k-CATALAN STRUCTURES

SILVIA HEUBACH[1], NELSON Y. LI[2], TOUFIK MANSOUR[3]

[1]Dept. of Mathematics, California State University Los Angeles, Los Angeles, CA 90032, USA.

[2,3]Center for Combinatorics, LPMC, Nankai University, 300071 Tianjin, P.R. China.

[2]The High School Affiliated to Renmin University of China, Renmin University of China, 100080 Beijing, P.R. China.

[3]Department of Mathematics, Haifa University, 31905 Haifa, Israel.

[1]`sheubac@calstatela.edu`, [2]`nelsonli@eyou.com`, [3]`toufik@math.haifa.ac.il`

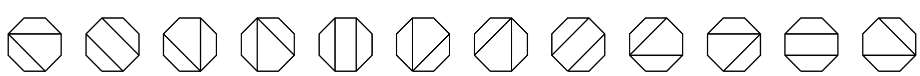
This list is a compilation of combinatorial structures enumerated by the k-Catalan numbers

$$C_n^k = \frac{1}{kn+1}\binom{kn+1}{n} = \frac{1}{(k-1)n+1}\binom{kn}{n}$$

for any positive integers k and n. We start by giving known combinatorial structures, together with references to where a proof can be found. The remainder of the list consists of new structures that are generalizations of combinatorial objects enumerated by the Catalan numbers. First we list combinatorial objects for which the "Catalan proof" generalizes easily or for which there exists an easy bijection to another object enumerated by the k-ary numbers. Some proofs may already exist in the folklore, others are straightforward generalizations of the Catalan proofs. Our main goal is to start collecting these objects in one place. We will provide "hints" to indicate how to provide a proof when the proof is easy, and provide complete proofs when the Catalan proof does not generalize.

Rather than give very technical definitions for each of the structures, we give illustrations of all the objects for the case $k = n = 3$, hoping that the pictures speak for themselves, except for some instances when pictures alone may not be enough, or combinatorial objects are not widely known. Since these structures are generalizations of Catalan type structures given in [7, Ex. 6.19] and [8], we will also indicate Stanley's labels for the respective Catalan object at the end of each description, where for example [(a)] refers to [7, Ex. 6.19 (a)]. Items labeled (a) through (nnn) appear in [7, Ex. 6.19], and items (ooo) through (e^6) appear in [8]. Note that the labels in [8] are changing as new objects are added to the list, so the labels given here may not remain accurate. The labels we have used in this list are the ones of the version of October 30, 2005.

1. Subdividing a convex polygon into n disjoint $(k+1)$-gons by means of non-intersecting diagonals [(a)], (see [4] and [7, Proposition 6.2.1]):

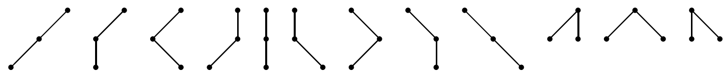


2. k-ary parenthesizations of a string of $(k-1)n+1$ letters [(b)], (see [2]):

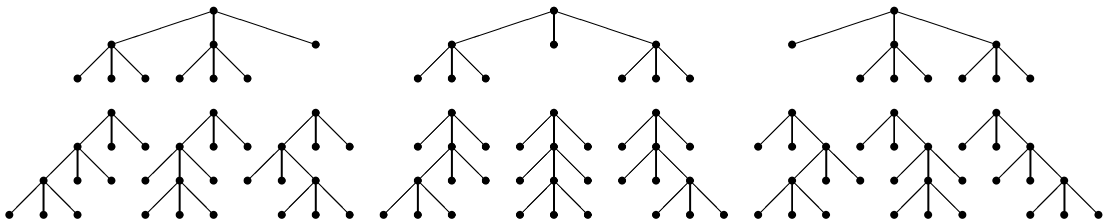
$$\begin{array}{llllll} ((xxx)xx)xx & (x(xxx)x)xx & (xx(xxx))xx & x((xxx)xx)x & x(x(xxx)x)x & x(xx(xxx))x \\ xx((xxx)xx) & xx(x(xxx)x) & xx(xx(xxx)) & (xxx)(xxx)x & (xxx)x(xxx) & x(xxx)(xxx) \end{array}$$

Date: July 30, 2005.

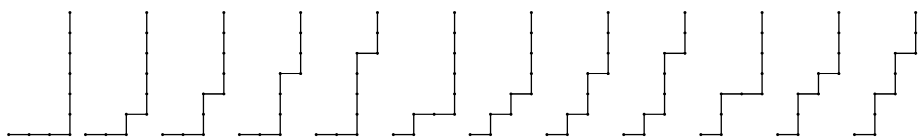
3. k-ary trees with n vertices [(c)], (see [7, Proposition 6.2.1]):



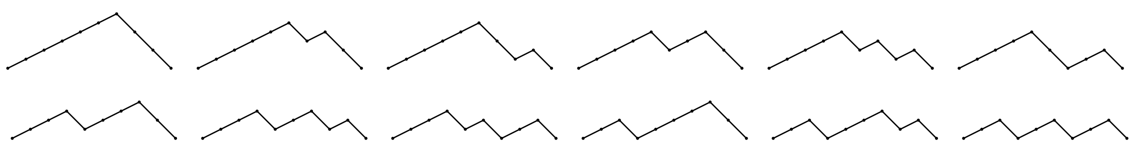
4. Plane k-ary trees with $kn+1$ vertices (or $(k-1)n+1$ endpoints) [(d)], (see [4]):



5. Lattice paths from $(0,0)$ to $(n,(k-1)n)$ with steps $(0,1)$ or $(1,0)$, never rising above the line $y=(k-1)x$, also called k-good paths [(h)], (see [4]):



6. k-ary paths of length kn, i.e., lattice paths from $(0,0)$ to $(kn,0)$ with steps $(1,\frac{1}{k-1})$ and $(1,-1)$, never falling below the x-axis [(i)], (see [7, Proposition 6.2.1]):



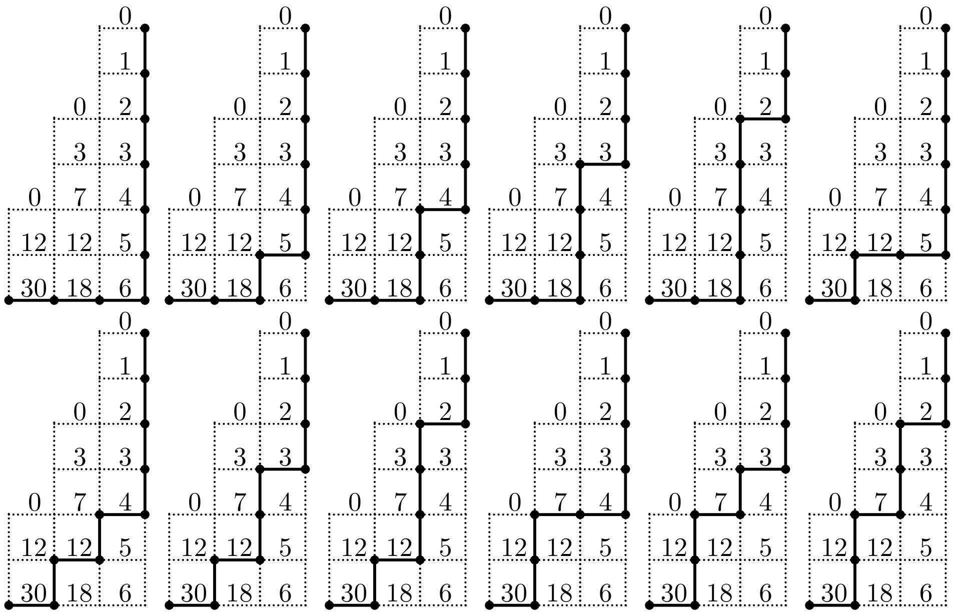
7. Sequences of $(k-1)n$ $\left(\frac{1}{k-1}\right)$s and n (-1)s such that every partial sum is nonnegative and the total sum is 0 (with $\frac{1}{k-1}$ and -1 denoted simply as $+$ and $-$, respectively, below) [(r)], (see [7, Proposition 6.2.1]):

$++++++---$	$+++++-+--$	$++++-++--$	$+++-+++--$
$++-++++--$	$+++++--+-$	$++++-+-+-$	$+++-++-+-$
$++-+++-+-$	$++++--++-$	$+++-+-++-$	$++-++-++-$

8. The ranking sequence of a k-good path (see (5)) ([6], Section 4):

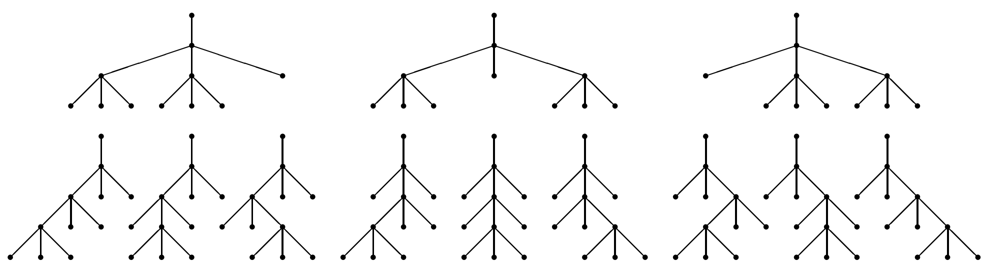
$6,18,30$	$5,18,30$	$4,18,30$	$3,18,30$	$2,18,30$	$5,12,30$
$4,12,30$	$3,12,30$	$2,12,30$	$4,7,30$	$3,7,30$	$2,7,30$

which is obtained as illustrated below. The rightmost column is filled from top to bottom with the integers $0,...,n(k-1)$. Then the columns are filled from right to left by assigning a 0 as the topmost value in each column. The remaining values are computed as the sum of the value above and the value to the right of it. The ranking labels are the ones above the east steps of the path, read from right to left.



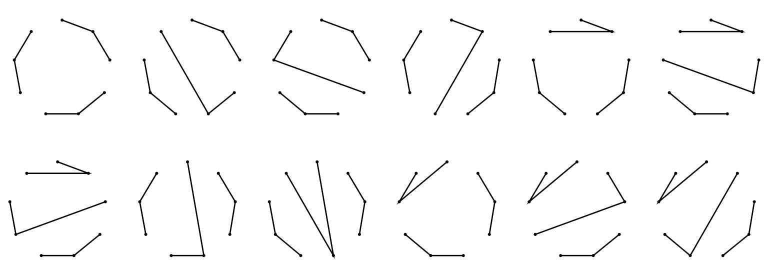
The following objects are new generalizations of combinatorial structures counted by the Catalan numbers. "New" indicates that we are not aware of a general proof that these structures are enumerated by C_n^k for any n and k. In some instances, the object described has been defined elsewhere (e.g. set-valued Young tableaux), but the enumeration proof was not given, or given only

9. Planted (i.e., root has degree one) k-valent plane k-ary trees with $kn+2$ vertices [(f)]:



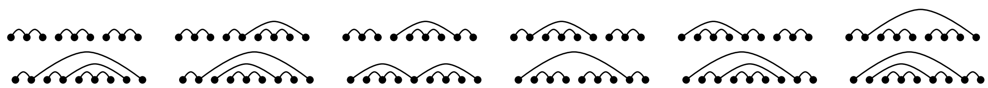
Hint: When the root is removed we obtain the trees of (4).

10. n nonintersecting chord paths of length $k-1$ joining kn points on the circumference of a circle [(n)]:



Hint: Fix a vertex v. Starting clockwise from v, at each vertex write 1 if encountering a vertex belonging to the chord path containing v, and $2, 3, \cdots$ for vertices in the 2nd, 3rd, ... chords encountered. This gives a bijection with (12).

11. Ways of connecting kn points in the plane lying on a horizontal line by n nonintersecting (k-1)-arc sequences, where each (k-1)-arc sequence connects k of the points and lies above the points [(o)]:



Hint: Define a k-ary path (6) as follows. Reading the points on the line from left to right, append a down step if a point is an end point of a (k-1)-arc sequence, otherwise append an up step. For example, the first structure in (11) is mapped to *UUDUUDUUD* and the second structure in (11) is mapped to *UUDUUUUDD*. To create the $(k-1)$-arc sequences from the k-ary path P of length kn, let $i_1 < i_2 < \cdots < i_n$ be the sequence of the positions of the down steps D in the path P. Connect the vertex i_j, $j = 1, 2, \ldots, n$, with the closest $k-1$ non-connected vertices on its left side. For example, the 3-ary path *UUDUUUUDD* has a sequence $3 < 8 < 9$ of down steps, and maps to second structure in (11). This gives a bijection between (6) and (11). Alternatively, we can obtain a bijection between (11) and (10) by cutting the circle in (10) between two fixed vertices (e.g., the topmost and the one to its left) and "straighten out".

12. Permutations $a_1 a_2 \cdots a_{kn}$ of the multiset $\{1^k, 2^k, \cdots, n^k\}$ such that: (i) the first occurrences of $1, 2, \cdots, n$ appear in increasing order, and (ii) there is no subsequence of the form $\alpha\beta\alpha\beta$:

111222333 111223332 111233322 112221333 112223331 112233321
112333221 122211333 122213331 122233311 122333211 123332211

Hint: There is an obvious bijection between these sequences and the nonintersecting arcs of (11). (Label the vertices and connect those that have the same label by a $(k-1)$-arc sequence.)

13. Sequences $1 = a_1 \leq a_2 \cdots \leq a_n$ of integers with $a_{i-1} \leq a_i \leq (i-1)(k-1)+1$ [(s)]:

111 112 113 114 115 122 123 124 125 133 134 135

Hint: Consider a lattice path P of the type (5). Let $a_i = j$ if the i-th $(1,0)$ step is on the line $y = j-1$.

14. Sequences $a_1, a_2, \cdots, a_n$ of integers such that $a_1 = 0$ and $0 \leq a_{i+1} \leq a_i + k - 1$ [(u)]:

000 001 002 010 011 012 013 020 021 022 023 024

Hint: Let $b_{n+1-i} = a_i - a_{i+1} + k - 1$. Replace a_i with b_i copies of $\frac{1}{k-1}$ followed by one -1 for $1 \leq i \leq n$ (with $a_{n+1} = 0$) to get (7). A more geometric bijection is to map the sequences of (14) to the k-good paths in (6) as the vertical distances of the left end points of the east steps to the diagonal $(k-1)x$. For example, the path with height sequence (of the east steps) 012 has distance sequence 013.

15. Sequences $a_1, a_2, \cdots, a_{kn}$ of nonnegative integers with $a_1 = 1$, $a_{kn} = 0$ and $a_i - a_{i-1} = 1$ or $-(k-1)$ [(t^4)]:

123456420 123453420 123423420 123123420 120123420 123453120
123423120 123123120 120123120 123420120 123120120 120120120

Hint: $(k-1)$ times the partial sums of the sequences in (7).

16. Sequences of $a_1 a_2 \cdots a_{kn-k}$ of $(k-1)(n-1)$ $\left(\frac{1}{k-1}\right)$'s and $n-1$ (-1)'s such that $a_1 + a_2 + \cdots + a_i \geq -1$ for any $1 \leq i \leq kn-k$ and $a_1 + a_2 + \cdots + a_{kn-k} = 1$ (with $\frac{1}{k-1}$ and -1 denoted simply as $+$ and $-$, respectively, below) [(v^4)]:

$++++--$ $+++-+-$ $++-++-$ $+-+++-$ $-++++-$ $+++--+$
$++-+-+$ $+-++-+$ $-+++-+$ $++--++$ $+-+-++$ $-++-++$

Hint: Place $k-1$ $\left(\frac{1}{k-1}\right)$'s at the beginning and a -1 at the end of each of these sequences and concatenate, yielding a bijection with (7).

17. Sequences of $(k-1)n-1$ $\left(\frac{1}{k-1}\right)$'s and any number of -1's such that every partial sum is nonnegative (with $\frac{1}{k-1}$ and -1 denoted simply as $+$ and $-$, respectively, below) [(u^4)]:

$$\begin{array}{llll} +++++ & +++++- & ++++-+ & +++-++ \\ ++-+++ & +++++-- & ++++-+- & +++-++- \\ ++-+++- & ++++--+ & +++-+-+ & ++-++-+ \end{array}$$

Hint: In (28) replace an up step with $\frac{1}{k-1}$ and a down step with -1.

18. Sequences $(a_1,\cdots,a_n)$ of nonnegative integers satisfying $a_1+\cdots+a_i \geq (k-1)i$ and $\sum a_j = (k-1)n$ [(r^4)]:

222 231 240 312 321 330 402 411 420 501 510 600

Hint: Add $k-1$ to the terms of the sequences of (24).

19. Sequences of $(a_1,\cdots,a_n) \in \mathbb{N}^n$ for which there exists a distributive lattice of rank $(k-1)(n-1)+1$ with a_i join-irreducibles of rank i, $1 \leq i \leq n$ [(a^5)]:

500 410 320 230 140 401 311 221 131 302 212 122

Hint: The sequences $1, 1+a_n, 1+a_n+a_{n-1}, \cdots, 1+a_n+a_{n-1}+\cdots+a_2$ coincide with those of (13).

20. Sequences $a_1 < a_2 < \cdots < a_{n-1}$ of integers satisfying $1 \leq a_i \leq ki$ [(t)]:

12 13 14 15 16 23 24 25 26 34 35 36

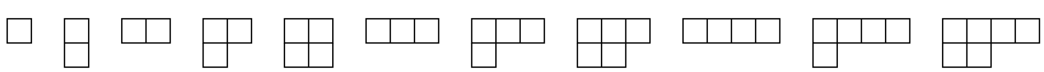
Hint: Subtract $i-1$ from a_i and append a one at the beginning to get (13).

21. Column-strict plane partitions of shape $((k-1)(n-1),(k-1)(n-2),\cdots,k-1)$, such that each entry in the i-th row is equal to $n-i$ or $n-i+1$ [(yy)]:

$$\begin{array}{cccccccccccc} 3333 & 3332 & 3322 & 3333 & 3332 & 3322 & 3222 & 3333 & 3332 & 3322 & 3222 & 2222 \\ 22 & 22 & 22 & 21 & 21 & 21 & 21 & 11 & 11 & 11 & 11 & 11 \end{array}$$

Hint: Let b_i be the number of entries in row i that are equal to $n-i+1$ (so $b_n=0$). Then sequences $b_n, b_{n-1}, \cdots, b_1$ obtained in this way are in bijection with the paths in (5) where b_i is the height of the i-th east step of the path.

22. Young diagrams that fit in the shape $((k-1)(n-1),(k-1)(n-2),\cdots,k-1)$ [(vv)]:

$\emptyset$ 

Hint: If $\lambda = (\lambda_1,\cdots,\lambda_{n-1}) \subseteq ((k-1)(n-1),(k-1)(n-2),\cdots,k-1)$, then the sequences $(0,\lambda_{n-1},\cdots,\lambda_1)$ are in bijection with the paths in (5) where λ_i is the height of the i-th east step of the path.

23. Sequences $a_1,a_2,\cdots,a_{n-1}$ of integers such that $a_i \leq k-1$ and all partial sums are nonnegative [(v)]:

$0,0$ $0,1$ $0,2$ $1,-1$ $1,0$ $1,1$ $1,2$ $2,-2$ $2,-1$ $2,0$ $2,1$ $2,2$

Hint: Take the first differences of the sequences in (14).

24. Sequences of $a_1, a_2, \cdots, a_n$ of integers such that $a_i \geq -(k-1)$, all partial sums are nonnegative, and $a_1 + a_2 + \cdots + a_n = 0$ [(v)]:

$$\begin{array}{cccccc} 0,0,0 & 0,1,-1 & 0,2,-2 & 1,-1,0 & 1,0,-1 & 1,1,-2 \\ 2,-2,0 & 2,-1,-1 & 2,0,-2 & 3,-2,-1 & 3,-1,-2 & 4,-2,-2 \end{array}$$

Hint: The sequences $a_1 + \cdots + a_n, a_1 + \cdots + a_{n-1}, \cdots, a_1$ coincide with those of (14).

For the following structures we provide a definition, as set-valued Young tableaux may not (yet) be widely used. Note that this definition is similar to the one given by Buch [1], except that he allows weakly increasing sets across rows, i.e., the definition by Buch is for semi-standard set-valued Young tableaux.

Definition 1. *A standard set-valued Young tableau with shape $(\lambda_1, \lambda_2, \ldots, \lambda_r)$ and weights $w_{i,j}$ is a tableau in which the cell at position (i,j) is filled with an integer set $B_{i,j}$, where*

- $|B_{i,j}| = w_{i,j}$ *and* $\bigcup B_{i,j} = [\ell]$, *with* $\ell = \sum_{i,j} w_{i,j}$,
- $B_{i,j} \bigcap B_{i',j'} = \emptyset$ *unless* $i = i'$ *and* $j = j'$,
- $max(B_{i,j}) < min(B_{i+1,j})$ *and* $max(B_{i,j}) < min(B_{i,j+1})$, *for* $j < \lambda_i$

where $max(B_{i,j})$ ($min(B_{i,j})$) stands for the maximum (minimum) integer in $B_{i,j}$. If the weights across a row are the same, then we give the weights as $(w_1, w_2, \ldots, w_r)$, where w_i is the weight for row i.

25. Standard set-valued Young tableaux of shape (n,n) with weight $(k-1,1)$ (see [5] for $k=3$) (or equivalently, of shape $(n, n-1)$ with weight $(k-1,1)$) [(ww)]:

$$\begin{array}{|c|c|c|}\hline 12&34&56\\\hline 7&8&9\\\hline\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&34&57\\\hline 6&8&9\\\hline\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&34&58\\\hline 6&7&9\\\hline\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&34&67\\\hline 5&8&9\\\hline\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&34&68\\\hline 5&7&9\\\hline\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&34&78\\\hline 5&6&9\\\hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline 12&35&67\\\hline 4&8&9\\\hline\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&35&68\\\hline 4&7&9\\\hline\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&35&78\\\hline 4&6&9\\\hline\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&45&67\\\hline 3&8&9\\\hline\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&45&68\\\hline 3&7&9\\\hline\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&45&78\\\hline 3&6&9\\\hline\end{array}$$

or

$$\begin{array}{|c|c|c|}\hline 12&34&56\\\hline 7&8&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&34&57\\\hline 6&8&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&34&58\\\hline 6&7&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&34&67\\\hline 5&8&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&34&68\\\hline 5&7&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&34&78\\\hline 5&6&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 12&35&67\\\hline 4&8&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&35&68\\\hline 4&7&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&35&78\\\hline 4&6&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&45&67\\\hline 3&8&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&45&68\\\hline 3&7&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&45&78\\\hline 3&6&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}$$

Hint: Given a standard set-valued Young tableau T of shape (n,n), define $a_1 a_2 \cdots a_{kn}$ by $a_i = U$ if i appears in row 1 of T, while $a_i = D$ if i appears in row 2. This sets up a bijection with (6), where $U = (1, \frac{1}{k-1})$ and $D = (1, -1)$. Note that the entries in the first row of the tableau of shape (n,n) give the positions of the up steps, and the second row gives the position of the down steps of the path.

26. Standard set-valued Young tableaux with at most two rows and with first row of length n, $w_{1,j} = k-1$ for $1 \leq j \leq n-1$, $w_{1,n} = k-2$ and $w_{2,j} = 1$ for $1 \leq j \leq n$ [(r^5)]:

$$\begin{array}{|c|c|c|}\hline 12&34&5\\\hline\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&34&5\\\hline 6&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&34&5\\\hline 6&7&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&34&6\\\hline 5&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&34&6\\\hline 5&7&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&34&7\\\hline 5&6&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}$$

$$\begin{array}{|c|c|c|}\hline 12&35&6\\\hline 4&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&35&6\\\hline 4&7&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&35&7\\\hline 4&6&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&45&6\\\hline 3&\multicolumn{1}{c}{}&\multicolumn{1}{c}{}\\\cline{1-1}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&45&6\\\hline 3&7&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}\quad
\begin{array}{|c|c|c|}\hline 12&45&7\\\hline 3&6&\multicolumn{1}{c}{}\\\cline{1-2}\end{array}$$

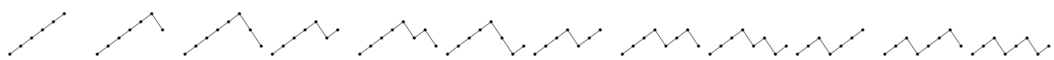
Hint: Given a standard set-valued Young tableau T of the type being counted, construct a k-ary path of length kn as follows. For each entry $1, 2, \cdots m$ of T, if i appears in row 1 then draw an up step, while if i appears in row 2 then draw a down step. Afterwards draw an up step followed by down steps to the x-axis. This sets up a bijection with (6).

27. Standard set-valued Young tableaux with at most two rows and with first row of length n, $w_{1,j} = k-1$ and $w_{2,j} = 1$, such that for all i the ith entry of row 2 is not ki [(s^5)]:

12 34 56	12 34 56	12 34 56	12 34 57	12 34 57	12 34 58
	7	7 8	6	6 8	6 7
12 34 67	12 34 67	12 34 68	12 35 67	12 35 67	12 35 68
5	5 8	5 7	4	4 8	4 7

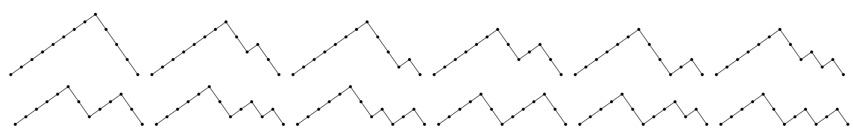
Hint: Prepend $k-2$ up steps to the bijection of (25) to obtain an *elevated* k-ary path, i.e., a k-ary path of length $kn+k$ which never touches the x-axis except at the beginning and end. Remove the first $k-1$ (up) steps and the last (down) step to get a bijection with (6).

28. Left factors L of k-ary paths such that L has $(k-1)n-1$ up steps (i.e., the part of the path that is to the left of the $(k-1)n$-th up step) [(sss)]:



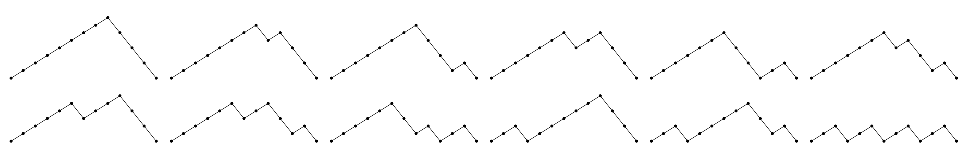
Hint: Add one further up step and then down steps until reaching $(kn, 0)$. This gives a bijection with the k-ary path of (6).

29. k-ary paths (as defined in (6)) of length $k(n+1)$ whose first down step is followed by another down step [(ttt)]:



Hint: Deleting the first UUD gives a bijection with (6) (k-ary paths of length kn).

30. k-ary paths (as defined in (6)) from $(0,0)$ to $(k(n+1), 0)$ with no peaks at height $\frac{k}{k-1}, \frac{k+1}{k-1}, \cdots, 2$ [(k)]:



Proof: Let $g(x)$ be the generating function for the number of k-ary paths from $(0,0)$ to $(kn, 0)$, i.e., lattice paths from $(0,0)$ to $(kn,0)$ with steps $(1, \frac{1}{k-1})$ and $(1,-1)$, never falling below the x-axis. Let $F_j(x)$ be the generating function for the number of k-ary paths of length kn with no peaks at height $\frac{i}{k-1}$ for all $i = k-1, k, \ldots, j$. Let such a path P be decomposed according to the the first-last return decomposition of k-ary paths as $P = UP_1UP_2 \ldots UP_{k-1}DP_0$. Then P does not have a peak at level $\frac{j}{k-1}$ if and only if P_i does not have a peak at level $\frac{j-i}{k-1}$. Therefore, in the generating function F_j each P_i contributes F_{j-i} for $i = 1, \ldots, j-(k-1)$, $P_{j-k+2}, \ldots, P_{k-2}$ each contribute a factor of g, P_{k-1} contributes $g-1$ (to avoid peaks at level 1) and P_0 contributes F_j. Soving for F_j, we obtain

$$F_j(x) = \frac{1}{1 - xF_{j-1}(x)F_{j-2}(x)\cdots F_{k-1}(x)g^{2k-3-j}(x)(g(x)-1)},$$

for all $j = k-1, k, \ldots, 2k-3$. By induction on i we can prove for all $i = 0, 1, 2, \ldots, k-2$,

$$F_{k-1+i}(x) = \frac{1 - xg^{k-1-i}(x)(g(x)-1)\sum_{j=0}^{i-1} g^i(x)}{1 - xg^{k-2-i}(x)(g(x)-1)\sum_{j=0}^{i} g^i(x)},$$

and therefore,

$$\prod_{i=0}^{k-2} F_{k-1+i} = \frac{1}{1 - xg^0(g-1)(1+g+g^2+\cdots+g^{k-2})} = \frac{1}{1+x-xg^{k-1}}.$$

Now, let $h(x)$ be the generating function for the number of k-ary paths from $(0,0)$ to $(kn, 0)$ with no peaks at height $\frac{k}{k-1}, \frac{k+1}{k-1}, \cdots, 2$. Again, the first-last return decomposition of k-ary paths gives

$$h(x) = 1 + xF_{k-1}(x)F_k(x)\cdots F_{2k-3}(x)h(x).$$

Hence,

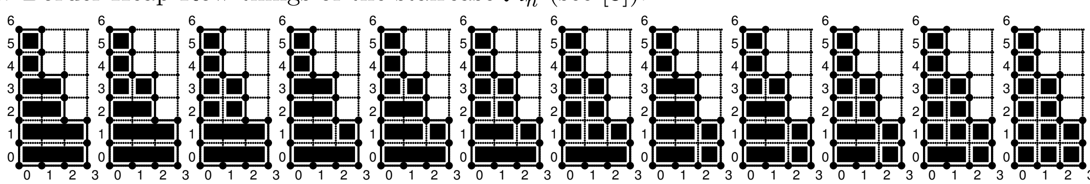
$$h(x) = \frac{1}{1 - x\prod_{i=0}^{k-2} F_{k-1+i}(x)} = \frac{1}{1 - \frac{x}{1+x-xg^{k-1}}} = 1 + \frac{x}{1 - xg^{k-1}(x)}.$$

Since $g(x) = 1 + xg^k(x)$ (using the first-last return decomposition), we have that $\frac{g-1}{xg} = g^{k-1}$, and therefore,

$$h(x) = 1 + \frac{x}{1 - \frac{g-1}{g}} = 1 + \frac{xg}{g-g+1} = 1 + xg(x),$$

which shows that the number of k-ary paths from $(0,0)$ to $(k(n+1), 0)$ with no peaks at height $\frac{k}{k-1}, \frac{k+1}{k-1}, \cdots, 2$ equals the number of k-ary paths from (0,0) to $(kn, 0)$, as requested. ■

31. Border Heap Row-tilings of the staircase $\mathcal{A}_n^k$ (see [3]):



Last updated on July 29, 2007.

References

[1] A. S. Buch, A Littlewood-Richardson rule for the K-theory of Grassmannians, *Acta Math.*, **189**, no. 1, (2002), 37–78.

[2] J. Cigler, Some remarks on Catalan families, *Europ. J. Combin.* **8**, no. 3, (1987) 261–267.

[3] S. Heubach, Nelson Y. Li, T. Mansour, *Staircase Tilings and k-Catalan Structures*, preprint.

[4] P. Hilton, J. Pedersen, Catalan numbers, their generalization, and their uses, *Math. Intell.* **13** (1991) 64-75.

[5] D. S. Hough, L. W. Shapiro, The noncrossing descent matrix is Riordan, *Congr. Numer.*, **162** (2003) 83–96.

[6] F. Ruskey, Generating t-ary trees lexicographically, *SIAM J. Comput.* **7** (1978) 424-439.

[7] R.P. Stanley, *Enumerative combinatorics, Vol. 2*, Cambridge University Press, Cambridge, 1999.

[8] R.P. Stanley, Catalan addendum, available at *http://www-math.mit.edu/∼rstan/ec/*.