_{n,k}q^k$, it corresponds to give a q-color to the statistic of Narayana distribution in Catalan structures, i.e., each parameter in the statistic can be colored by one of q colors.

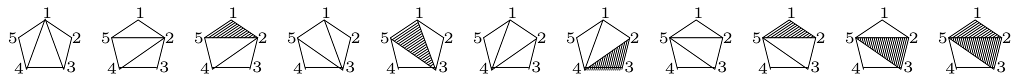
Besides, a subdivision of super-Catalan number is also studied in [8, 17], from which we get $s_n = \sum_{k=0}^{n-1} \frac{1}{n+k+2}\binom{n+k+2}{k+1}\binom{n-1}{k}$. The number $\frac{1}{n+k+2}\binom{n+k+2}{k+1}\binom{n-1}{k}$ has been studied by many people [3, 15, 17, 22, 27, 34], and appears in [31, Sequence A033282]. It counts dissections of a convex $(n + 2)$-gon with k diagonals not intersecting in their interiors [15, 34], standard Young tableaux of shape $(k + 1, k + 1, 1^{n-1-k})$ [34], integer sequences $(a_1, a_2, \cdots, a_{n+k+1})$ such that either $a_i = -1$ or $a_i \geqslant 1$, exactly n terms are equal to -1, $a_1 + a_2 + \cdots + a_i \geqslant 0$ for all i, and $a_1 + a_2 + \cdots + a_{n+k+1} = 0$ [15] and so on (see [17]).

In this paper, we main to give a list of super-Catalan numbers occurences.

2 A list of super-Catalan structures

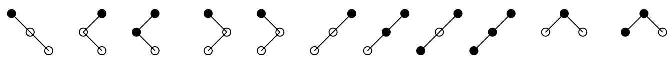
Now, we give a list of super-Catalan structures and most of them are the correspondings from [34, 35, Exercise 6.19] and reference therein and in each figure below, we represent the statistic colored black (resp. white) by dotted (resp. solid) lines in paths, partitions and diagrams while by bold (resp. non-bold) body in permutations and Young tableaux.Totally, we have more than **160** structures (see below), which made us believe it is also a supernatural sequence like Catalan numbers.

(a) For triangulations of a convex $(n + 2)$-gon labelled with $\{1, 2, \cdots, n + 2\}$ clockwise into n triangles by $n - 1$ diagonals that do not intersect in their interiors, color the triangle with the minimum edge (i.e., both of the two ends of the edge are smaller than at least one end of any other edge) with black or white if the minimum vertex is not on any diagonal, then delete the edges of this triangle which is not in common with other triangles. Otherwise, color the triangle white. By omitting the isolated vertex created by the above procedure, color the triangles by reconsidering the remaining convex polygons recursively until we get the last triangle which is colored white uniquely (we use the shadowed triangle to represent the black color).



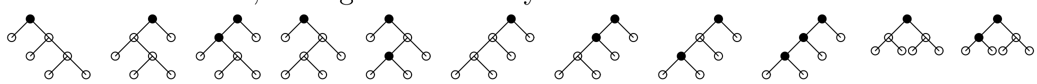
Hint. Let $A(x)$ be the generating function of such colored triangulations of convex polygon with the last triangle colored by white uniquely, and $B(x)$ be the generating function of such colored triangulations of convex polygon with the last triangle can be colored by black or white. Then $A = 1 - x + 2xA + x(A-1)B$ and $B = 1 + xB + xB^2$. Solving the two equations, the coefficient of A, which is super-Catalan numbers, corresponds to the enumeration of triangulations in (a).

(b) Color the binary trees of n vertices with two colors, black and white, such that the root must be black and the right child of any vertex must be white.



Hint. See [34, Exercise 6.39(d)] and [17], where such trees are called small-binary trees in [17].

(b') Color the full binary trees of $2n+1$ vertices with two colors, black and white, such that the root must be black , the right child of any vertex and the leaves must be white.



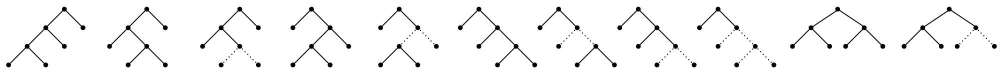
Hint. There is an obvious bijection between (b) and (b').

(c) For the binary parenthesizations of a string of $n+1$ letters(including adding a pair of parentheses containing all the elements), color two pairs of parentheses with their left part consecutive black or white (we use square brackets to represent the color of the parentheses being black). Otherwise, color the pair of parentheses white.

$$(((xx)x)x) \quad [[(xx)x]x] \quad ([[xx]x]x) \quad [[[xx]x]x] \quad (x((xx)x)) \quad (x[[xx]x])$$
$$((x(xx))x) \quad [[x(xx)]x] \quad (x(x(xx))) \quad ((xx)(xx)) \quad [[xx](xx)]$$

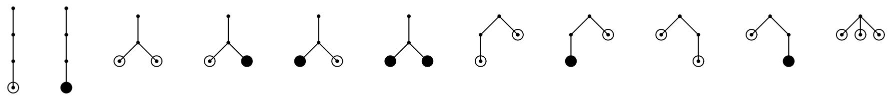
Hint. In (b'), label each leaf by x and each internal vertex (except the root) by a pair of parentheses. For any vertex v, have the string corresponding to the tree rooted at v in the parentheses corresponding to v, we get a bijection between (b') and (c).

(d) Color each pair of edges at the same level and have a common vertex with black or white in the plane binary trees with $2n+1$ vertices (or $n+1$ endpoints) if the common vertex is the right child of some vertex, and color it white otherwise.



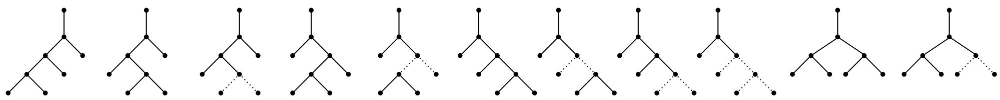
Hint. This is a clear result from (b').

(e) For the plane trees with $n+1$ vertices, color the leaves with black or white except the leaves at level one being colored only with white.



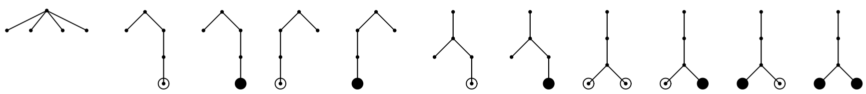
Hint. See [18] for a bijection between small binary trees in (b) and (e).

(f) For the planted (i.e., root has degree one) trivalent plane trees with $2n+2$ vertices, color each pair of edges at the same level and have a common vertex with black or white if the common vertex is the right child of some vertex, and color it white otherwise.



Hint. Adding a planted edge for each tree in (d), we get (f).

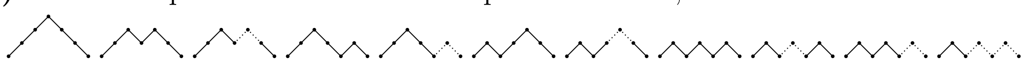
(g) For the plane trees with $n+2$ vertices such that the rightmost path of each subtree of the root has even length, consider each subtree of the root and color the leaves of the first subtree at level greater than 1 with black or white while color all the leaves of any other subtree with black or white (the root of the subtree is not considered as a leaf).



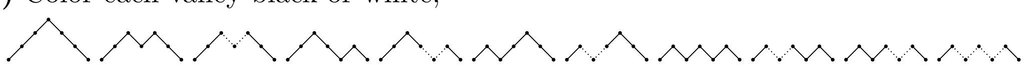
Hint. There is a bijection between (e) and (g). For a plane tree with $n+2$ vertices in (g), we get an ordered forest by deleting the root. Adding a planted edge for each tree in the forest except the first tree, and then combine the root of all trees in order, we get a plane tree with $n+1$ vertices. Having the corresponding leaves the same color, (g) holds from (e).

(h) For the Dyck paths from $(0,0)$ to $(2n,0)$, i.e., lattice paths with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis, we also have several cases to get super-Catalan number.

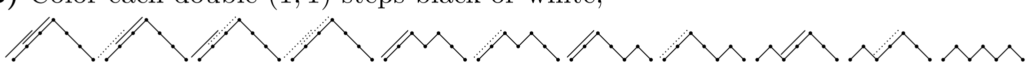
(1) Color each peak black or white except the first one;



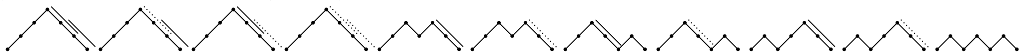
(2) Color each valley black or white;



(3) Color each double $(1,1)$ steps black or white;



(4) Color each double $(1,-1)$ steps black or white;

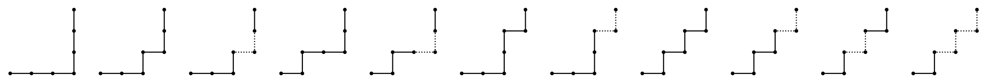


(5) Color each peak with height greater than 1 black or white.

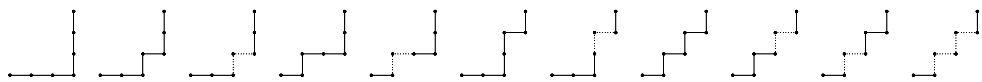
Hint. Dyck paths with given number of peaks, valleys, double ascents and double descents are enumerated by Narayana numbers [9]. By the equation $s_n = \sum_{k=1}^{n} 2^{k-1} N_{n,k}$, where $N_{n,k}$ is the Narayana number $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$, we can get the desired result. Besides, there are many other statistics having the Narayana distribution, such as high peaks, evenly positioned ascents and nonfinal maximal constant subpaths of length greater than one (See [32]).

(i) For the lattice paths from $(0,0)$ to (n,n) with steps $(0,1)$ or $(1,0)$, never rising above the line $y=x$, we have four cases to obtain super-Catalan numbers.

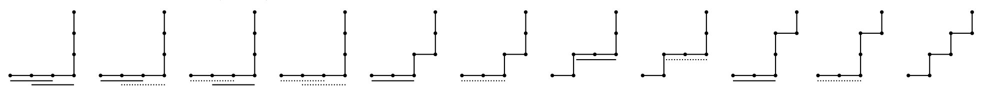
(1) Color each peak black or white except the first one;



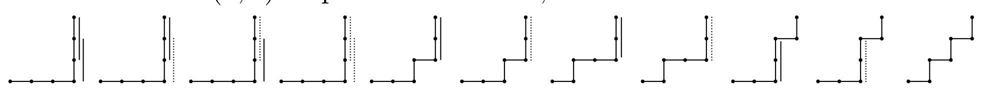
(2) Color each valley black or white;



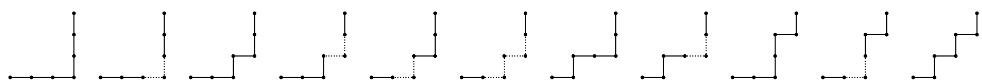
(3) Color each double $(1,0)$ steps black or white;



(4) Color each double $(0,1)$ steps black or white;

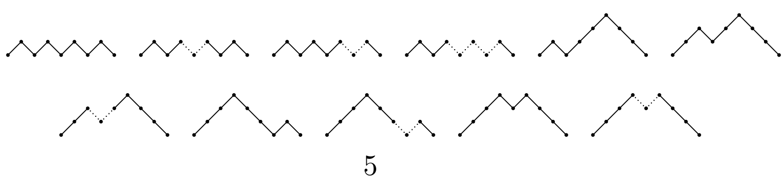


(5) Color each peak with height greater than 1 black or white.



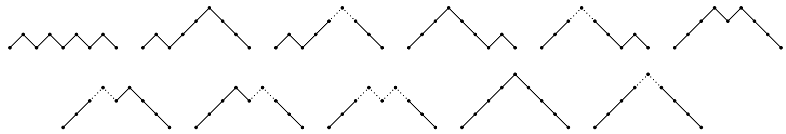
Hint. There is an obvious bijection between the corresponding structures in (i) and (h).

(j) For Dyck paths (as defined in (h)) from $(0,0)$ to $(2n+2,0)$ such that any maximal sequence of consecutive steps $(1,-1)$ ending on the axis has odd length, color each valley black or white if the $(1,-1)$ steps in the valley is not following the first $(1,1)$ step.



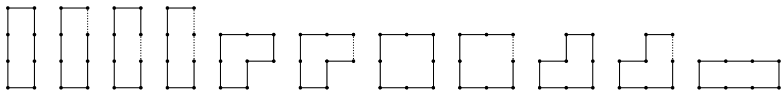
Hint. There is a bijection between (h2) and (j). In the bijection, we need to consider the subpath ending with the rightmost maximal sequence of consecutive steps $(1,-1)$ ending on the axis has even length. If it is not empty, add a $(1,1)$ at the beginning and a $(1,-1)$ at the end of this subpath, otherwise, add a peak at the beginning. Then we can get a Dyck path satisfying the condition in (j) and each valley in (h) corresponds to a valley not followed by the first $(1,1)$ steps in (j), and furthermore, we get a bijection by restricting corresponding valley the same color.

(k) For Dyck paths (as defined in (h)) from $(0,0)$ to $(2n+2,0)$ with no peaks at height two, color the peaks at height more than 2 black or white.



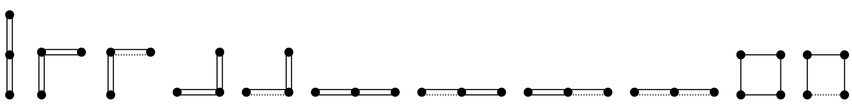
Hint. Let $A(x)$ (resp. $B(x)$) be the generating function of Dyck paths with no peak at level two (resp. no peak at level one) and each peak at level greater than 1 colored black or white, then $A(x) = 1 + xA(x) + x(B(x) - 1)A(x)$ and $B(x) = 1 + xB(x)(C(x) - 1)$, where $C(x)$ is the generating function of Dyck path with each peak colored black or white. Therefore, $C(x) = 1 + 2xC(x) + x(C(x) - 1)C(x)$.

(l) For (unordered) pairs of lattice paths with $n+1$ steps each, starting at $(0,0)$, using steps $(1,0)$ or $(0,1)$, ending at the same point, and only intersecting at the beginning and end, color each $(0,1)$ steps black or white except the first one for the lattice path below. Another way, each column of the Zebra (figure) can be colored black and white sucth that the first colored black.



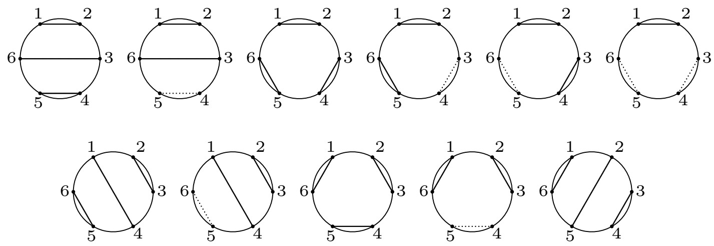
Hint. see math.boisestate.edu/ sulanke/paper1/PergolaSulanke/node3.html.

(m) For (unordered) pairs of lattice paths with $n-1$ steps each, starting at $(0,0)$, using steps $(1,0)$ or $(0,1)$, ending at the same point, such that one path never rises above the other path, color each $(1,0)$ step on the lattice path below black or white.



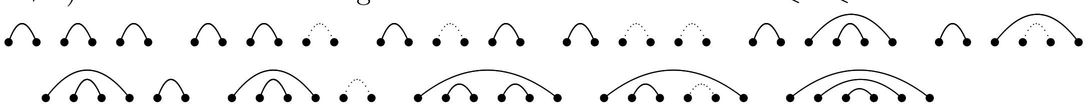
Hint. Regarding a path as a sequence of steps, remove the first and last steps from the two paths in (l). This variation was suggested by L. W. Shapiro (see [34, solution for (m)]).

(n) For n nonintersecting chords joining $2n$ points which are labelled from 1 to $2n$ clockwise on the circumference of a circle, color the chords $(i, i+1)$ from $i = 1$ to $i = 2n-1$ black or white except the first one.



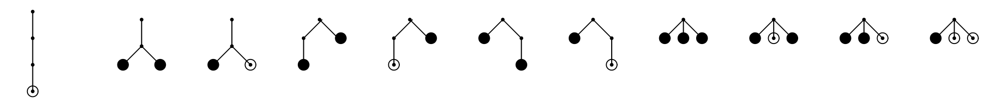
Hint. Starting clockwise from 1, at each vertex draw an up step (1,1) if encountering a chord for the first time and a down step (1,-1) otherwise. This gives a bijection with (h)(1)(see [34, solution for (n)]). Have each chord (i,i+1) the same color as its corresponding peak.

(o) Ways of connecting $2n$ points in the plane lying on a horizontal line by n nonintersecting arcs, each arc connecting two of the points and lying above the points, and the edges $(i, i+1)$ not the first one being colored black or white where $1 \leqslant i \leqslant n-1$.



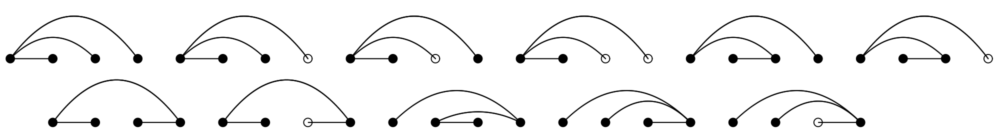
Hint. Cut the circle in (n) between 1 and 6 and "straighten out" (see [34, solution for (o)]).

(o') Ordered trees with n edges such that the first leaf when we transfer the tree in pre-order is colored black and all other leaves are colored black or white.



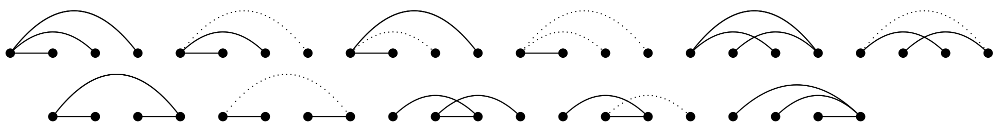
Hint. An easy involution on plane trees is given in [18], by which we can get (o') from (e).

(p) Ways of drawing in the plane $n+1$ points lying on a horizontal line L and n arcs connecting them such that (1) the arcs do not pass below L, (2) the graph thus formed is a tree, (3) no two arcs intersect in their interiors (i.e., the arcs are noncrossing), (4) at every vertex, all the arcs exit in the same direction (left or right), and (5) each end point not the first one is colored black or white.



Hint. This is direct from (o').

(q) Ways of drawing in the plane $n+1$ points lying on a horizontal line L and n arcs connecting them such that (1) the arcs do not pass below L, (2) the graph thus formed is a tree, (3) no arc (including its endpoints) lies strictly below another arc, (4) at every vertex, all the arcs exit in the same direction (left or right), and (5) for each vertex, the edges from it on its right (but not the bottom) are colored black or white.



Hint. For a vertex v, assume there are r_v neighbors on its right. Considering the vertices from left to right, we can get a Dyck path by drawing r_v up steps followed by a down steps in order. This is a bijection with (h3) (see to [34, solution for (q)]).

(r) Sequences of n 1's and n -1's such that (1) every partial sum is nonnegative and (2) each string 11 can be colored black or white (with -1 denoted simply as - below).

111--- **1**1**1**--- 1**11**--- **111**--- 11-1-- **11**-1-- 11--1- **11**--1- 1-11-- 1-**11**-- 1-1-1-

Hint. There is an easy bijection between (h3) and (r) by considering 1 to be an up step and -1 to be a down step.

(s) Sequences $1 \leqslant a_1 \leqslant \cdots \leqslant a_n$ of integers satisfying $a_i \leqslant i$ and each a_i with $a_i > a_{i-1} (i \geqslant 2)$ can be colored black or white.

111 112 11**2** 113 11**3** 122 1**2**2 123 1**2**3 12**3** 1**23**

Hint. Consider a lattice path P of the type (i)(ii) in (h). Let $a_i - 1$ be the area above the x-axis between $x = i-1$ and $x = i$, and below P, from which a_i with $a_i > a_{i-1}$ corresponds to a valley in P. This sets up a bijection (see [34, solution for (s)]).

(t) Sequences $0 = a_0 < a_1 < a_2 < \cdots < a_{n-1}$ of integers satisfying $1 \leqslant a_i \leqslant 2i$, $1 \leq i \leq n-1$, and each a_i with $a_i > a_{i-1} + 1$ can be colored black or white.

012 013 01**3** 014 01**4** 023 0**23** 024 0**2**4 02**4** 0**24**

Hint. Subtract $i - 1$ from a_i for $i \geq 1$ and append a 0 at the beginning to get a bijection with (s) (see [34, solution for (t)]), in which we have the corresponding statistics the same color.

(u) Sequences $a_1, a_2, \cdots, a_n$ of integers such that (1) $a_1 = 0$, (2) $0 \leqslant a_{i+1} \leqslant a_i + 1$, and (3) each a_i with $a_i > a_{i-1}$ can be colored black or white.

000 00**1** 001 0**1**0 010 0**1**1 011 0**12** 01**2** 01**2** 012

Hint. Let $b_i = a_i - a_{i+1} + 1$. Replace a_i with one 1 followed by b_i -1's for $1 \leq i \leq n$(with $a_{n+1} = 0$) to get (r) (see [34, solution for (u)]), where each string 11 in (r) corresponds to a a_i with $a_i > a_{i-1}$ in (u). We can also get the conclusion from (xxx). In the tree T of (xxx), label the root by 0 and its three children by 0,1 and 1. Then for vertex labelled by i, it has $i+3$ children and label them by $0, 1, 2, \cdots, i, i+1, i+1$. Then a saturated chain from the root to a vertex at level $n-1$ is thus labelled by a sequence $(a_1, a_2, \cdots, a_n)$ and the occurrence number of a sequence $(a_1, a_2, \cdots, a_n)$ is 2^j, where j is its ascent number.

(v) Sequences of $a_1, a_2, \cdots, a_{n-1}$ of integers such that (1) $a_i \leqslant 1$, (2) all partial sums are nonnegative, and (3) each $a_i = 1$ can be colored black or white.

$$00 \quad 0\mathbf{1} \quad 01 \quad \mathbf{1}-1 \quad 1-1 \quad \mathbf{1}0 \quad 10 \quad \mathbf{11} \quad \mathbf{1}1 \quad 1\mathbf{1} \quad 11$$

Hint. Take the first differences of the sequence in (u), and each 1 corresponds to an a_i with $a_i > a_{i-1}$ (see [34, solution for (v)]).

(w) Sequences $a_1, a_2, \cdots, a_n$ of integers such that (1) $a_i \geqslant -1$, (2) all partial sums are nonnegative and $a_1 + a_2 + \cdots + a_n = 0$, and (3) each -1 can be colored black or white.

000 01-**1** 01-1 10-**1** 10-1 1-**1**0 1-10 2-**1**-**1** 2-**1**-1 2-1-**1** 2-1-1

Hint. Do a depth-first search through a leaves-colored tree with $n+1$ vertices as in (e) from right to left. When a vertex is encountered for the first time, record one less than its number of successors, except that the leftmost leaf is ignored, and color each -1 the same color as its corresponding leaf. This gives a bijection with (e) (see [34, solution for (w)]).

(x) Sequences $a_1, a_2, \cdots, a_n$ of integers such that (1) $0 \leqslant a_i \leqslant n-i$, (2) if $i < j$, $a_i > 0, a_j > 0$ and $a_{i+1} = a_{i+2} = \cdots = a_{j-1} = 0$ then $j - i > a_i - a_j$,and

(3) each nonzero element can be colored black or white.

000 010 0**1**0 100 **1**00 200 **2**00 110 **1**10 1**1**0 **11**0

(3') each zero element not the first can be colored black or white.

000 00**0** 0**0**0 0**00** 010 01**0**100 10**0** 200 20**0** 110

Hint. Proof of (3). For each 321-avoiding permutation $\pi_1\pi_2\cdots\pi_n$ in (ee), define a sequence $a_1a_2\cdots a_n$ with a_i is the number of element π_j such that $j > i$ and $\pi_j < \pi_i$ (see [34, solution for (x)]). Then we get sequences in (x). From this bijection, each excedence in the permutation of (ee) corresponds to a nonzero element in the sequence of (x). So we get the desired result.

Proof of (3'). Let $w = a_1a_2\ldots a_n$ be any sequence satisfies (1)-(2), then there exists s minimal such that $w' = a_1 \ldots a_{s-1}$ is a sequence satisfies (1)-(2), $a_s = 0$ and $w'' = a_{s+1} \ldots a_n$ is a sequence satisfies (1)-(2). In such a case w' is said *minimal sequence.* Let $A(x)$ be the generating function for the number of such sequences of length n. Then $A(x) = \frac{1}{1-xA(x)}$. This leads to $A(x)$ is the generating function for the Catalan numbers, as required.

(y) Sequences $a_1, a_2, \cdots, a_n$ of integers such that (1) $i \leqslant a_i \leqslant n$, (2) $a_j \leqslant a_i$ if $i \leqslant j \leqslant a_i$ and (3) each a_i with $a_i \geqslant a_{i+1}$ can be colored black or white.

123 1**3**3 133 **2**23 223 **3**23 323 **33**3 **3**33 3**3**3 333

Hint. If we replace a_i by $n - a_i$ and color each a_i with $a_i \leq a_{i+1}$ black or white, then consider each a_i is just the number of element π_j with $j > i$ and $\pi_j < \pi_i$ in 213-avoiding permutations. We get a bijection between (y) and 213-avoiding permutations with each ascent colored black or white. Such colored permutations are in obvious bijection with the colored 312-avoiding permutations of (ee). Then we get (y).

(z) Sequences $a_1, a_2, \cdots, a_n$ of integers such that (1) $1 \leqslant a_i \leqslant i$, (2) $a_{i-r} \leqslant j - r$ for all $1 \leqslant r \leqslant j - 1$ if $a_i = j$ and (3) each $a_i = 1$ except the last one (or the first one) can be colored black or white.

1**1**1 111 **1**11 **11**1 **1**12 112 **1**13 113 **1**21 121 123

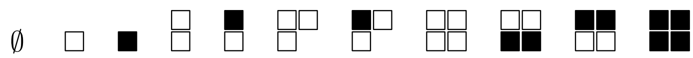
Hint. Given a sequence $a_1, \cdots, a_n$ of type being counted, define recursively a binary tree $T(a_1, \cdots, a_n)$ as follows. Set $T(\emptyset) = \emptyset$. If $n > 0$, then let the left subtree of the root of $T(a_1, \cdots, a_n)$ be $T(a_1, \cdots, a_{n-a_n})$ and the right subtree of the root be $T(a_{n-a_n+1}, a_{n-a_n+2}, \cdots, a_{n-1})$. This sets up a bijection with (c) (see [34, solution for (z)]), and we can see each 1 except the last one corresponds to a left child of some vertex. Having them the same color, we obtain (z).

(aa) For equivalence classes B of words in the alphabet $[n-1]$ such that any three consecutive letters of any word in B are distinct, under the equivalence relation $uijv \sim ujiv$ for any words u, v and any $i, j \in [n-1]$ satisfying $|i - j| \geqslant 2$, color the second letter of each two consecutive letters xy with $x < y$ (assume a 0 at the first position in the word is omitted for nonempty word) black or white.

$\emptyset$ 1 **1** 2 **2** 12 **1**2 1**2** **12** 21 **2**1

Hint. Use the bijection with (ee) which appeared in [34, Solution for (aa)], we can get the desired the result. We color the second letter y of each two consecutive letters xy with $x < y$ (assume a 0 at the first position in the word is omitted for nonempty word) black or white just like that we color each excedance in (ee).

(bb) For the Young diagram of partitions $\lambda = (\lambda_1, \cdots, \lambda_{n-1})$ with $\lambda_1 \leq n - 1$ (so the diagram of λ is contained in an $(n-1) \times (n-1)$ square), such that if $\lambda' = (\lambda'_1, \lambda'_2, \cdots)$ denotes the conjugate partition to λ then $\lambda'_i \geq \lambda_i$ whenever $\lambda_i \geq i$, color each row of the Durfee squares black or white and other rows white.



Hint. There is a bijection between (bb) and (m)(see [34, solution for (bb)]), which maps each row in the Durfee square of a diagram in (bb) to a (0,1) step on the path above in its corresponding lattice paths. As (1,0) steps and (0,1) steps have the same distribution in (m), we get the solution easily.

(cc) Permutations $a_1a_2\cdots a_{2n}$ of the multiset $\{1^2, 2^2, \cdots, n^2\}$ such that: (i) the first occurrences of $1, 2, \cdots, n$ appear in increasing order, and (ii) there is no subsequence of the form $\alpha\beta\alpha\beta$ with each pair of consecutive letters of pattern $\alpha\alpha$ can be colored black or white except the first one.

112233 11**22**33 1122**33** 11**2233** 112332 112**33**2 122331 122**33**1 123321 122133 1221**33**

Hint. Obviously there is a bijection between such sequences and nonintersecting arcs of (o). If we remove the first occurrence of each number, what remains is a permutation w of $[n]$ that uniquely determines the original sequence. These permutations are precisely the reverse ones in (ff).

(dd) Permutations $a_1a_2\cdots a_{2n}$ of the set $[2n]$ such that: (i) $2, 4, \cdots, 2n$ appear in increasing order, and (ii) $2i-1$ appears before $2i$, $1 \leq i \leq n$, with the latter of each two consecutive even numbers can be colored black or white.

123456 123546 12354**6** 132456 132**4**56 132546 13254**6** 135246 1352**4**6 13524**6** 1352**46**

Hint. Replace each odd number by an up step and even number by a down step to get a bijection with Dyck path which corresponds each two consecutive even number to double down steps (see [34, solution for (dd)]). Since double up steps in Dyck path and double down steps have the same distribution, we get the it by coloring each two consecutive even number from (h).

(ee) 321-avoiding permutations of $[n]$ with each letter $a_i > i$ colored black or white, and all others are colored white.

123 213 **2**13 132 1**3**2 312 **3**12 231 **2**31 2**3**1 **23**1

Hint. See [14], the number of 321-avoiding permutations with given number excedances (i.e., $a_i > i$) are counted by Narayana numbers, then we get the desired result.

(ff) 312-avoiding permutations with each a_i, $1 \leq i \leq n-1$, satisfying $a_i > a_{i+1}$ colored black or white.

123 132 1**3**2 213 **2**13 231 2**3**1 321 **3**21 3**2**1 **32**1

Hint. See the notes in (cc), where we have given a proof of this result.

(gg) Permutations w of $[2n]$ with n cycles of length two, such that the product $(1, 2, \cdots, 2n)\cdot w$ has $n+1$ cycles, with each cycle $(x, x+1)$ except the first can be colored black or white.

$(1,2)(3,4)(5,6)$ $(1,2)(3,6)(4,5)$ $(1,2)(\mathbf{3},\mathbf{6})(4,5)$ $(1,4)(2,3)(5,6)$ $(\mathbf{1},\mathbf{4})(2,3)(5,6)$
$(1,6)(2,3)(4,5)$ $(\mathbf{1},\mathbf{6})(2,3)(4,5)$ $(1,6)(2,5)(3,4)$ $(\mathbf{1},\mathbf{6})(2,5)(3,4)$ $(1,6)(\mathbf{2},\mathbf{5})(3,4)$
$(\mathbf{1},\mathbf{6})(\mathbf{2},\mathbf{5})(3,4)$

Hint. The involutions here are the same as those in (kk)(see [34, solution for (gg)]).

(hh) **(open)**

Pairs (u,v) of permutations of $[n]$ such that u and v have a total of $n+1$ cycles and $uv=(1,2,\cdots,n)$, with each cycle of the permutation u colored by black or white except the first one.

$(1)(2)(3)\cdot(1,2,3)$ $(1)(\mathbf{2})(3)\cdot(1,2,3)$ $(1)(2)(\mathbf{3})\cdot(1,2,3)$ $(1)(\mathbf{2})(\mathbf{3})\cdot(1,2,3)$
$(1)(2,3)\cdot(1,2)(3)$ $(1)(\mathbf{2},\mathbf{3})\cdot(1,2)(3)$ $(12)(3)\cdot(13)(2)$ $(12)(\mathbf{3})\cdot(13)(2)$
$(13)(2)\cdot(1)(23)$ $(13)(\mathbf{2})\cdot(1)(23)$ $(123)\cdot(1)(2)(3)$

(ii) Permutations $a_1a_2\cdots a_n$ of $[n]$ that can be put in increasing order on a single stack, defined recursively as follows: If $\emptyset$ is the empty sequence, then let $S(\emptyset)=\emptyset$. If $w=unv$ is a sequence of distinct integers with largest term n, then $S(w)=S(u)S(v)n$. A *stack-sortable* permutation w is one for which $S(w)=w$. (231-avoiding permutations) After that, each element who is a_i of an consecutive increasing subsequence a_ia_{i+1} can be colored black or white.

123 **1**23 1**2**3 1**23** 132 **1**32 213 2**1**3 312 3**1**2 321

Hint. See [34, solution for (ii)] for the bijection with (r), from which we get a bijection between (ii) and Dyck path. Draw an up step when a_i is put on the stack, and draw a down step when a_i is taken off, then we know each a_i with $a_i<a_{i+1}$ in permutation corresponds to a valley in Dyck path. We can also get (ii) directly from (ff) as the permutations in (ii) are just 231-avoiding permutations.

(jj) For Permutations $a_1a_2\cdots a_n$ of $[n]$ that can be put in increasing order on two parallel queues, (321-avoiding permutations) with each letter $a_i>i$ colored black or white, and all others colored white.

123 213 **2**13 132 1**3**2 312 **3**12 231 **2**31 2**3**1 **23**1

Hint. Same set as (ee).

(kk) Fixed-point free involutions w of $[2n]$ such that if $i<j<k<l$ and $w(i)=k$, then $w(j)\neq l$ (in other words, 3412-avoiding fixed-point free involutions), color each cycle $(x\,x+1)$ not the first by black and white.

(12)(34)(56) (12)(34)(**56**) (12)(**34**)(56) (12)(**34**)(**56**) (12)(36)(45) (12)(36)(**45**)
(14)(23)(56) (14)(23)(**56**) (16)(23)(45) (16)(23)(**45**) (16)(25)(34)

Hint. Obviously there is a bijection with matchings of (o).

(ll) **(Open)**

Cycles of length $2n+1$ in $\mathfrak{S}_{2n+1}$ with descent set $\{n\}$ is enumerated by Catalan number (see Theorem 9.4 in [19]). Then whether we can get a corresponding structure counted by super-Catalan number by coloring each two letters xy at the right position of n such that $y-x>1$ black or white, such as

2371456 237**14**56 2571346 257**13**46 25713**46** 257**1346**
3471256 3471**256** 3671245 3671**24**5 5671234

for $n=3$, if "yes", is there a bijective proof?

(mm) Baxter permutations of $[2n]$ (or of $[2n+1]$) such that (1) they are reverse alternating and whose inverses are reverse alternating, and (2) each π_i, $2 \le i \le 2n-1$, with $\pi_i > \pi_{i+1} > \pi_{i-1}$ in the permutation $\pi_1\pi_2\cdots\pi_{2n}$ can be colored black or white (or similarly each π_i, $3 \le i \le 2n$, with $\pi_i > \pi_{i-1} > \pi_{i+1}$ in $\pi_1\pi_2\cdots\pi_{2n+1}$).

132546 1**3**2546 132**5**46 1**3**2**5**46 153426 1**5**3426
354612 3**5**4612 561324 561**3**24 563412

or

1325476 1327564 13275**6**4 1534276 153**4**276 1735462
17354**6**2 1756342 175**6**342 17563**4**2 175**6**3**4**2

Hint. This result is due to the bijection given by O. Guibert and S. Linusson [21]. From their bijection, we can see each Baxter permutation corresponds to a complete binary tree and each π_i satisfying the condition (2) corresponds to a left internal vertex in complete binary tree, i.e., which induces a bijection between (mm) and (b').

(nn) Permutations w of $[n]$ such that if w has ℓ inversions then for all pairs of sequences $(a_1, a_2, \ldots, a_\ell), (b_1, b_2, \ldots, b_\ell) \in [n-1]^\ell$ satisfying $w = s_{a_1}s_{a_2}\ldots s_{a_\ell} = s_{b_1}s_{b_2}\ldots s_{b\ell}$ where s_j is the adjacent transposition $(j, j+1)$, we have that the multisets $\{a_1, \ldots, a_\ell\}$ and $\{b_1, \ldots, b_\ell\}$, and then color each letter $\pi_i > i$ black or white, and all others white.

123 213 **2**13 132 1**3**2 312 **3**12 231 **2**31 2**3**1 **23**1

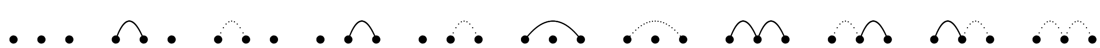
Hint. Same set as (ee).

(oo) 132-avoiding permutations of $[n]$ with each $a_i > i$ colored black or white, and all others colored white.

123 213 **2**13 231 **2**31 2**3**1 **23**1 312 **3**12 321 **3**21

Hint. See [14], the number of 132-avoiding permutations with given number excedances (i.e., $a_i > i$) are counted by Narayana numbers, then we get the desired result. Also the 132-avoiding permutations with given number descents (i.e., $a_i < i$) are also counted by Narayana numbers.

(pp) Noncrossing partitions of $[n]$, i.e., partitions $\pi = \{B_1, \cdots, B_k\} \in \Pi_n$ such that (1) if $a < b < c < d$ and $a, c \in B_i$ and $b, d \in B_j$, then $i = j$, and (2) each edge can be colored black or white.



Hint. There is a bijection θ between (pp) and small-binary trees. For a noncrossing partition P of $[n]$, traverse each element from left to right, for the element i encountered, we have two cases:

Case 1: if 1 is an initiated vertex of an edge $e = (1, j)$, set the partition covered by e is P' and P'' is the partition got from P by omitting P' and the element 1, then $\theta(P) = LrR$, where $L = \theta(P'')$ and $R = \theta(P')$ (see the left side of Figure 1);

Case 2: if 1 is an isolated vertex and P' is the partition got from P by omitting 1, then $\theta(P) = rR$, where $R = \theta(P')$ (see the right side of Figure 1).

Then we can see the end vertex of each edge in partition corresponds to a left child in the tree, color this left child the same color as that of the edge. From the rule, we can see it is really a one to one correspondence.

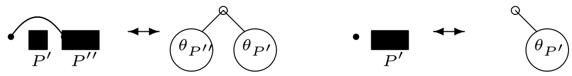
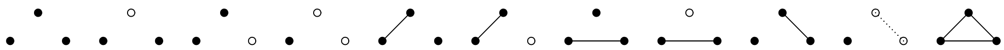


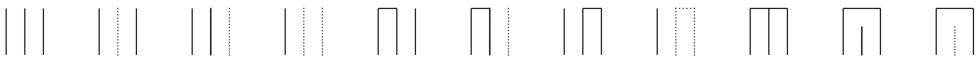
Figure 1: bijection θ.

(qq) Partitions $\{B_1, \cdots, B_k\}$ of $[n]$ such that (1) if the numbers $1, 2, \cdots, n$ are arranged in order around a circle, then the convex hulls of the blocks $B_1, \cdots, B_k$ are pairwise disjoint, and (2) each part except the first part (the part containing the first node) can be colored black or white and the first part is colored black.



Hint. If we number the nodes in clockwise order, then we can give a bijection between them and (pp) in the following way: (1) two nodes are connected by an edge are corresponding to that two isolated nodes in (pp); (2) two adjacent nodes (i, j) without an edge between them are corresponding to an arc between (i, j) in (pp). Then color each part by the same way.

(rr) Noncrossing Murasaki diagrams with n vertical lines and each connected line can be colored black or white and the first one colored black.



Hint. Obviously there is a bijection between such diagrams and partitions in (qq). Each vertical line is corresponding to a node of (qq).

(ss) Noncrossing partitions of some set $[k]$ with $n + 1$ blocks, such that (1) any two elements of the same block differ by at least three, and (2) each edge (i, j) colored black and white.

1-2-3-4 14-2-3-5 **14**-2-3-5 15-2-3-4 **15**-2-3-4 25-1-3-4
25-1-3-4 16-25-3-4 **16**-25-3-4 16-**25**-3-4 **16**-**25**-3-4

Hint. Let $g(x)$ be the generating function for the number of such noncorssing partitions with n blocks. Then, if we consider the number letters in the first block then we arrive at

$$g(x) = 1 + xg(x) + x\sum_{j\geq 1}(2(g(x) - 1 - x))^j g(x) = 1 + \frac{xg(x)}{1 - 2(g(x) - 1 - x)}.$$

Hence, $g(x) = 1 + x\frac{1+x-\sqrt{x^2-6x+1}}{4x}$, as required.

(tt) Noncrossing partitions of $[2n+1]$ into $n+1$ blocks, such that (1) no block contains two consecutive integers, and (2) each block at least two numbers except the first can be colored black or white and the first one is colored white.

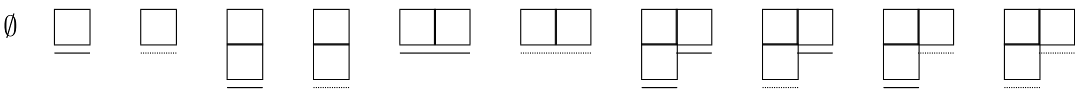
137-46-2-5 137-**46**-2-5 1357-2-4-6 157-24-3-6 157-**24**-3-6 17-246-3-5
17-**246**-3-5 17-26-35-4 17-**26**-35-4 17-26-**35**-4 17-**26**-**35**-4

Hint. Label the vertices $1, 2, \cdots, 2n+1$ of a tree in (b') in preorder. Define i and j to be in the same block of $\pi \in \Pi_{2n+1}$ if j is a right child of i(see [34, solution for (tt)]). Then we can see each block with at least two numbers corresponds to a left internal vertex except the block containing 1 corresponds to the root, then we obtain a bijection by requiring the corresponding statistics the same color.

(uu) Nonnesting partitions of $[n]$, i.e., partitions of $[n]$ such that (1) if a, e appear in a block B and b, d appear in a different block B' where $a < b < d < e$, then there is $ac \in B$ satisfying $b < c < d$, and (2) each edge can be colored black or white.

Hint. For a given noncrossing partition, consider its standard linear representation. Omitting the singleton blocks, we can get a noncrossing matching from which we can get a nonnesting partition by bijection(see [35, Solution for (g^4)]). Then have the singleton block into the nonnesting partition, which have the same label as before. Since the noncrossing partitions on $[n]$ with m edges has the same distribution as the nonnesting partitions on $[n]$ with m edges, (uu) counted by super-Catalan number is obvious.

(vv) Young diagrams that fit in the shape $(n-1, n-2, \cdots, 1)$ and each horizontal step is colored black or white.



Hint. For a given Young diagram of shape $(\lambda_1, \lambda_2, \cdots, \lambda_{n-1})$ which fits in the shape $(n-1, n-2, \cdots, 1)$, we can get a Dyck Path of length $2n$ starts with an up step followed by λ_{n-1} down steps, then has, for each $2 \le j \le n-1$, one up step followed by $\lambda_{n-i} - \lambda_{n-i+1}$ down steps, and finally ends with an up steps followed by $n - \lambda_1$ down steps. This gives a bijection between the Young diagram and Dyck Path, from which we can see the distinct parts in the corresponding partition of the conjugate Young diagram corresponds to the peaks in Dyck path except the first one. Therefore (vv) holds by having these two statistics the same color from (h) (the statistics peaks and double up steps has the same distribution in (vv)).

(ww) Standard Young tableaux of shape (n, n) (or equivalently, of shape $(n, n-1)$), and each segment $(i, i+1)$ in the first row can be colored black or white (the same for that of shape $(n, n-1)$).

123	**12**3	1**23**	**123**	124	**12**4	125	**12**5	134	1**34**	135
456	456	456	456	356	356	346	346	256	256	246

or

123	**12**3	1**23**	**123**	124	**12**4	125	**12**5	134	1**34**	135
45	45	45	45	35	35	34	34	25	25	24

Hint. We can get (ww) directly by applying a bijection between Dyck path and standard Young tableaux (see [34, Solution for (ww)]). From the bijection, set the ith step $(1, 1)$ if i is in the first row and $(1, -1)$ if i is in the second row for a (n, n) standard Young tableaux, then we get a Dyck path. It is clear double $(1, 1)$ steps in Dyck path corresponds to a segment $(i, i + 1)$ in the first row of standard Young tableaux. So we can have them the same color.

(xx) For pairs (P, Q) of standard Young tableaux of the same shape, each with n squares and at most two rows, where each number i of the first row of P or number j of the second row of Q can be colored black and white if $i - 1$ appears in the second row of P or $2n - j$ appears in the second row of P or $j + 1$ appears in the first row of Q.

123,123	12,12	12,12	12,13	12,13	13,12	1**3**,12	13,13	1**3**,13	13,13	1**3**,13
	3 3	3 **3**	3 2	3 **2**	2 3	2 **3**	2 2	2 2	2 **2**	2 **2**

Hint. See solution of [34, Exercise 6.19(ff)] for the bijection between (ww) and (xx) without color for Catalan number. From the bijection, we can see each number i of the first row with $i-1$ in the second row of P and number j of the second row of Q with $2n-j$ in the second row of P or $j + 1$ in the first row of Q corresponds to number i of the first row in (ww) with $i - 1$ in the second row. From the bijection between the Young tableaux and Dyck path in solution of (ww), we can see the number i of the first row with $i - 1$ in the second row has the same distribution with $(i, i + 1)$ segment in the Young tableaux, then we get (xx) easily from (ww).

(yy) Column-strict plane partitions of shape $(n - 1, n - 2, \cdots, \cdots, 1)$, such that (1) each entry in the ith row is equal to $n - i$ or $n - i + 1$, and (2) row i can be colored black or white if the number of occurrences of the letter $n - i + 1$ in row i is greater than the number of occurrence of the letter $n - i$ in row $i + 1$.

33	**33**	33	**33**	**33**	33	32	32	**32**	32	22
2	**2**	**2**	2	1	1	**2**	2	1	1	1

Hint. Use the bijection with (s) given by Stanley [34, Solution for (yy)]: Let b_i be the number of entries in row i that are equal to $n - i + 1$ (so $b_n = 0$). The sequences $b_n + 1, b_{n-1} + 1, \cdots, b_1 + 1$ obtained in this way are in bijection with (s). Then color the corresponding row as that color the letter in (s).

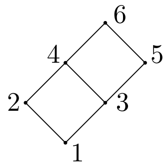
(zz) **(how to color is unsolved)**

Convex subsets S of the poset $\mathbb{Z} \times \mathbb{Z}$, up to translation by a diagonal vector (m, m), such that if $(i, j) \in S$ then $0 < i - j < n$.

$$\emptyset \quad \{(1, 0)\} \quad \{(2, 0)\} \quad \{(1, 0), (2, 0)\} \quad \{(2, 0), (2, 1)\}$$

Hint. (zz) is (zz) in Stanley's homepage.

(aaa) Linear extensions of the poset $\mathbf{2} \times \mathbf{n}$ with each element a_i, $1 \le i \le 2n-1$, satisfying $a_i > a_{i+1}$ colored by black or white.

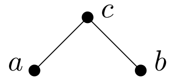


123456   123546   123**5**46   132456   1**3**2456   132546

1**3**2546   132**5**46   1**3**2**5**46   135246   13**5**246

Hint. For each linear extension, consider each number from left to right, and draw an up step for each odd number while a down step for each even number. Since each a_i with $a_i > a_{i+1}$ corresponds to an up step of a peak at level greater than 1, color it the same color as the peak, then we get a bijection between (aaa) and (h5).

(bbb) Order ideals of $\mathrm{Int}(n-1)$, the poset of intervals of the chain $n-1$, then color the maximal element of each poset black or white, where x is maximal if there is no v in the poset such that $x < v$.



Int(2)

$\emptyset$ a   **a**   b   **b**   ab   **a**b   a**b**   **ab**   abc   ab**c**

Hint. According to [34, Solution for (bbb)] there is an obvious bijection with order ideals I of $Int(\mathbf{n})$ that contain every one-element interval of $\mathbf{n}$ where each maximal element of I colored black and white. But the upper boundary of the Hasse diagram of I looks like the Dyck paths of (h) where each peak not at high one colored black and white. So the result follows from (h)(5).

(ccc) **(how to color is unsolved)**

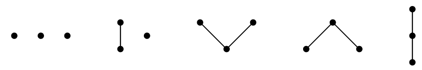
Order ideals of the poset A_n obtained from the poset $(n-1) \times (n-1)$ by adding the relations $(i,j) < (j,i)$ if $i > j$.

$$\emptyset \quad \{11\} \quad \{11,21\} \quad \{11,21,12\} \quad \{11,21,12,22\}$$

Hint.(ccc) is (ccc) is Stanley's homepage.

(ddd) **(how to color is unsolved)**

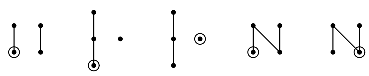
Nonisomorphic n-element posets with no induced subposet isomorphic to $\mathbf{2}+\mathbf{2}$ or $\mathbf{3}+\mathbf{1}$.



Hint.(ddd) is (ddd) is Stanley's homepage.

(eee) **(how to color is unsolved)**

Nonisomorphic $(n+1)$-element posets that are a union of two chains and that are not a (nontrivial) ordinal sum, rooted at a minimal element.



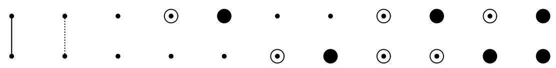
Hint.(eee) is (eee) is Stanley's homepage.

(fff) Relations R on $[n]$ that are reflexive (iRi), symmetric ($iRj \Rightarrow jRi$), and such that if $1 \leq i < j < k \leq n$ and iRk, then iRj and jRk (in the example below we write ij for the pair (i, j), and we omit the pairs ii). Then color the pair ij black or white such that $i > j$, ij have the biggest differences for i, pairs $(i-1, j), \cdots, jj$ are also in this relation set, and no pairs jk such that $j > k$ appear in this set (see the Hint).

$\emptyset$ $\{12, 21\}$ $\{12, \mathbf{21}\}$ $\{23, 32\}$ $\{23, \mathbf{32}\}$ $\{12, 21, 23, 32\}$ $\{12, \mathbf{21}, 23, 32\}$
$\{12, 21, 13, 31, 23, 32\}$ $\{12, \mathbf{21}, 13, 31, 23, 32\}$ $\{12, 21, 13, \mathbf{31}, 23, 32\}$ $\{12, \mathbf{21}, 13, \mathbf{31}, 23, 32\}$

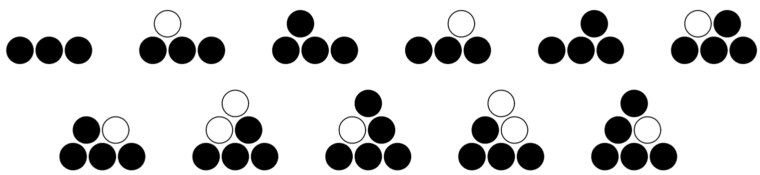
Hint. Without considering the colorings, there is a bijection between present item and (s). Let a_i denote the smallest element j such that iRj and first adding pairs ii on each relation set. Then we get the bijection obviously. We already know there is a bijection between (s) and Dyck paths. So we obtain a bijection between present item and Dyck paths. The colorings of the paris are just the colorings of subsequence ii of (s) or double ascents on Dyck paths.

(ggg) After joining some of the vertices of a convex $(n-1)$-gon by disjoint line segments, and circling a subset of the remaining vertices, color each edge and each circle black or white while others white.



Hint. If $f(x)$ is the generating function for the number of such figures on n vertices, then $f(x) = 1 + 3xf(x) + 2x^2 f^2(x)$, which gives $f(x) = \frac{1-3x-\sqrt{1-6x+x^2}}{4x^2}$. Hence, the number of such figures on $n-1$ vertices equals the n-th super-Catalan number. In fact, from the expression of $f(x)$, we can see it is the same as that of $(2,3)$-Motzkin path. So there is a bijection between (ggg) and (yyy). Initiating at a fixed vertex of given $(n-1)$-gon, draw a horizontal steps for each isolated vertex, and draw a up step for encountering an edge the first time while a down step the second time. There is three possibilities for each isolated vertex, i.e., uncircled, circled by white or black circle. And for each initiating vertex of an edge, there is two possibilities corresponding to the edge is black or white. Therefore, we can get the bijection obviously.

(hhh) Ways to stack black and white coins in the plane such that the bottom row consisting of n consecutive black coins and there is no adjacent coins in the same row (except the bottom one) being the same color.



Hint. See [18, solution for (k)].

(iii) For n-tuples $(a_1, a_2, \cdots, a_n)$ of integers $a_i \geq 2$ such that in the sequence $1a_1a_2\cdots a_n1$, each a_i divides the sum of its two neighbors, with each a_j, $1 \leq j \leq n-1$, satisfying $a_j > a_{j+1}$ colored black or white.

14321 1**4**321 14**3**21 1**43**21 13521 13**5**21 13231 1**3**231 12531 12**5**31 12341

Hint. For a sequence in (iii), find the rightmost number a_i with $a_i = a_{i-1} + a_{i+1}$ for $1 \leq i \leq n$ and set $a_0 = 1$, $a_{n+1} = 1$. Insert a mark before a_{i-1}. Now consider the same sequence by omitting this a_i, and do the above process recursively. Then we get a sequence with n marks. By replacing the mark and each number by 1 and -1 respectively, we get a bijection between (iii) and (r) without color and given by J. H. van Lint [25]. From the bijection, we can see each descent in (iii) corresponds to two consecutive marks (i.e., 11). Therefore, (iii) is desired from (i).

(jjj) **(Unproved)**

n-element multisets on $\mathbb{Z}/(n+1)\mathbb{Z}$ whose elements sum to 0 with each x followed by another x immediately colored black or white.

000 **0**00 0**0**0 **00**0 013 022 0**2**2 112 **1**12 233 2**3**3

Hint.(jjj) is (jjj) in Stanley's homepage.

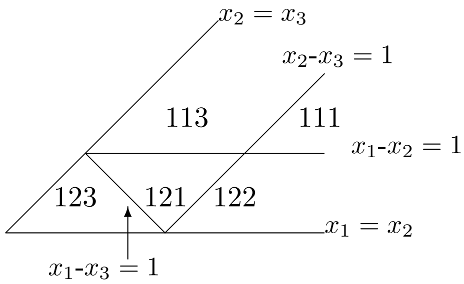
(kkk) **(Unproved)**

n-elements subsets S of $\mathbb{N} \times \mathbb{N}$ such that if $(i, j) \in S$ then $i \geq j$ and there is a lattice path from $(0, 0)$ to (i, j) with steps $(0, 1), (1, 0)$, and $(1, 1)$ that lies entirely inside S, with each element (x, y), satisfying $(x-1, y-1) \in S$, colored black or white.

$$\{(0,0),(1,0),(2,0)\} \quad \{(0,0),(1,0),(1,1)\} \quad \{(0,0),(1,0),(\mathbf{1},\mathbf{1})\} \quad \{(0,0),(1,0),(2,1)\}$$
$$\{(0,0),(1,0),(\mathbf{2},\mathbf{1})\} \quad \{(0,0),(1,1),(2,1)\} \quad \{(0,0),(\mathbf{1},\mathbf{1}),(2,1)\} \quad \{(0,0),(1,1),(2,2)\}$$
$$\{(0,0),(\mathbf{1},\mathbf{1}),(2,2)\} \quad \{(0,0),(1,1),(\mathbf{2},\mathbf{2})\} \quad \{(0,0),(\mathbf{1},\mathbf{1}),(\mathbf{2},\mathbf{2})\}$$

Hint.(kkk) is (kkk) in Stanley's homepage.

(lll) Regions into which the cone $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n$ in $\mathbb{R}_n$ is divided by the hyperplanes $x_i - x_j = 1$, for $1 \leqslant i < j \leqslant n$ (the diagram below shows the situation for $n = 3$, intersected with the hyperplane $x_1 + x_2 + x_3 = 0$), then we give a labelling to each region by $1, \cdots, n$ by the method in [1]. Obviously the labelling is a partition. Then we can color each edge of the matching for each region black or white.



Hint. We use the bijection used in [1] to give the methods to color the regions of hyperplanes.

(mmm) For positive integer sequences $a_1, a_2, \cdots, a_{n+2}$ for which there exists an integer array (necessarily with $n+1$ row)

$$\begin{array}{cccccccccccccccccccc}
1 & & 1 & & 1 & & \cdots & & 1 & & 1 & & 1 & & \cdots & & 1 & & 1 \\
& a_1 & & a_2 & & a_3 & \cdots & a_{n+2} & & a_1 & & a_2 & \cdots & a_{n-1} & & & & & \\
& & b_1 & & b_2 & & b_3 & \cdots & b_{n+2} & & b_1 & \cdots & b_{n-2} & & & & & & \\
& & & & & & & \vdots & & & & & & & & & & & \\
& & & & r_1 & & r_2 & & r_3 & \cdots & r_{n+2} & & r_1 & & & & & & \\
& & & & & 1 & & 1 & & 1 & & \cdots & 1 & & & & & &
\end{array}$$

such that any four neighboring entries in the configuration $s_u^r t$ satisfy $st = ru + 1$, color each minimal element in the sequence black or white and others white, where we define the minimal element as below. For a Convex $(n+2)$-gon P with vertices $1, 2, \cdots, n+2$ labelled in clockwise order. Let T be a triangulation of such polygon, and let a_i be the number of triangles incident to i, which gives a bijection with (a). From the color process in (a), we get a sequence of graphs initiating with a $(n+2)$-gon and ending with a triangular. Considering vertex i in the sequence, if there exists a graph except the last one in the sequence such that i is on the minimal edge(defined in (a)) as a minimum number and it is not on any diagonal, then a_i is called the minimum element in (mmm).

12213 **1**2213 1**2**213 **12**213 22131 2**2**131 21312 **2**1312 13122 **1**3122 31221

Hint. From the narration of (mmm), we can know the objects enumerated by super-Catalan number obviously from (a). The coloring in (mmm) is somehow difficult, so it is a challenge to find out a direct coloring method to get the structures enumerated by super-Catalan numbers.

(nnn) **(how to color is unsolved)**

n-tuples $(a_1, \cdots, a_n)$ of positive integers such that the tridiagonal matrix

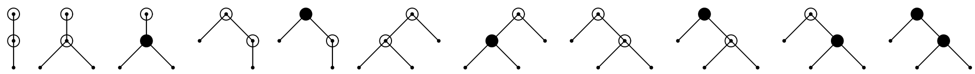
$$\begin{bmatrix}
a_1 & 1 & 0 & 0 & \cdots & 0 & 0 \\
1 & a_2 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & a_3 & 1 & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & a_{n-1} & 1 \\
 & & & & \vdots & & \\
0 & 0 & 0 & 0 & \cdots & 1 & a_n
\end{bmatrix}$$

is positive definite with determinant one

131 122 221 213 312

Hint. (nnn) is (nnn) in Stanley's homepage.

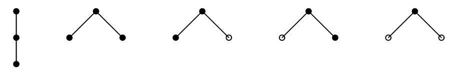
(ooo) For the plane trees with $n-1$ internal nodes (including the root), each having degree 1 or 2, such that nodes of degree 1 occur only on the rightmost path, each internal vertex of degree 2 with its left child vertex is not internal can be colored black or white while other internal vertices white.



Hint. Traverse the tree in preorder. When going down an edge (i.e., away from the root) record 1 if this edge goes to the left or straight down, and record -1 if this edge goes to the right. Then each vertex of degree 2 corresponds to a 1 followed by a -1 immediately, we can get a bijection by requiring the corresponding statistic the same color(see [33, solution for (ooo)]).

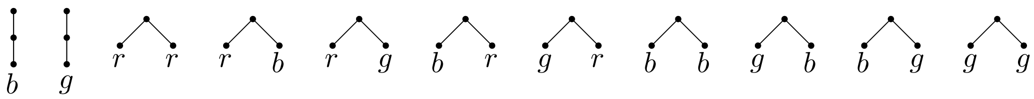
(ppp) **(how to color is unsolved)**

Plane trees with n vertices, such that going from left to right all subtrees of the root first have an even number of vertices and then an odd number of vertices, with those subtrees with an odd number of vertices colored either red or blue.



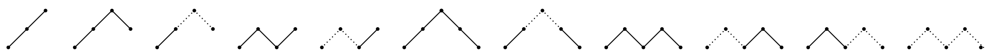
Hint. This (ppp) corresponds to (ppp) in Stanley's homepage.

(qqq) Plane trees with n vertices whose leaves at height 1 are colored by red or blue or green while the leaves at height greater than 1 are colored by blue or green.



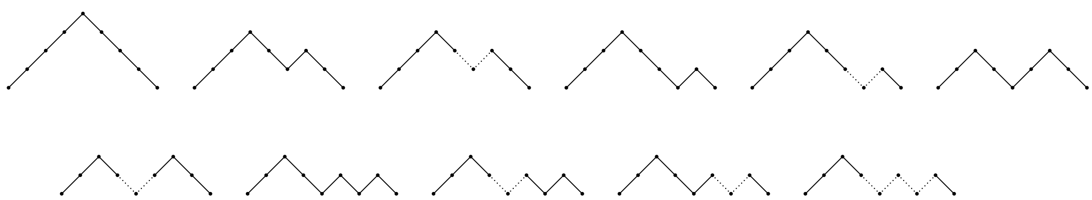
Hint. See [18] for the enumeration of such leaves-colored trees.

(rrr) Left factors L of Dyck paths such that L has $n-1$ up steps with each peak can be colored black or white.



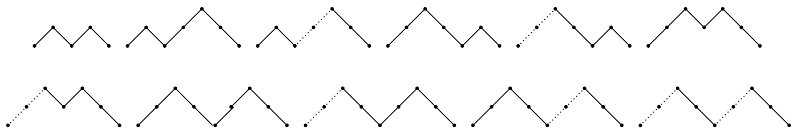
Hint. Add one further up step and then down steps until reaching (2n,0). This gives a bijection with (h1)(the peaks except the first one and the peaks except the last one have the same distribution)(see [33, solution for (sss)]).

(sss) Dyck paths of length $2n+2$ whose first downstep is followed by another downstep with each valley (or peak except the first one) colored by black or white.



Hint. Deleting the first peak gives a bijection with (h2)(see [33, solution for (ttt)]).

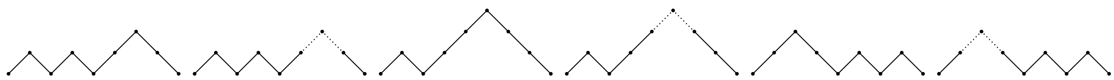
(ttt) For Dyck paths with $n-1$ peaks and without three consecutive up steps or three consecutive down steps, color each double $(1,1)$ steps (or double $(1,-1)$ steps) black or white.



Hint. In the (2,3)-Motzkin paths of (yyy), replace the step (1,1) with the sequence of steps $(1,1)+(1,1)+(1,-1)$, the step (1,-1) with the sequence of steps $(1,1)+(1,-1)+(1,-1)$, the red step (1,0) with $(1,1)+(1,-1)$, and the other (1,0) step with $(1,1)+(1,1)+(1,-1)+(1,-1)$(see [33, solution for (uuu)]). Color each double (1,1) steps white if its corresponding edge in (yyy) is red or green and double (1,1) steps black if its corresponding edge is blue.

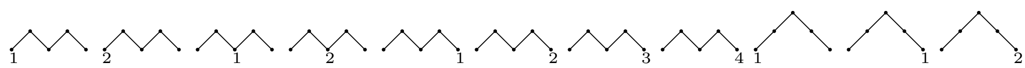
(uuu) **(Unproved)**

Dyck paths P from $(0,0)$ to $(2n+2,0)$ such that (1) there is no horizontal line segment L with endpoints (i,j) and $(2n+2-i,j)$, with $i>0$, such that the endpoints lie on P and no point of L lies above P, (2) all the peaks in the same block (a block of a Dyck path is a segment of the path with only two endpoints being on x-axis) with height at least 2 can be colored black and white while others white.



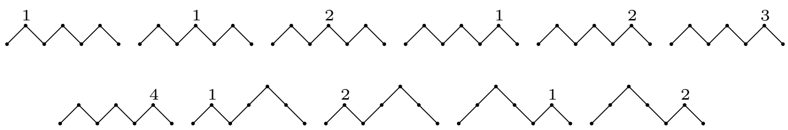
Hint. This (uuu) corresponds to (vvv) in Stanley's homepage.

(vvv) For the points of the form $(m,0)$ on all Dyck paths from $(0,0)$ to $(2n-2,0)$, color one of such points uniquely on a Dyck path with s colors, where $s=2^{\#\ of\ valleys\ of\ the\ path+1}$ for the last point and $s=2^{\#\ of\ valleys\ of\ the\ path}$ for the others (we denote s colors for each point by $\{1,2,\cdots,s\}$).



Hint. To obtain a bijection with the Dyck paths of (h) add a (1,1) step immediately following a path point (m,0) and a (1,-1) step at the end of the path(see [33, solution for (www)]). From this bijection, we can get a path P in (h) with the same number of valleys as its corresponding path in (vvv) if (m,0) is not the last point on x-axis while P has one more valley than its corresponding path in (vvv) if (m,0) is the last point. As the valleys in Dyck Path have the Narayana distribution, we get (vvv).

(www) Dyck paths from $(0,0)$ to $(2n,0)$ having peaks at height one, with only one peak can by colored by 2^i colors, where $i=\#$ of peaks at height one before the colored peak $+\#$ of peaks with height at least 2.

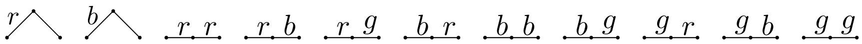


Hint. Denote the generating function for the number of Dyck paths of length $2n$ with r peaks at high one and s peaks at high at leats 2 by $C(x;p,q)$. Then, $C(x;p,q)$ satisfies $C(x;p,q)=1+xpC(x;p,q)+xC(x;p,q)(C(x;q,q)-1)$. If $F(x)$ is the generating function for the number of Dyck paths in the question of length $2n$, then it is not hard to see that $F(x)=1+C(x;2,2)-C(x;1,2)=\frac{1+x-\sqrt{1-6x+x^2}}{4x}$, as required.

(xxx) Vertices of height n of the tree T defined by the property that the root has degree 3, and if the vertex x has degree k, then the children of x have degrees $3,4,\ldots,k-1,k,k+1,k+1$.

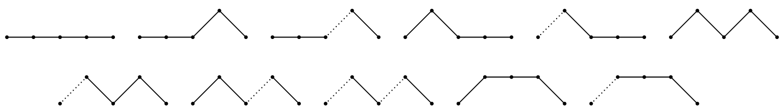
Hint. See math.boisestate.edu/∼sulanke/paper1/PergolaSulanke/node3.html.

(yyy) Motzkin paths with the steps $(1,1)$ can be colored red or blue while the steps $(1,0)$ can be colored by red, blue or green, i.e., $(2,3)$-Motzkin paths (see [17]).



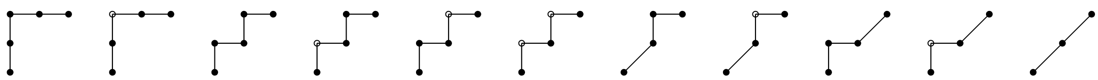
Hint. $(2,3)$-Motzkin paths have been studied in [17, 38].

(zzz) Motzkin paths $a_1,\cdots,a_{2n-2}$ from $(0,0)$ to $(2n-2,0)$ such that each odd step a_{2i+1} is either $(1,0)$ (straight) or $(1,1)$ (up), and each even step a_{2i} is either $(1,0)$ (straight) or $(1,-1)$ (down), with each step (1,1) can be colored black or white.



Hint. See the bijection between Motzkin paths and noncrossing partitions given in [35, solution for (d^4)]. Then by Coloring the up step in Motzkin path the same color as the edge initiatin at the vertex corresponding to the up step, we sets up a bijection between (zzz) and (pp).

(a^4) Lattice paths from $(0,0)$ to $(n-1,n-1)$ with steps $(0,1),(1,0)$, and $(1,1)$, never going below the line $y=x$, such that the steps $(1,1)$ only appear on the line $y=x$, each peak being colored black or white.



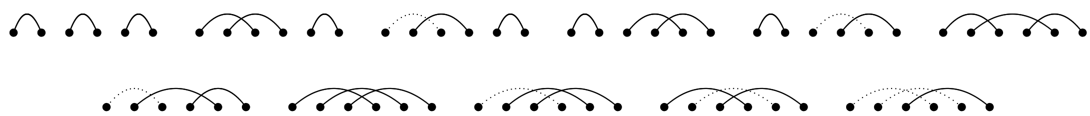
Hint. We know that such paths are called Schröder paths. Changing each peak to be a horizontal step, and each horizontal step $(1,1)$ to be a peak, we get a Schröder path from $(0,0)$ to $(n-1, n-1)$ without high peaks, which is just as the same as that of (r^6).

(b^4) Lattice paths of length $n-1$ from $(0,0)$ to the x-axis with steps $(\pm 1, 0)$ and $(0, \pm 1)$, never going below the x-axis, and each right and up steps are colored black or white. (or left and down, or right and down, or left and up steps.)

$$(-1,0)+(-1,0) \quad (-1,0)+(1,0) \quad (-1,0)+(\mathbf{1},\mathbf{0}) \quad (0,1)+(0,-1) \quad (\mathbf{0},\mathbf{1})+(0,-1)$$
$$(1,0)+(-1,0) \quad (\mathbf{1},\mathbf{0})+(-1,0) \quad (1,0)+(1,0) \quad (\mathbf{1},\mathbf{0})+(1,0) \quad (1,0)+(\mathbf{1},\mathbf{0}) \quad (\mathbf{1},\mathbf{0})+(\mathbf{1},\mathbf{0})$$

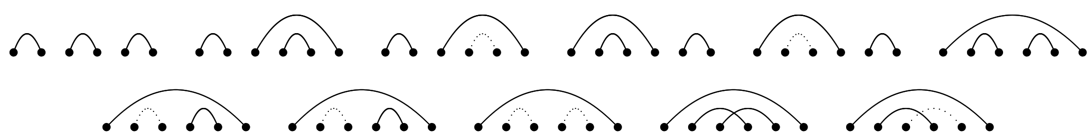
Hint. Let $A(x)$ be the generating function of such lattice paths with each right and up step colored black or white. Then $A = 1 + 3xA + 2x^2A^2$, from which we can get the desired result.

(c^4) For the *Nonnesting matchings* on $[2n]$, i.e., ways of connecting $2n$ points in the plane lying on a horizontal line by n arcs, each arc connecting two of the points and lying above the points, such that no arc is contained entirely below another, color each edge $e = (i, j)$ if there is exist an edge $e' = (i+1, k)$ with $j < k$.



Hint. For the left end of each arc on a matching, draw a up step and down one for each right end, we get a bijection between it and Dyck path. Color each double ascent (two consecutive up steps) black or white, we get the desired result (see (h3)).

(d^4) For the ways of connecting 2n points in the plane lying on a horizontal line by n arcs, each arc connecting two of the points and lying above the points, such that the following condition holds: for every edge e let $n(e)$ be the number of edges e' that nest e (i.e., e lies below e'), and let $c(e)$ be the number of edges e' that begin to the left of e and that cross e, then $n(e) - c(e) = 0$ or 1, color each edge with $n(e) - c(e) = 1$ black or white.



Hint. In [33] Stanley gave a bijection with (r), where $f(i) = \lceil \frac{i}{2} \rceil$ and (r) is, in fact, a Dyck path. And then color each high peak black or white.

(e^4) Ways of connecting any number of points in the plane lying on a horizontal line by non-intersecting arcs lying above the points, such that the total number of arcs and isolated points is n. Then color each arc $(i, i+1)$ black or white and the other arcs black.

Hint. Reading the points from left-to-right, replace each isolated point and each point which is the left endpoint of an arc with 1, and replace each point which is the right endpoint of an arc with -1. We obtain a bijection with (p^4). This is the method used in [33], then color arcs by the same way of that in (p^4).

(f^4) Ways of connecting n points in the plane lying on a horizontal line by noncrossing arcs above the line such that if two arcs share an endpoint p, then p is a left endpoint of both the arcs. Then color each edge by black or white.

Hint. Label the points $1, 2, \cdots, n$ from left-to-right. Given a noncrossing partition of $[n]$ as in (pp), draw an arc from the first element of each block to the other elements of that block yielding a bijection with the current item. We get the same number of arcs as that of (pp). Color the arcs black or white we get such objects enumerated by the super-Catalan numbers.

(g^4) Ways of connecting $n+1$ points in the plane lying on a horizontal line by noncrossing arcs above the line such that no arc connects adjacent points and the right endpoints of the arcs are all distinct. Then each edge is colored black or white.

Hint. Use the bijection with binary trees mentioned in [33], we get the desired result by the objects given in [34, Exercise 6.39(d)] which are counted by super-Catalan numbers.

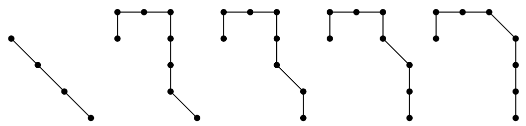
(h^4) Lattice paths in the first quadrant with n steps from $(0,0)$ to $(0,0)$, where each step is of the form $(\pm 1, \pm 1)$, or goes from $(2k, 0)$ to $(2k, 0)$ or $(2(k+1), 0)$, or goes from $(0, 2k)$ to $(0, 2k)$ or $(0, 2(k+1))$, with (1) for the second lattice point, if it is $(1,1)$, color it black or white, otherwise, color it black, (2) for the other lattice point (a, b) with $(a'', b'') \to (a', b') \to (a, b)$, color it by 2^i colors, where $i = \chi(a = 1, a' = 0 \textit{ or } a'' = a = a' + 1) + \chi(b = 1, b' = 0 \textit{ or } b'' = b = b' + 1)$ and χ is the characteristic function of a and b.

$$
\begin{array}{c}
(0,0) \to (0,0) \to (0,0) \to (0,0) \\
(0,0) \to (0,0) \to (1,1) \to (0,0) \\
(0,0) \to (0,0) \to (\mathbf{1},1) \to (0,0) \\
(0,0) \to (0,0) \to (1,\mathbf{1}) \to (0,0) \\
(0,0) \to (0,0) \to (\mathbf{1{,}1}) \to (0,0) \\
(0,0) \to (1,1) \to (0,0) \to (0,0) \\
(0,0) \to (\mathbf{1{,}1}) \to (0,0) \to (0,0) \\
(0,0) \to (2,0) \to (1,1) \to (0,0) \\
(0,0) \to (2,0) \to (1,\mathbf{1}) \to (0,0) \\
(0,0) \to (0,2) \to (1,1) \to (0,0) \\
(0,0) \to (0,2) \to (\mathbf{1},1) \to (0,0)
\end{array}
$$

Hint. Elizalde gives a bijection between (h^4) and (h) without color in Proposition 3.5.3(1) of [14], from which we can get the valleys in (h) corresponds to the points with condition (1) or (2) in the Lattice path of (h^4). Therefore, (h^4) holds from (h2).

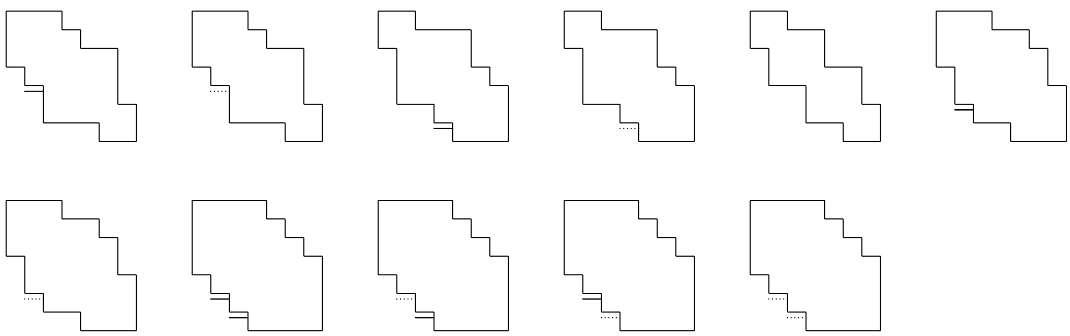
(i^4) **(how to color is unsolved)**

Lattice paths from $(0,0)$ to $(n,-n)$ such that (α) from a point (x,y) with $x < 2y$ the allowed steps are $(1,0)$ and $(0,1)$, (β) from a point (x,y) with $x > 2y$ the allowed steps are $(0,-1)$ and $(1,-1)$, (γ) from a point $(2y,y)$ the allowed steps are $(0,1),(0,-1)$, and $(1,-1)$, and (δ) it is forbidden to enter a point $(2y+1,y)$.



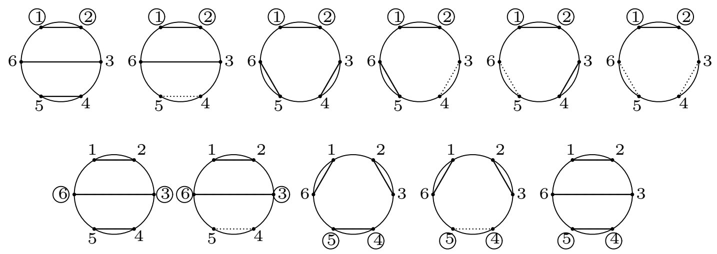
Hint. This i^4 corresponds to m^4 in Stanley's homepage.

(j^4) Symmetric parallelogram polyominos of perimeter $4(2n+1)$ such that the horizontal (equivalently, vertical) boundary steps on each level form an unbroken line. Then each horizontal step of length one except the first one in the bottom polyomino can be colored black or white, and the first one is colored black.



Hint. See [33, solution for (n^4)] for a bijection between such polyominos and Dyck paths. Here, let us linearly order the maximal horizontal line segments on the boundary of given polyomino according to the level of their rightmost step. Replace such a line segment appearing on the left-hand (resp. right-hand) path of the boundary of the polyomino by an up (resp. a down) step, while omit the final line segment which is always on the left. Then we can see each horizontal step of length creates a double up steps. So we get a bijection with (h3).

(k^4) All the nonintersecting chord diagrams of (n) containing at least one horizontal chord and only one of them are distinguished such that (1) the labels of vertices of the chords are the same as (n); (2) if $(n+1,n+2)$ is not the distinguished chord, then each chord $(i,i+1)$ except the first one can be colored black or white; (3) if $(n+1,n+2)$ is the distinguished chord and $(2n,1)$ is not a chord, then each $(i,i+1)$ except the first one and $(n+1,n+2)$ can be colored black or white; (4) if $(n+1,n+2)$ is the distinguished chord and $(2n,1)$ is also a chord, then each chord $(i,i+1)$ except the first one can be colored black or white; (5) all the other chords colored white(we use the circled labels of two endpoints to represent the distinguished horizontal chord).



Hint. There is a simple bijection between (k^4) and (n) without color, i.e., for any given nonintersecting chord diagram with a distinguished horizontal chord K, rotate the chords so that the left-hand endpoint of K is 1 (see [33, solution for (o^4)]). Then from the bijection, the number of chords $(i, i+1)$ in (k^4) is the same as that in (n) for cases (2) and (4) while increases by 1 in case (3). Therefore, (k^4) holds from (n).

(l^4) **(open)** Kepler towers with n bricks, i.e., sets of concentric circles, with "bricks" (arcs) placed on each circle, as follows: the circles come in sets called *walls* from the center outwards. The circles (or *rings*) of the ith wall are divided into 2^i equal arcs, numbered $1, 2, \cdots, 2^i$ clockwise from due north. Each brick covers an arc and extends slightly beyond the endpoints of the arc. No two consecutive arcs can be covered by bricks. The first (innermost) arc within each wall has bricks at positions $1, 3, 5, \cdots, 2^i - 1$. Within each wall, each brick B not on the innermost ring must be supported by another brick B' on the next ring toward the center, i.e., some ray from the center must intersect both B and B'. Finally, if $i > 1$ and the ith wall is nonempty, then wall $i-1$ must also be nonempty. Such Kepler towers are enumerated by Catalan number. The question is what is the corresponding statistic from Viennot and Knuth's bijection [23] to give it a bi-color and get the structures for super-Catalan number, and whether the statistic that bricks in the left half side is right. If the answer is "yes" for the second question, can you give a bijective proof?

(m^4) Compositions of n whose parts equal to k are colored with one of S_k colors (colors are indicated by subscripts below), where S_k is the k-th large Schöder number.

$$1_a + 1_a + 1_a \quad 1_a + 2_a \quad 1_a + 2_b \quad 2_a + 1_a \quad 2_b + 1_a \quad 3_a \quad 3_b \quad 3_c \quad 3_d \quad 3_e \quad 3_f$$

Hint. In [18], the authors gave a bijection with small-binary trees to prove this result.

(n^4) Sequences $(a_1, \cdots, a_n)$ of nonnegative integers satisfying $a_1 + \cdots + a_i \geq i$ and $\Sigma a_j = n$ with each 0 can be colored black or white.

111 120 12**0** 210 21**0** 201 2**0**1 300 3**0**0 30**0** **300**

Hint. Add 1 to the terms of the sequences of (w), we get the desired result. This is the solution given by Stanley in [33], where he also gave inverse of the bijection.

(o^4) Sequences $a_1, \cdots, a_{2n}$ of nonnegative integers with $a_1 = 1$, $a_{2n} = 0$ and $a_i - a_{i-1} = \pm 1$ with each $a_i = a_{i-1} + 1 = a_{i-2} + 2$, $2 \le i \le 2n$, (assuming $a_0 = 0$,) colored by black or white.

$$123210\quad 1\mathbf{2}3210\quad 12\mathbf{3}210\quad 1\mathbf{23}210\quad 121210\quad 1\mathbf{2}1210\quad 121010\quad 1\mathbf{2}1010\quad 101210\quad 101\mathbf{2}10\quad 101010$$

Hint. Partial sums of the sequences in (r). We consider the sequences in (r) as Dyck path, then color each double ascents blakc or white.

(p^4) Sequences of $n-1$ $1's$ and any number of $-1's$ such that every partial sum is nonnegative with each 1 followed by a -1 can be colored black or white.

$$1,1\quad 1,1,-1\quad 1,\mathbf{1},-1\quad 1,-1,1\quad \mathbf{1},-1,1\quad 1,1,-1,-1$$
$$1,\mathbf{1},-1,-1\quad 1,-1,1,-1\quad \mathbf{1},-1,1,-1\quad 1,-1,\mathbf{1},-1\quad \mathbf{1},-1,\mathbf{1},-1$$

Hint. In (rrr) replace an up step with 1 and a down step with -1.

(q^4) Sequences $a_1 a_2 \cdots a_{2n-2}$ of $n-1$ $1's$ and $n-1$ $-1's$ such that if $a_i = -1$ then either $a_{i+1} = a_{i+2} = \cdots = a_{2n-2} = -1$ or $a_{i+1} + a_{i+2} + \cdots + a_{i+j} > 0$ for some $j \ge 1$ with each 1 followed by a -1 can be colored black or white.

$$1,1,-1,-1\quad 1,\mathbf{1},-1,-1\quad 1,-1,1,-1\quad \mathbf{1},-1,1,-1\quad 1,-1,\mathbf{1},-1\quad \mathbf{1},-1,\mathbf{1},-1$$
$$-1,1,1,-1\quad -1,1,\mathbf{1},-1\quad -1,1,-1,1\quad -1,\mathbf{1},-1,1\quad -1,-1,1,1$$

Hint. See the bijection given in [33].

(r^4) Sequences $a_1 a_2 \cdots a_n$ of integers such that (1) $a_1 = 1, a_n = \pm 1, a_i \neq 0$ for $1 \le i \le n$, (2) $a_{i+1} \in \{a_i, a_i + 1, a_i - 1, -a_i\}$, and each a_{i+1} with $a_{i+1} = a_i$ or $|a_{i+1}| = |a_i| + 1$ can be colored black and white for $1 \le i \le n-1$ while other elements colored white.

$$1,1,1\quad 1,\mathbf{1},1\quad 1,1,\mathbf{1}\quad 1,\mathbf{1},\mathbf{1}\quad 1,1,-1\quad 1,\mathbf{1},-1$$
$$1,-1,1\quad 1,-1,-1\quad 1,-1,\mathbf{-1}\quad 1,2,1\quad 1,\mathbf{2},1$$

Hint. There is a bijection between (l) and (r^4)(see [33, solution for (w^4)]) without coloring. From the bijection, we can get each new row corresponds to b_i with $b_{i+1} = b_i + 1$, or $b_{i+1} = b_i$ such that the top lattice square in b_{i+1} squares above the top lattice square in b_i squares, which gives a bijection between (l) and (r^4).

(s^4) Sequences $a_1 a_2 \cdots a_n$ of nonnegative integers such that $a_j = \#\{i : i < j, a_i < a_j\}$ for $1 \leqslant j \leqslant n$, and each a_i, $1 \le i \le n-1$, with $a_i \ge a_{i+1}$ can be colored black or white.

$$000\quad \mathbf{0}00\quad 0\mathbf{0}0\quad \mathbf{00}0\quad 002\quad \mathbf{0}02\quad 010\quad 0\mathbf{1}0\quad 011\quad 0\mathbf{1}1\quad 012$$

Hint. For a 312-avoiding permutation $a_1 a_2 \cdots a_n$ in (ff), we can get a sequence $b_1 b_2 \cdots b_n$ in (s^4) by considering $b_j = \#\{i : i < j, a_i < a_j\}$. From this bijection, we can see each a_i with $a_i > a_{i+1}$ in 312-avoiding permutation corresponds to a b_i with $b_i \ge b_{i+1}$ in the sequence of (s^4).

(t^4) Sequences $a_1a_2\cdots a_{n-1}$ of nonnegative integers such that each nonzero term initiates a factor (subsequence of consecutive elements) whose length is equal to its sum, and each 0 can be colored black or white.

00 **0**0 0**0** **00** 01 **0**1 10 1**0** 11 20 2**0**

Hint. Given a plane tree with n edges, traverse the edges in preorder and record for each edge except the first pendant one the degree of the vertex terminating the edge. This is a similar bijection with (o') as that in [33]. The color of each letter is the same.

(u^4) For sequences $a_1a_2\cdots a_{2n+1}$ of positive integers such that $a_{2n+1}=1$, some $a_i=n+1$, the first appearance of $i+1$ follows the first appearance of i, no two consecutive terms are equal, no pair ij of integers occur more than once as a factor (i.e., as two consecutive terms), and if ij is a factor then so ij, color each a_i such that $a_i>a_{i-1}, a_i>a_{i+1}$ black or white except the first one, and all other letters are colored white.

1213141 121**3**141 12131**4**1 121**3**1**4**1 1213431 1213**4**31
1232141 12321**4**1 1232421 1232**4**21 1234321

Hint. Traverse a plane tree with $n+1$ vertices in preorder. Do a depth first search through the tree and write down the vertices in the order they are visited (including repetitions). This establishes a bijection with (o').

(v^4) Sequences $(a_1,\ldots,a_n)\in\mathbb{N}^n$ for which there exists a distributive lattice of rank n with a_i join-irreducible of rank i, $1\le i\le n$, where colored black to white if $i>1$.

300 210 2**1**0 120 1**2**0 201 20**1** 111 11**1** 1**1**1 1**11**

Hint. The sequences $1, 1+a_n, 1+a_n+a_{n-1}, \cdots, 1+a_n+a_{n-1}+\cdots+a_2$ coincide with those of (s) if we do not consider the colors. Then color the corresponding letter or part black or white as that of (s).

(w^4) Pairs of sequences $1\leqslant i_1<i_2<\cdots<i_k\leqslant n-1$ and $2\leqslant j_1<\cdots<j_k\leqslant n$ such that $i_r<j_r$ for all r, and each element of the 2ed sequence (not the $\emptyset$) can be colored black or white.

$(\emptyset,\emptyset)$ $(1,2)$ $(1,\mathbf{2})$ $(1,3)$ $(1,\mathbf{3})$ $(2,3)$ $(2,\mathbf{3})$ $(12,23)$ $(12,\mathbf{2}3)$ $(12,2\mathbf{3})$ $(12,\mathbf{23})$

Hint. As Stanley mentioned in [33], there is a bijection with 321-avoiding permutations, we color the corresponding second sequence of this item black or white as that in (ee).

(x^4) For the ways two persons can each start with 0 and alternating add positive integers to their numbers so that they first have equal numbers when that number is n (notation such as $1,2;4,3;5,5$ means that the first person adds 1 to 0 to obtain 1, then the second person adds 2 to 0 to obtain 2, then the first person adds 3 to 1 to obtain 4, etc.), color each segment (a_i,a_{i+1}) black or white with $a_{i+1}>a_i$ after setting the sequence $a_1,a_2;a_3,a_4;a_5\cdots$ and other segments only white.

$3,3$ $2,3;3$ $\mathbf{2},\mathbf{3};3$ $2,1;3,3$ $2,\mathbf{1};\mathbf{3},3$ $1,2;3,3$
$\mathbf{1},\mathbf{2};3,3$ $1,\mathbf{2};\mathbf{3},3$ $\mathbf{1},\mathbf{2};\mathbf{3},3$ $1,3;3$ $\mathbf{1},\mathbf{3};3$

Hint. See [33, solution for (x^4)] for a bijection between (c^5) and lattice paths as in (i), from which the segments (a_i, a_{i+1}) with $a_{i+1} > a_i$ and valleys in lattice paths of (i) have the same distribution.

(y^4) The representative sequences of cyclic equivalence classes (or necklaces) of sequences of $n+1$ 1's and n 0's in which any first factor has more 1 than 0. Then each two consecutive 00 can be colored black or white.

1111000 1111**00**0 11110**00** 1111**000** 1110100 11101**00**
1110010 111**00**10 1101100 11011**00** 1101010

Hint. The sequence must initiate with 1 since any first factor has more 1 than 0. By omitting the first 1, draw an up step for each 1 and draw a down step for each 0, then we can get a Dyck path, and each 00 corresponds to double down steps, which is a bijection with (h4). In fact, we can also color 11 or 01 except the first one or just 01, which correspond to (h3), (h2) and (h1) respectively, to get the structures enumerated by super-Catalan numbers.

(z^4) **(how to color is unsolved)**

Integer partitions which are both n-cores and $(n+1)$-cores, in the terminology of Exercise 7.59(d).

$\emptyset$ 1 2 11 311

Hint. This z^4 corresponds to the solution of (e^5) in Stanley's homepage.

(a^5) Equivalence classes of the equivalence relation on the set $S_n = \{a_1, \cdots, a_n) \in \mathbb{N}^n : \sum a_i = n\}$ generated by $(\alpha, 0, \beta) \sim (\beta, 0, \alpha)$ if β (which may be empty) contains no 0's. Then each sequence of the class except the first can be colored black or white, and the first is colored black.

$\{300, 030, 003\}$ $\{300, \mathbf{030}, 003\}$ $\{300, 030, \mathbf{003}\}$ $\{300, \mathbf{030}, \mathbf{003}\}$
$\{210, 021\}$ $\{210, \mathbf{021}\}$ $\{120, 012\}$ $\{120, \mathbf{012}\}$ $\{201, 102\}$ $\{201, \mathbf{102}\}$ $\{111\}$

Hint. Each equivalence class contains a unique element $(a_1, \cdots, a_n)$ satisfying $a_1 + a_2 + \cdots a_i \geq i$ for $1 \leq i \leq n$(see [33, Exercise 6.19(f^5)]), which gives a proof followed from (n^4) since the number of sequences in an equivalent class is $1 + \# \ of \ zeros$ in any representative element.

(b^5) Pairs (α, β) of compositions of n with the same number of parts, such that $\alpha \geqslant \beta$ (dominance order, i.e., $\alpha_1 + \cdots + \alpha_i \geqslant \beta_1 + \cdots + \beta_i$ for all i), and each two consecutive elements of the 2ed composition can be colored black or white.

$(111, 111)$ $(111, \mathbf{11}1)$ $(111, 1\mathbf{11})$ $(111, \mathbf{111})$ $(12, 12)$ $(12, \mathbf{12})$
$(21, 21)$ $(21, \mathbf{21})$ $(21, 12)$ $(21, \mathbf{12})$ $(3, 3)$

Hint. Define a Dyck path by going up α_1 steps then down β_1 steps, then up α_2 steps, then down β_2 steps, etc. This gives a bijection with (i) if we do not consider the colors of letters. The number of peaks minus 1 equals the number of consecutive elements of composition β. By (i1), we know such pairs of compositions are counted by super-Catalan numbers.

(c[5]) **(open)**

Weak ordered partitions (P, V, A, D) of $[n]$ into four blocks such that there exists a permutation $w = a_1a_2\cdots a_n \in \mathfrak{S}_n$ (with $a_0 = a_{n+1} = 0$) satisfying

$$\begin{array}{ll} P = \{a_i \in [n] : a_{i-1} < a_i > a_{i+1}\}, & V = \{a_i \in [n] : a_{i-1} > a_i < a_{i+1}\} \\ A = \{a_i \in [n] : a_{i-1} < a_i < a_{i+1}\}, & D = \{a_i \in [n] : a_{i-1} > a_i > a_{i+1}\}. \end{array}$$

Then each $x \in P \cup A$ not the first one can be colored black or white.

$$(3, \emptyset, 12, \emptyset) \quad (3, \emptyset, \mathbf{1}2, \emptyset) \quad (3, \emptyset, 1\mathbf{2}, \emptyset) \quad (3, \emptyset, \mathbf{12}, \emptyset) \quad (3, \emptyset, 1, 2) \quad (3, \emptyset, \mathbf{1}, 2)$$
$$(23, 1, \emptyset, \emptyset) \quad (2\mathbf{3}, 1, \emptyset, \emptyset) \quad (3, \emptyset, 2, 1) \quad (3, \emptyset, \mathbf{2}, 1) \quad (3, \emptyset, \emptyset, 12)$$

(d[5]) **(open)** Permutations $\omega \in \mathfrak{S}_n$ satisfying the following conditions: let $\omega = R_{s+1}R_s\cdots R_1$ be the factorization of ω into maximal ascending runs (so $s = des(\omega)$, the number of descents of ω). Let m_k and M_k be the smallest and largest elements in the run R_k. Let n_k be the number of symbols in R_k for $1 \le k \le s+1$; otherwise set $n_k = 0$. Define $N_k = \sum_{i<k} n_i$ for all $k \in \mathbb{Z}$. Then $m_{s+1} > m_s > \cdots > m_1$ and $M_i \le N_{i+1}$ for $1 \le i \le s+1$. For example, when $n = 3$, the permutations are

$$123 \quad 213 \quad 231 \quad 312 \quad 321.$$

Such permutations are enumerated by Catalan number, given in [33, Exercies 6.19]. It is a challenge to find the corresponding statistic to give a color and get the structure counted by super-Catalan number.

(e[5]) **(open)** Permutations $\omega \in \mathfrak{S}_n$ satisfying, in the notation of (d[5]) above, $m_{s+1} > m_s > \cdots > m_1$ and $m_{i+1} > N_{i-1} + 1$ for $1 \le i \le s$, such as

$$123 \quad 213 \quad 231 \quad 312 \quad 321$$

for n=3, are enumerated by Catalan number given in [33, Exercise 6.19]. Then it is a challenge to find the corresponding statistic to give a color and obtain the structure counted by super-Catalan number.

(f[5]) 321-avoiding permutations $w \in \mathfrak{S}_{2n+1}$ such that i is an excedance of w (i.e., $w(i) > i$) if and only if $i \neq 2n+1$ and $w(i) - 1$ is not an excedance of w (so that w has exactly n excedances), and each two consecutive execedances can be colored black or white.

$$4512736 \quad \mathbf{45}12736 \quad 3167245 \quad 31\mathbf{67}245 \quad 3152746$$
$$4617235 \quad \mathbf{46}17235 \quad 5671234 \quad \mathbf{56}71234 \quad 5\mathbf{67}1234 \quad \mathbf{567}1234$$

Hint. Replace an excedance of w with an up step and a nonexcedance with a down step, except for the nonexcedance $2n + 1$ at the end of w. This sets up a bijection with (i3), whose double ascents are colored black or white.

(g^5) For the 321-avoiding alternating permutations $\pi_1\pi_2\cdots\pi_{2n}$ in $\mathfrak{S}_{2n}$, color each π_{2i-1}, $1 \leq i \leq n-1$, satisfying $\pi_{2i+1} - \pi_{2i-1} > 1$ black or white.

214365 **2**14365 21**4**365 **2**1**4**365 215364 **2**15364
314265 31**4**265 315264 **3**15264 415263

Hint. There is a bijection with (ww) without color(see [33, Exercise 6.19(l^5)]). Let $a_1a_2\cdots a_{2n}$ be a permutation being counted, and we can get a standard Young tableaux by associating it with the array

$$\begin{array}{ccccc} \pi_2 & \pi_4 & \pi_6 & \cdots & \pi_{2n} \\ \pi_1 & \pi_3 & \pi_5 & \cdots & \pi_{2n-1} \end{array}$$

And then draw an up step for the i-th step if i is in the first a row while draw a down step if i is in the second row, from which we can see each π_{2i-1} with $\pi_{2i+1} - \pi_{2i-1} > 1$ corresponds to a valley in the Dyck path. Therefore, we get a bijection between (g^5) and (h2).

(g^{5}') For 321-avoiding alternating permutation $\pi_1\pi_2\cdots\pi_{2n-1}$ in $\mathfrak{S}_{2n-1}$, color each π_{2i-1}, $1 \leq i \leq n-2$, satisfying $\pi_{2i+1} - \pi_{2i-1} > 1$ black or white, and also π_{2n-3} iff $\pi_{2n-1} - \pi_{2n-3} = 1$.

21435 **2**1435 21**4**35 **2**1**4**35 21534 **2**1534 31425 31**4**25 31524 **3**1524 41523

Hint. Associate the standard Young tableaux with the array

$$\begin{array}{cccccc} \pi_2 & \pi_4 & \pi_6 & \cdots & \pi_{2n-2} & \pi_{2n-1} \\ \pi_1 & \pi_3 & \pi_5 & \cdots & \pi_{2n-3} & 2n \end{array}$$

Then by the same rule as (g^5), we can get a bijection with (h2).

(h^5) 321-avoiding fixed-point-free involutions $\pi_1\pi_2\cdots\pi_{2n}$ of $[2n]$ with each π_i, $1 \leq i \leq 2n-1$, satisfying $\pi_i + 1$ on its right can be colored black or white except the first one.

214365 21**4**365 2143**6**5 21**4**3**6**5 215634 215**6**34341265 3412**6**5 351624 351**6**24 456123

Hint. By the RSK algorithm, we can get a bijection between this item and standard Young tableaux of (ww) without color(see [33, Exercise 6.19(m^5)]). Then for each Young tableaux obtained, draw the i-th steps an (1,1) if i is in the first row while draw a (1,-1) if i is in the second row, from which each π_i with $\pi_i + 1$ on its right in the involution corresponds to a peak in Dyck path, i.e., a bijection between (h^5) and (h1).

(i^5) 321-avoiding involutions of $[2n-1]$ with one fixed point, and then find subsequence $1 = a_1 < a_2 < \cdots < a_n$ in $\pi \in \mathfrak{S}_{2n-1}$ and color each form of a_ia_{i+1} such that $a_{i+1} = a_i + 1$ black or white.

13254 **132**54 14523 **1**45**2**3 145**23** **1**45**23**21354 21**3**5**4** 21435 34125 34**12**5

Hint. The solution of (n^5) in [33] gave a bijection with standard Young tableaux of shape $(n, n-1)$ of (ww). Then as in (h^5), we can also get a bijection with (h3) such that each a_ia_{i+1} such that $a_{i+1} = a_i + 1$ corresponds to a double up steps.

(j[5]) For 213-avoiding fixed-point-free involutions $\pi_1\pi_2\cdots\pi_{2n}$ of $[2n]$, color each π_i, $1 \le i \le n-1$, with $\pi_i < \pi_{i+1}$ black or white.

456123 **4**56123 4**5**6123 **45**6123 465132 **4**65132564312 **5**64312 645231 6**4**5231 654321

Hint. The solution to (o[5]) of [33] gave a bijection with Dyck path of (i).

(k[5]) For 213-avoiding involutions $\pi_1\pi_2\cdots\pi_{2n-1}$ of $[2n-1]$ with one fixed point, color each π_i, $1 \le i \le n$, with $\pi_i < \pi_{i+1}$ black or white.

14523 **1**4523 1**4**523 **13**523 15432 **1**5432 45312 **4**5312 52431 5**2**431 54321

Hint. Similar to (j[5]).

(l[5]) 3412-avoiding (or noncrossing) involutions of a subset of $[n-1]$, with each left to right maximal element colored black or white.

$\emptyset$ 1 **1** 2 **2** 12 **1**2 1**2** **12** 21 **2**1

Hint. Obvious bijection with (aa).

(m[5]) For the standard Young tableaux with at most two rows and with first row of length $n-1$, color each x in the second row black or white if $x-1$ is in the first row.

$$\begin{array}{ll}1&2\\&\end{array}\quad\begin{array}{ll}1&2\\3&\end{array}\quad\begin{array}{ll}1&2\\\mathbf{3}&\end{array}\quad\begin{array}{ll}1&3\\2&\end{array}\quad\begin{array}{ll}1&3\\\mathbf{2}&\end{array}\quad\begin{array}{ll}1&2\\3&4\end{array}$$

$$\begin{array}{ll}1&2\\\mathbf{3}&4\end{array}\quad\begin{array}{ll}1&3\\2&4\end{array}\quad\begin{array}{ll}1&3\\\mathbf{2}&4\end{array}\quad\begin{array}{ll}1&3\\2&\mathbf{4}\end{array}\quad\begin{array}{ll}1&3\\\mathbf{2}&\mathbf{4}\end{array}$$

Hint. Given a standard Young tableau T of the type being counted, construct a Dyck path of length $2n$ as follows. For each entry $1, 2, \cdots, m$ of T, if i appears in row 1 then draw an up step, while if i appears in row 2 then draw a down step. Afterwards draw an up step followed by down steps to the x-axis. This bijection is given in [33], where we color the letter corresponding to the peaks except the first one black or white, as the same as that (i1).

(n[5]) For the standard Young tableaux with at most two rows and with first row of length n, such that for all i the ith entry of of row 2 is not $2i$, and color each x in the second row black or white if $x-1$ is in the first row.

$$\begin{array}{lll}1&2&3\\&&\end{array}\quad\begin{array}{lll}1&2&3\\4&&\end{array}\quad\begin{array}{lll}1&2&3\\\mathbf{4}&&\end{array}\quad\begin{array}{lll}1&2&4\\3&&\end{array}\quad\begin{array}{lll}1&2&4\\\mathbf{3}&&\end{array}\quad\begin{array}{lll}1&2&3\\4&5&\end{array}$$

$$\begin{array}{lll}1&2&3\\\mathbf{4}&5&\end{array}\quad\begin{array}{lll}1&2&4\\3&5&\end{array}\quad\begin{array}{lll}1&2&4\\\mathbf{3}&5&\end{array}\quad\begin{array}{lll}1&2&4\\3&\mathbf{5}&\end{array}\quad\begin{array}{lll}1&2&4\\\mathbf{3}&\mathbf{5}&\end{array}$$

Hint. The bijection of (m[5]) yields a Dyck path of length $2n+2$ which never touches the x-axis except first and last steps. Remove them get a bijection with (i1). This was given in [33].

(o[5]) For the standard Young tableaux of shape $(2n+1, 2n+1)$ such that adjacent entries have opposite parity, color each x followed by y with $y - x > 1$ in the first row black or white.

$$\begin{bmatrix}1&2&3&4&5&6&7\\8&9&10&11&12&13&14\end{bmatrix}\quad\begin{bmatrix}1&2&3&4&5&8&9\\6&7&10&11&12&13&14\end{bmatrix}$$

$$\begin{bmatrix}1&2&3&4&\mathbf{5}&8&9\\6&7&10&11&12&13&14\end{bmatrix}\quad\begin{bmatrix}1&2&3&4&5&10&11\\6&7&8&9&12&13&14\end{bmatrix}$$

$$\begin{bmatrix}1&2&3&4&\mathbf{5}&10&11\\6&7&8&9&12&13&14\end{bmatrix}\quad\begin{bmatrix}1&2&3&6&7&8&9\\4&5&10&11&12&13&14\end{bmatrix}$$

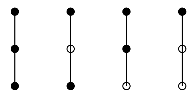
$$\begin{bmatrix}1&2&\mathbf{3}&6&7&8&9\\4&5&10&11&12&13&14\end{bmatrix}\quad\begin{bmatrix}1&2&3&6&7&10&11\\4&5&8&9&12&13&14\end{bmatrix}$$

$$\begin{bmatrix}1&2&\mathbf{3}&6&7&10&11\\4&5&8&9&12&13&14\end{bmatrix}\quad\begin{bmatrix}1&2&3&6&\mathbf{7}&10&11\\4&5&8&9&12&13&14\end{bmatrix}$$

$$\begin{bmatrix}1&2&\mathbf{3}&6&\mathbf{7}&10&11\\4&5&8&9&12&13&14\end{bmatrix}$$

Hint. Remove all entries except $3, 5, 7, \cdots, 4n+1$ and shift all the entries to the left in the same row after replacing $2i+1$ by i. This is a bijection with standard Young tableaux of shape (n,n) in (ww)(see [33, Exercise 6.19(t[5])]). By the definition of the item in (o[5]), we can see each $2i+1$ (except 1) must follow $2i$ immediately. Then for a tableaux in (o[5]), if there is $y - x > 1$ in the first row, there must be x' followed by y' with $y' - x' > 1$ in the first of the corresponding Young tableaux. From the bijection for the solution of (ww), we derived a bijection between (o[5]) and Dyck path, where such x' corresponds to a valley, i.e., a bijection between (o[5]) and (h)(2).

(p[5]) Plane partitions with largest part at most two and contained in a rectangle of perimeter $2(n-1)$ (including degenerate $0 \times (n-1)$ and $(n-1) \times 0$ rectangles). Then color each row by black or white. (the line or point in the top is the 0th row.)

 $\boxed{0}$ $\boxed{\mathbf{0}}$ $\boxed{1}$ $\boxed{\mathbf{1}}$ $\boxed{2}$ $\boxed{\mathbf{2}}$ 

Hint. Given a plane partition π, let L be the lattice path from the lower left to upper right that has only 2's above it and no 2's below. Similarly, let L' be the lattice path from the lower left to upper right that has only 0's below it and no 0's above, then we get a bijection between (p[5]) and (m) without color(see [33, Exercise 6.19(u[5])]), from which each row in (p[5]) corresponds to a (0,1) step on the lattice path below in (m). Since the (1,0) steps and (0,1) steps on the lattice paths below in (m) have the same distribution, we can get the required conclusion.

(q[5]) For triples (A, B, C) of pairwise disjoint subsets of $[n-1]$ such that $\#A = \#B$ and if write the element of A and B in increasing order, every element of A is less than the corresponding element of B, color each element in A and C black or white.

$$(\emptyset,\emptyset,\emptyset)\quad(\emptyset,\emptyset,1)\quad(\emptyset,\emptyset,\mathbf{1})\quad(\emptyset,\emptyset,2)\quad(\emptyset,\emptyset,\mathbf{2})\quad(\emptyset,\emptyset,12)$$
$$(\emptyset,\emptyset,\mathbf{1}2)\quad(\emptyset,\emptyset,1\mathbf{2})\quad(\emptyset,\emptyset,\mathbf{12})\quad(1,2,\emptyset)\quad(\mathbf{1},2,\emptyset)$$

Hint. In the $(2, 3)$-Motzkin paths of (yyy), number the steps $1, 2, \cdots, n-1$ from left to right. Place the up red or blue steps $(1, 1)$ in A with white or black letter respectively, the steps $(1, -1)$ in B, and the red or blue steps $(1, 0)$ in C with white or black letter respectively.

(r^5) For subsets $S_n = \mathbb{N} - T_n$ of $\mathbb{N}$ such that 0 in $\mathbb{N}$ and such that if i in S_n then $i+n, i+n+1$ in S_n, each number a in T_n can be colored black and white if and only if there no numbers b, c in T_n with b greater than a and $a = b+n+1+c+n$.

$$\mathbb{N} \quad \mathbb{N}-\{1\} \quad \mathbb{N}-\{\mathbf{1}\} \quad \mathbb{N}-\{2\} \quad \mathbb{N}-\{\mathbf{2}\}$$
$$\mathbb{N}-\{12\} \quad \mathbb{N}-\{\mathbf{1}2\} \quad \mathbb{N}-\{1\mathbf{2}\} \quad \mathbb{N}-\{\mathbf{12}\} \quad \mathbb{N}-\{125\} \quad \mathbb{N}-\{12\mathbf{5}\}$$

Hint. The solution to (w^5) in [33] also gives a bijection between present item and (bbb).

(s^5) $(n+1)$-element multisets on $\mathbb{Z}/n\mathbb{Z}$ whose elements sum to 0 and each string of the form xx except the first one can be colored black and the first one is colored white.

$$0000 \quad 0\mathbf{00}0 \quad 00\mathbf{00} \quad 0\mathbf{000} \quad 0012 \quad 0111 \quad 01\mathbf{11} \quad 0222 \quad 02\mathbf{22} \quad 1122 \quad 11\mathbf{22}$$

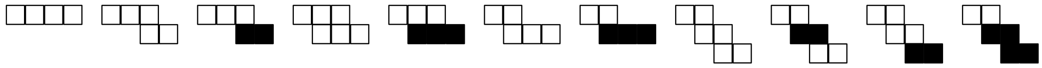
Hint. Similarly as that of (jjj).

(t^5) Ways to write $(1, 1, \cdots, 1, -n) \in \mathbb{Z}^{n+1}$ as a sum of vectors $e_i - e_{i+1}$ and $e_j - e_{n+1}$, without regard to order, where e_k is the kth unit coordinate vector in $\mathbb{Z}^{n+1}$, and if set $a_0 = 0$ and a_i is the coefficient of $e_i - e_{i+1}$, $1 \le i \le n-1$, each a_i with $a_i > a_{i-1}$ can be colored black or white while other coefficients colored white.

$$\begin{array}{ll}
1(1,-1,0,0)+2(0,1,-1,0)+3(0,0,1,-1) & 1(1,-1,0,0)+\mathbf{2}(0,1,-1,0)+3(0,0,1,-1) \\
\mathbf{1}(1,-1,0,0)+2(0,1,-1,0)+3(0,0,1,-1) & \mathbf{1}(1,-1,0,0)+\mathbf{2}(0,1,-1,0)+3(0,0,1,-1) \\
1(1,0,0,-1)+1(0,1,-1,0)+2(0,0,1,-1) & 1(1,0,0,-1)+\mathbf{1}(0,1,-1,0)+2(0,0,1,-1) \\
1(1,-1,0,0)+2(0,1,0,-1)+1(0,0,1,-1) & \mathbf{1}(1,-1,0,0)+2(0,1,0,-1)+1(0,0,1,-1)
\end{array}$$
$$1(1,-1,0,0)+1(0,1,-1,0)+1(0,1,0,-1)+2(0,0,1,-1)$$
$$\mathbf{1}(1,-1,0,0)+1(0,1,-1,0)+1(0,1,0,-1)+2(0,0,1,-1)$$
$$1(1,0,0,-1)+1(0,1,0,-1)+1(0,0,1,-1)$$

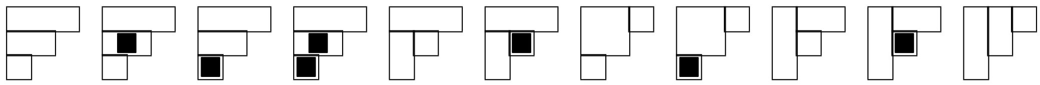
Hint. This is a direct result from the bijection with (u) given by A. Postnikov and R. Stanley(see [33, Exercise 6.19(y^5)]).

(u^5) Horizontally convex polyominoes of width $n+1$ such that each (1) row begins strictly to the right of the beginning of the previous row and ends strictly to the right of the end of the previous row, and (2) each row except the first row are colored black or white and the first row is colored white.



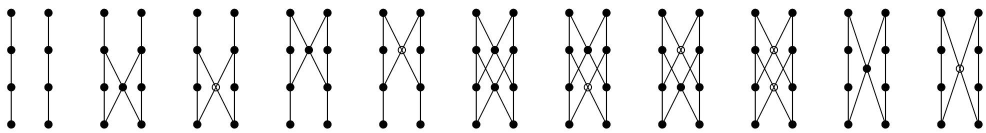
Hint. The bijection in [33, solution for (z^5)] maps each peak except the first one in Dyck paths to a row in the polyomino. Setting the corresponding peak and row the same color sets up a bijection between (u^5) and Dyck paths of length $2n$ with colored peaks.

(v[5]) Tilings of the staircase shape $(n, n-1, \cdots, 1)$ with n rectangles. Then each rectangle of shape $(1, n)$ except the first one can be colored black or white, and the first one is colored white.



Hint. There is a simple bijection with binary trees with root colored white, and no left vertex colored white, similar as (b), which are also counted by super-Catalan numbers. The root of T corresponds to the rectangle containing the upper right-hand corner of the staircase. Remove this rectangle we get two smaller staircase tilings on its left and bottom, which are corresponding to the left subtree and right one of the root, making the bijection obviously.

(w[5]) Nonisomorphic $2(n+1)$-element posets that are a union of two chains, that are not a (nontrivial) ordinal sum, and that have a nontrivial automorphism. Then each crossing is colored black or white.



Hint. For a crossing, define its crossing length to be the number of edges between the two end-vertices of the crossing on the right line. Then for each vertex of the right line from top to bottom, if it is an end vertex of a crossing and below the crossing, record it by the crossing length minus 1, otherwise record a 0 and delete the first two element of the final sequence(the first two element must be 00). For example, in the sixth figure above, the corresponding sequence is 11. Then we can get a bijection with sequences $b_1 b_2 \cdots b_{n-2}$ such that $b_1 + b_2 + \cdots + b_i \leq i$. Besides, a sequence $c_0 c_2 \cdots c_{n-1}$ is obtained by defining $c_0 = 0$ and $c_i = \sum_{j=1}^{i} b_j + i$ for $1 \leq i \leq n-1$. Then obviously, $0 = c_0 < c_1 < c_2 < \cdots < c_{n-1}$ satisfies $1 \leq c_i \leq 2i$ (except $c_0 = 0$) and each a_i with $a_i > a_{i-1} + 1$ corresponds to a crossing in (w[5]), and such sequences are just the sequences in (t).

(x[5]) $n \times n$ $\mathbb{N}$-matrices $M = (m_{ij})$ where $m_{ij} = 0$ unless $i = n$ or $i = j$ or $i = j - 1$, with row and column sum vector $(1, 2, \cdots, n)$, and if set $m_{00} = 0$, each m_{ii}, $1 \leq i \leq n-1$, with $m_{ii} > m_{i-1,i-1}$ can be colored black or white while other entries in the matrix colored white.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \mathbf{2} & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{2} & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & \mathbf{1} & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Hint. Let

$$(1,1,\cdots,1,-n)=\sum_{i=1}^{n-1}(a_i(e_i-e_{i+1})+b_i(e_i-e_{n+1})+a_n(e_n-e_{n+1}))$$

in (t^5). Then set $m_{ii}=a_i$ and $m_{ni}=b_i$, which uniquely determines the matrix M and sets up a bijection with (t^5)(see [33, Exercise 6.19(c^6)]).

(y^5) Bounded regions into which the cone $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_{n+1}$ in $\mathbb{R}^{n+1}$ is divided by the hyperplanes $x_i - x_j = 1$, $1 \leqslant i < j \leqslant n+1$ (compare (lll), which illustrates the case $n=2$ of the present item), see that of (lll) for coloring.

Hint. See (lll) and [1].

(z^5) **(open)** The extreme rays of closed convex cone generated by a all flag f-vectors(i.e., the functions $\beta(P,S)$ of Section 3.12) of graded posets of rank n with $\hat{0}$ and $\hat{1}$ (the vectors below lie on the extreme rays, with the coordinates $\emptyset,\{1\},\{2\},\{1,2\}$ in that order), such as

$$(0,0,0,1)\quad(0,0,1,1)\quad(0,1,0,0)\quad(0,1,1,1)\quad(1,1,1,1),$$

for $n=3$, are enumerated by Catalan number(see [33, Exercise 6.19(e^6)]), then it is a challenge to find out the corresponding statistic and get the structures counted by super-Catalan numbers by coloring them.

(a^6) Bootstrap percolations of $(n+1)\times(n+1)$ permutation matrices such that we can get matrix with all elements of 1 by adding a 1 to the empty positions adjacent to at least two 1s recursively, and the 1 in the first column is in a higher position than the 1 in the second column, where a *permutation matrix* is a matrix such that each row and column has exactly one element of 1, and the others are all empty. (see [17, 30]).

$$
\begin{array}{|c|c|c|c|}\hline 1&&&\\\hline &1&&\\\hline &&1&\\\hline &&&1\\\hline\end{array}\quad
\begin{array}{|c|c|c|c|}\hline 1&&&\\\hline &1&&\\\hline &&&1\\\hline &&1&\\\hline\end{array}\quad
\begin{array}{|c|c|c|c|}\hline 1&&&\\\hline &&1&\\\hline &1&&\\\hline &&&1\\\hline\end{array}\quad
\begin{array}{|c|c|c|c|}\hline 1&&&\\\hline &&&1\\\hline &1&&\\\hline &&1&\\\hline\end{array}\quad
\begin{array}{|c|c|c|c|}\hline 1&&&\\\hline &&1&\\\hline &&&1\\\hline &1&&\\\hline\end{array}\quad
\begin{array}{|c|c|c|c|}\hline 1&&&\\\hline &&&1\\\hline &&1&\\\hline &1&&\\\hline\end{array}
$$

$$
\begin{array}{|c|c|c|c|}\hline &&1&\\\hline 1&&&\\\hline &1&&\\\hline &&&1\\\hline\end{array}\quad
\begin{array}{|c|c|c|c|}\hline &&&1\\\hline 1&&&\\\hline &1&&\\\hline &&1&\\\hline\end{array}\quad
\begin{array}{|c|c|c|c|}\hline &&&1\\\hline 1&&&\\\hline &&1&\\\hline &1&&\\\hline\end{array}\quad
\begin{array}{|c|c|c|c|}\hline &&1&\\\hline &&&1\\\hline 1&&&\\\hline &1&&\\\hline\end{array}\quad
\begin{array}{|c|c|c|c|}\hline &&&1\\\hline &&1&\\\hline 1&&&\\\hline &1&&\\\hline\end{array}
$$

Hint. If the 1 in the first column is not required to be higher than the 1 in the second column, they are called percolations. $(n+1)\times(n+1)$-percolations are enumerated by the n-th large Schröder number, which is proved by generating function method in [30]. In [17] they gave a bijection between percolations with the 1 in the first column being higher than the 1 in second column and small-binary trees (see (c)).

Remark. The next 10 objects are forbidden permutations, where the both two forbidden patterns are of length 4. Kremer [24] proved that there are exactly 10 equivalence classes of such pattern avoiding permutations, enumerated by the large Schröder numbers. In the following we only list one for each equivalence class. $(b^6), (c^6)$ are given by West [37], the d^6 is given by Gire [20], and $(e^6), (f^6), (g^6), (h^6), (i^6), (j^6)$ are given by Gibert.

(b^6) Permutations of $[n+1]$ avoiding both patterns 3142, 2413, i.e., $S_{n+1}(3142, 2413)$ such that the first number is larger than the second number (i.e. $\pi_1 > \pi_2$).

2134 2143 3124 4123 4132 3214 3241 4213 4231 4312 4321

Hint. We have known $S_{n+1}(3142, 2413) = S(n)$, where $S(n)$ is the n-th large schröder number. Then by the complement operation, $\pi_1\pi_2\pi_3\cdots\pi_{n+1} \in S_{n+1}(3142, 2413)$ iff $\pi'_1\pi'_2\pi'_3\cdots\pi'_{n+1} \in S_{n+1}(3142, 2413)$, where $\pi'_i = n+2-\pi_i$. Then as $\pi_1 > \pi_2$ iff $\pi'_1 < \pi'_2$, we get the desired result clearly.

(c^6) Permutations of $[n+1]$ avoiding both patterns 1423, 1324, i.e., $S_{n+1}(1423, 1324)$ such that the position of n is larger than the position of $n-1$ (i.e. $\pi^{-1}_{n+1} > \pi^{-1}_n$).

1234 1342 2314 2341 3412 3421 2134 3124 3142 3214 3241

Hint. We have known $S_{n+1}(1423, 1324) = S(n)$, where $S(n)$ is the n-th large schröder number. Then by the inverse operation, $S_{n+1}(1342, 1324) = S(n)$. Since $\pi_1\cdots\pi_{n-1}\pi_n\pi_{n+1} \in S_{n+1}(1342, 1324)$ iff $\pi_1\cdots\pi_{n-1}\pi_{n+1}\pi_n \in S_{n+1}(1342, 1324)$, we get the desired result clearly.

(d^6) Permutations of $[n+1]$ avoiding both patterns 3124, 3214, i.e., $S_{n+1}(3124, 3214)$ such that the position of 2 is larger than the position of 1 (i.e. $\pi^{-1}_2 > \pi^{-1}_1$).

1234 1243 1324 1342 1423 1432 3412 3142 4123 4132 4312

Hint. We have known $S_{n+1}(3124, 3214) = S(n)$. Then by the inverse operation, we have $S_{n+1}(2314, 3214) = S(n)$. Since $\pi_1\pi_2\pi_3\cdots\pi_{n+1} \in S_{n+1}(2314, 3214)$ if and only if $\pi_2\pi_1\pi_3\cdots\pi_{n+1} \in S_{n+1}(2314, 3214)$, we get the desired result clearly.

(e^6) Permutations of $[n+1]$ avoiding both patterns 1234, 2134, i.e., $S_{n+1}(1234, 2134)$ such that the first number is larger than the second number (i.e. $\pi_1 > \pi_2$).

2143 3124 3142 4123 4132 3214 3241 4213 4231 4312 4321

Hint. We have known $S_{n+1}(1234, 2134) = S(n)$. Since $\pi_1\pi_2\pi_3\cdots\pi_{n+1} \in S_{n+1}(1234, 2134)$ iff $\pi_2\pi_1\pi_3\cdots\pi_{n+1} \in S_{n+1}(1234, 2134)$, we get the result.

(f^6) Permutations of $[n+1]$ avoiding both patterns 1324, 2134, i.e., $S_{n+1}(1324, 2134)$ such that the first number is larger than the second number (i.e. $\pi^{-1}_1 > \pi^{-1}_2$).

2314 2341 2413 2431 3421 2143 3214 3241 4213 4231 4321

Hint. Many people have used generating trees to study the permutations with forbidden patterns [20, 24]. We know the root for such generating tree is the permutation 1, and the permutations at level one are 21, 12, and the following levels are the same by adding $(n+1)$ in the active positions of permutations of $[n]$ if we don't consider the forbidden patterns. When we face a Schröder generating tree [24], we just consider the right subtree of the root since $\pi^{-1}_1 > \pi^{-1}_2$. We can get that the generating tree for $S_{n+1}(1324, 2134)$ such that $\pi^{-1}_1 > \pi^{-1}_2$ is just the right subtree of the root on the generating tree for $S_{n+1}(1324, 2134)$, easily obtaining the desired result.

(g^6) Permutations of $[n+1]$ avoiding both patterns $1342, 2341$, i.e., $S_{n+1}(1342, 2341)$ such that the position of 1 is larger than the position of 2 (i.e. $\pi_1^{-1} > \pi_2^{-1}$).

2134 2143 2314 2413 2431 3421 3214 3241 4213 4231 4321

Hint. We have known $S_{n+1}(1342, 2341) = S(n)$. By the inverse operation, we have $S_{n+1}(1423, 4123) = S(n)$. Since $\pi_1\pi_2\pi_3\cdots\pi_{n+1} \in S_{n+1}(1423, 4123)$ iff $\pi_2\pi_1\pi_3\cdots\pi_{n+1} \in S_{n+1}(1423, 4123)$, we get the desired result clearly.

(h^6) Permutations of $[n+1]$ avoiding both patterns $2134, 3124$, i.e., $S_{n+1}(2134, 3124)$ such that the position of 2 is larger than the position of 1 (i.e. $\pi_2^{-1} > \pi_1^{-1}$).

1234 1243 1324 1342 1423 1432 3412 3142 4123 4132 4312

Hint. Similar to (f^6), we can get that the generating tree for $S_{n+1}(2134, 3124)$ such that $\pi_2^{-1} > \pi_1^{-1}$ is just the left subtree of the root on the generating tree for $S_{n+1}(2134, 3124)$, easily obtaining the desired result.

(i^6) Permutations of $[n+1]$ avoiding both patterns $2314, 3124$, i.e., $S_{n+1}(2314, 3124)$ such that the position of 2 is larger than the position of 1 (i.e. $\pi_2^{-1} > \pi_1^{-1}$).

1234 1243 1324 1342 1423 1432 3412 3142 4123 4132 4312

Hint. Use the same method of the Solution for (h^6) by just considering the left subtree of the root the generating tree for $S_{n+1}(2314, 3124)$.

(j^6) Permutations of $[n+1]$ avoiding both patterns $3412, 3421$, i.e., $S_{n+1}(3412, 3421)$ such that the $(n+1)$th number is larger than the nth number (i.e. $\pi_{n+1} > \pi_n$).

1234 1324 1423 2134 2314 2413 3124 3214 4123 4213 4312

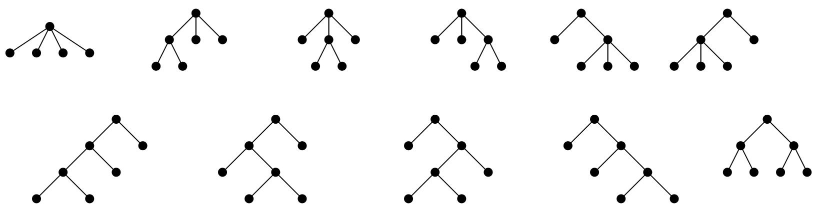
Hint. Since $\pi_1\cdots\pi_{n-1}\pi_n\pi_{n+1} \in S_{n+1}(3412, 3421)$ iff $\pi_1\cdots\pi_{n-1}\pi_{n+1}\pi_n \in S_{n+1}(3412, 3421)$, we get result similar as above.

(k^6) Permutations of $[n+1]$ avoiding both patterns $1342, 3142$, i.e., $S_{n+1}(1342, 3142)$ such that the 1th number is larger than the 2th number (i.e. $\pi_1 > \pi_2$).

2134 2143 3124 4123 4132 3214 3241 4213 4231 4312 4321

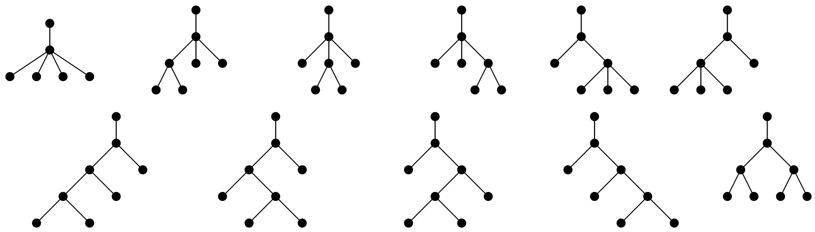
Hint. Since $\pi_1\pi_2\pi_3\cdots\pi_{n+1} \in S_{n+1}(1342, 3142)$ iff $\pi_2\pi_1\pi_3\cdots\pi_{n+1} \in S_{n+1}(1342, 3142)$, we get result similar as above.

(l^6) Ordered trees with $n+1$ leaves in which no node has outdegree equal to 1, i.e., bushes with $n+1$ leaves. (see [34, Exercise 6.39(b)]).



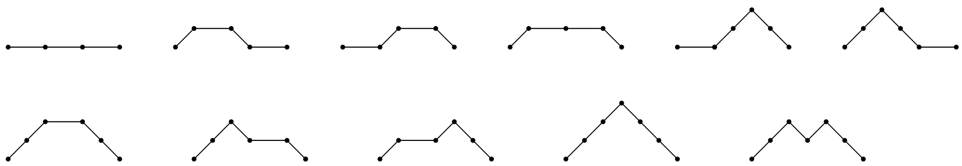
Hint. Bushes are studied in [10, 12]. In [10] Deutsch use a bijective proof to get this result. Bushes are also studied in [17], where they gave many applications.

(m^6) Ordered trees with $n+1$ leaves in which root has outdegree equal to 1 and no other vertex has outdegree equal to 1, i.e., planted bushes with $n+1$ leaves.



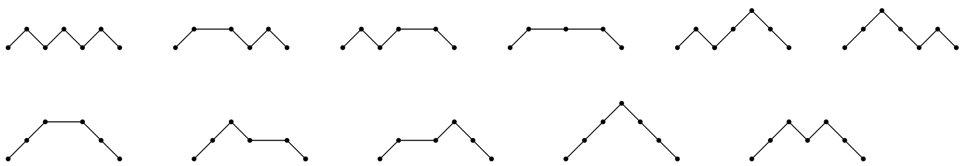
Hint. In fact, in [12], they called bushes for bushes in (l^6) and planted bushes, where called short bushes or tall bushes respectively. By (l^6) we can get this result just as that (d) to (f).

(n^6) Schröder paths of length $2n$ without hills.



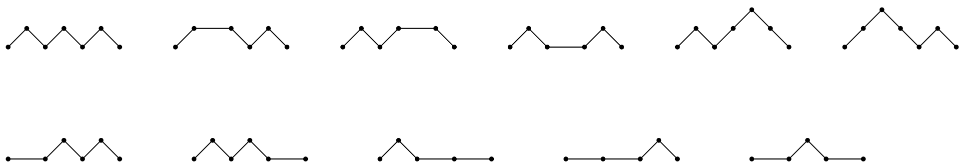
Hint. Schröder paths without hills are also called *hill-free Schröder paths.* They have been studied in [5, 17, 18].

(o^6) Schröder paths of length $2n$ without horizontal steps at x-axis.



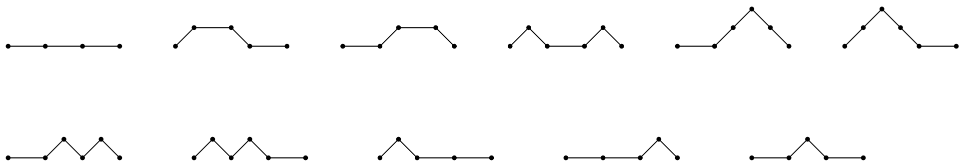
Hint. Schröder paths without horizontal steps at x-axis are also counted by super-Catalan numbers as that we can change each peak to be horizontal step of length 2.

(p^6) Schröder paths of length $2n$ with at least one hill.



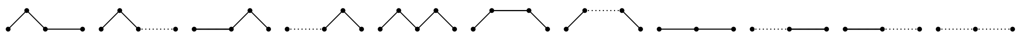
Hint. Since Schröder paths are enumerated by Schröder numbers, and by (n^6), we get the result.

(q^6) Schröder paths of length $2n$ with at least one horizontal step at x-axis.



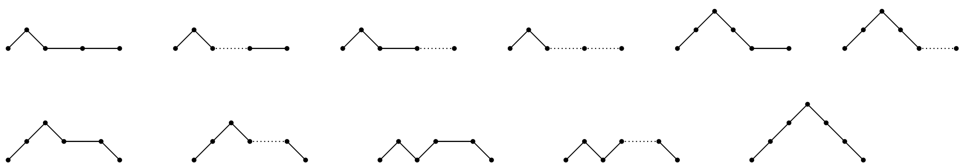
Hint. The same reason as that of (p^6) by (o^6).

(r^6) Schröder paths of length $2(n-1)$ without high peaks and the horizontal steps colored black or white, and all other steps colored black. (similar as that of (a^4)).



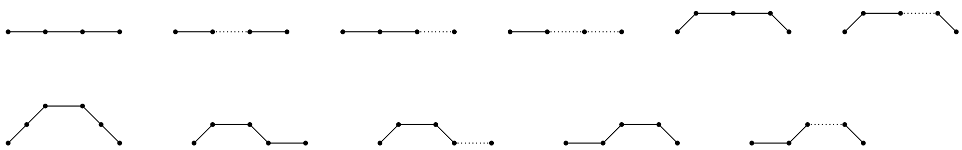
Hint. Use the bijection between leaves-colored plane trees and colored Schröder paths in [18], the Corollary of the bijection showed this result [18].

(s^6) Schröder paths of length $2n$ with only one peak, and all horizontal steps appearing after the peak and colored black or white. All other steps are colored black.



Hint. Use the bijection between leaves-colored plane trees and colored Schröder paths in [18], such paths are corresponding to leaves-colored plane trees as that of (o').

(t^6) Schröder paths of length $2n$ with no peak and all horizontal steps except the first one colored black or white. All other steps are colored black.



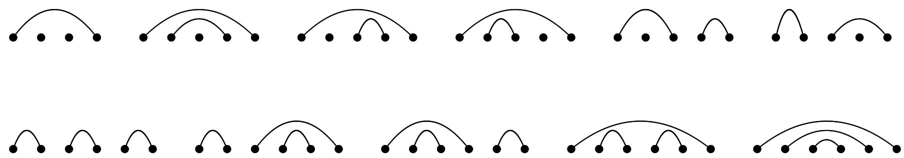
Hint. From (s^6), we change the first peak of each path into a horizontal step, the result is obvious.

(u^6) Matchings on $[2n]$ avoiding both patterns 12312 and 121323.

112233 112332 122133 122331 121233 112323
122313 121332 123231 123213 123321

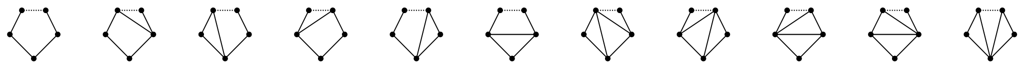
Hint. Matchings avoiding given pattern have been studied in [8], where gave the bijection between Schröder paths without hill and matchings avoiding both patterns 12312 and 121323. They are also studied in [17] with some applications.

(v^6) Hilly poor noncrossing partitions with n blocks.



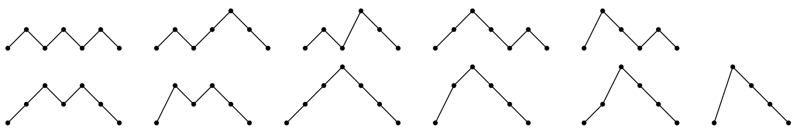
Hint. Hilly poor noncrossing partitions are studied in [8, 38], where in [38] they gave a bijection between themselves and $(2,3)$-Motzkin paths in (yyy).

(w^6) Dissections of a convex $(n+2)$-gon that do not intersect in their interiors (see [34, Exercise 6.39(h)].



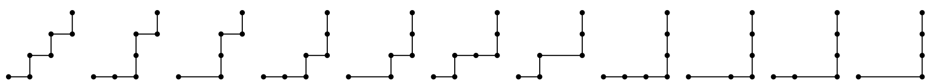
Hint. Dissections of a convex $(n+2)$-gon are studied in [35, 36], where in [36] Stanley also gave the history of Schröder numbers, and in [35] stated a bijection between dissections of convex polygons and Standard Young Tableaux.

(x^6) Lattice paths in the (x,y) plane from $(0,0)$ to the x-axis using steps $(1,k)$, where $k \in \mathbb{P}$ or $k=-1$, never passing below the x-axis, and with n steps of the form $(1,-1)$ (see [34, Exercise 6.39(e)]).



Hint. See [34, Solution to Exercise 6.39(e)].

(y^6) Lattice paths in the (x,y)-plane from $(0,0)$ to the (n,n) using steps $(k,0)$ or $(0,1)$ with $k \in \mathbb{P}$, and never passing above the line $y=x$ (see [34, Exercise 6.39(f)]).



Hint. See [34, Solution to Exercise 6.39(f)].

(z^6) Sequences $i_1 i_2 \cdots i_k$, where $i_j \in \mathbb{P}$ or $i_j = -1$ (and k can be arbitrary), such that $n = \#\{j : i_j = -1\}$, $i_1 + \cdots + i_j \geq 0$ for all j, and $i_1 + \cdots + i_k = 0$ (see [34, Exercise 6.39(i)]).

3-1-1-1   2-1-11-1   1-12-1-1   12-1-1-1   21-1-1-1   2-11-1-1
1-11-11-1   11-1-11-1   1-11-11-1   1-111-1-1   111-1-1-1

Hint. See [34, Solution to Exercise 6.39(i)].

(a^7) Graphs G (without loops and multiple edges) on the vertex set $[n+2]$ with the following two properties: (1) All of the edges $\{1, n+2\}$ and $\{i, i+1\}$ are edges of G, and (2) G is noncrossing, i.e., there are not both edges $\{a, c\}$ and $\{b, d\}$ with $a < b < c < d$ (see [34, Exercise 6.39(p)]).

Hint. See [34, Solution to exercise 6.39(p)].

(b^7) Ways to insert parentheses in a string of $n+1$ symbols. The parentheses must be balanced but there is no restriction on the number of pairs of parentheses. The number of letters inside a pair of parentheses must be at least 2. Parentheses enclosing the whole string are ignored. (see [34, Exercise 6.39(a)]).

$$(xx)xx \quad x(xx)x \quad xx(xx) \quad (xx)(xx) \quad (xxx)x \quad x(xxx)$$
$$((xx)x)x \quad (x(xx))x \quad x((xx)x) \quad x(x(xx)) \quad xxxx$$

Hint. See [34, Solution to Exercise 6.39(a)] though here we use different expressions for this kind of objects.

(c^7) Pairs (u;v) of same-length compositions of n, 0s allowed in u but not in v, and u dominates v (meaning $u_1 \geqslant v_1$, $u_1 + u_2 \geqslant v_1 + v_2$, and so on) (D. Callan).

$$(3;3) \quad (3,0;2,1) \quad (3,0;1,2) \quad (3,0,0;1,1,1) \quad (2,1;2,1) \quad (2,1;1,2) \quad (1,2;1,2)$$
$$(2,1,0;1,1,1) \quad (1,2,0;1,1,1) \quad (2,0,1;1,1,1) \quad (1,1,1;1,1,1)$$

Hint. See [31] (David Callan).

(d^7) Length of list produced by a variant of the Catalan producing iteration: replace each integer k by the list $0, 1, .., k, k{+}1, k, ..., 1, 0$ and get the length s_n of the resulting (flattened) list after $n+1$ iterations.

Hint. See [31] (Wouter Meeussen).

(e^7) Possible schedules for n time slots in the first-come first-served (FCFS) printer policy.

Hint. See [31].

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