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[ > #####case 1324,1342
[ > restart:
[ > aa1:=proc(n,i,j) option remember: if n=1 then return(1): fi: if
n=2 then if i=1 and j=2 then return(1): fi: if i=2 and j=1 then
return(q): fi: return(0): fi: if n=3 then if i=1 and j=2 then
return(1): fi: if (i=1 and j=3) or (i=2 and j=1) or (i=2 and j=3)
or (i=3 and j=1) then return(q): fi: if i=3 and j=2 then
return(q^2): fi: return(0): fi: if 0<j and j< i and i< n+1 then
return(add(q*aa1(n-1,j,ii),ii=1..n-1)): fi: if j=i+1 and i>0 and
i< n then return(add(aa1(n-1,i,jj),jj=1..n-1)): fi: if j=n and i>0
and i< n-2 then
return(q*aa1(n-1,i,n-1)+add(add(q*aa1(n-2,jj,ii),jj=1..i),ii=1..n-
2)): fi: if i=n-2 and j=n then
return(add(add(q^2*aa1(n-3,ii,jj),ii=1..n-3),jj=1..n-3)+add(add(q*
aa1(n-2,jj,ii),jj=1..n-3),ii=1..n-2)): fi: return(0): end:
[ > NN:=13:
AA:=(x,v,w)->add(add(add(aa1(n,i,j)*x^n*v^i*w^j,j=1..n),i=1..n),n=
2..NN):
AAP:=(x,v,w)->add(add(add(aa1(n,i,j)*x^n*v^i*w^j,j=i+1..n),i=1..n),
,n=2..NN):
AAN:=(x,v,w)->add(add(add(aa1(n,i,j)*x^n*v^i*w^j,j=1..i-1),i=1..n),
,n=2..NN):
CAA:=(x,v)->add(add(aa1(n,i,i+1)*v^i*x^n,i=1..n-1),n=2..NN):
DAA:=(x,v)->add(add(aa1(n,i,n)*v^i*x^n,i=1..n-1),n=2..NN):
[ > #EQ1:
[ > simplify(taylor(-AAN(x,v,w)+q*v^2*w*x^2+q*x/(1-v)*(v*AA(x,v*w,1)-v
^2*AA(v*x,w,1)),x,12));
O( $x^{12}$ )
[ > #EQ2
[ > simplify(taylor(-CAA(x,v)+v*x^2+x*AA(x,v,1),x,12));
O( $x^{12}$ )
[ > #EQ3:
[ > simplify(taylor(-DAA(x,v)+q*x*DAA(x,v)-q^2*v*x^3*AA(x*v,1,1)-q^2*v
^2*x^4+q*x^2/(1-v)*(AA(x,v,1)-AA(v*x,1,1))+0*q^2*v*x^3*AA(v*x,1,1)
+q*x^2*(AA(v*x,1,1)-0*q*v*x*AA(v*x,1,1))+q*v*x^2*(AA(v*x,1,1)+v*x+
1/q),x,12));
O( $x^{12}$ )
[ > #EQ4:
[ > simplify(taylor(-AAP(x,v,w)+w*CAA(x,v*w)+DAA(w*x,v)-q*x^2*v*w^2*AA
(v*w*x,1,1)-v*w^2*x^2-q*v^2*w^3*x^3,x,12));
O( $x^{12}$ )
[ > eq1:=-FAAN(x,v,w)+q*v^2*w*x^2+q*x/(1-v)*(v*FAA(x,v*w,1)-v^2*FAA(v*

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$\mathbf{x}, \mathbf{w}, 1)) ;$

$$eq1 := -\text{FAAN}(x, v, w) + q v^2 w x^2 + \frac{q x (v \text{FAA}(x, v w, 1) - v^2 \text{FAA}(v x, w, 1))}{1 - v}$$

> $\text{eq2} := -\text{FCAA}(\mathbf{x}, \mathbf{v}) + v^2 x^2 + x \text{FAA}(\mathbf{x}, \mathbf{v}, 1) ;$

$$eq2 := -\text{FCAA}(x, v) + v x^2 + x \text{FAA}(x, v, 1)$$

> $\text{eq3} := -\text{FDAA}(\mathbf{x}, \mathbf{v}) + q^2 x^2 \text{FCAA}(\mathbf{x}, \mathbf{v}, 1, 1) - q^2 v^2 x^3 \text{FAA}(\mathbf{x}^2, \mathbf{v}, 1, 1) - q^2 v^2 x^2 \text{FAA}(\mathbf{x}^2, \mathbf{v}, 1, 1) + q^2 v^2 x^3 \text{FAA}(\mathbf{v}^2, \mathbf{x}, 1, 1) + q^2 v^2 x^2 (\text{FAA}(\mathbf{v}^2, \mathbf{x}, 1, 1) - q^2 v^2 x^2 \text{FAA}(\mathbf{v}^2, \mathbf{x}, 1, 1)) + q^2 v^2 x^2 (\text{FAA}(\mathbf{v}^2, \mathbf{x}, 1, 1) + v^2 x^2 / q) ;$

$$eq3 := -\text{FDAA}(x, v) + q x \text{FDAA}(x, v) - q^2 v^2 x^4 + \frac{q x^2 (\text{FAA}(x, v, 1) - \text{FAA}(v x, 1, 1))}{1 - v} + q x^2 (\text{FAA}(v x, 1, 1) - q v x \text{FAA}(v x, 1, 1)) + q v x^2 \left(\text{FAA}(v x, 1, 1) + v x + \frac{1}{q} \right)$$

> $\text{eq4} := -\text{FAAP}(\mathbf{x}, \mathbf{v}, \mathbf{w}) + w \text{FCAA}(\mathbf{x}, \mathbf{v}^2 \mathbf{w}) + \text{FDAA}(\mathbf{w}^2 \mathbf{x}, \mathbf{v}) - q^2 v^2 w^2 \text{FAA}(v w x, 1, 1) - v w^2 x^2 - q v^2 w^3 x^3$

> **#eq 25,26 from the paper**

> $\text{eq25} := -\text{FAA}(\mathbf{x}, \mathbf{v}, \mathbf{w}) + (w^2 x + v^2 q^2 x / (1 - v)) * \text{FAA}(\mathbf{x}, \mathbf{v}^2 \mathbf{w}, 1) - v^2 q^2 x / (1 - v) * \text{FAA}(\mathbf{v}^2 \mathbf{x}, \mathbf{w}, 1) + v^2 w^2 q^2 x^2 / (1 - v) / (w^2 q^2 x - 1) * \text{FAA}(\mathbf{v}^2 \mathbf{w}^2 \mathbf{x}, 1, 1) - w^2 q^2 x^2 / (1 - v) / (w^2 q^2 x - 1) * \text{FAA}(\mathbf{w}^2 \mathbf{x}, \mathbf{v}, 1) + (v^2 w^2 q^2 x^2 - v^2 q^2 w^2) * v^2 w^2 x^2 / (w^2 q^2 x - 1) ;$

$$eq25 := -\text{FAA}(x, v, w) + \left(w x + \frac{v q x}{1 - v} \right) \text{FAA}(x, v w, 1) - \frac{v^2 q x \text{FAA}(v x, w, 1)}{1 - v} + \frac{v w^2 q x^2 \text{FAA}(v w x, 1, 1)}{(1 - v)(q w x - 1)} - \frac{w^2 q x^2 \text{FAA}(w x, v, 1)}{(1 - v)(q w x - 1)} + \frac{(q^2 v w x - q v - w) v w x^2}{q w x - 1}$$

> $\text{eq26} := \text{subs}(w=1, x=x/v, \text{eq25}) ;$

$$eq26 := -\text{FAA}\left(\frac{x}{v}, v, 1\right) + \left(\frac{x}{v} + \frac{q x}{1 - v}\right) \text{FAA}\left(\frac{x}{v}, v, 1\right) - \frac{v q x \text{FAA}(x, 1, 1)}{1 - v} + \frac{q x^2 \text{FAA}(x, 1, 1)}{v(1 - v)\left(\frac{q x}{v} - 1\right)} - \frac{q x^2 \text{FAA}\left(\frac{x}{v}, v, 1\right)}{v^2(1 - v)\left(\frac{q x}{v} - 1\right)} + \frac{(q^2 x - q v - 1)x^2}{v\left(\frac{q x}{v} - 1\right)}$$

> $\text{solve}(\text{coeff}(\text{eq26}, \text{FAA}(x/v, v, 1)) = 0, v) ;$

$$\mathbf{v0} := (1 + x + (4 * q^2 x^2 - 4 * q x^2 - 4 * q^2 x^2 - 4 * q^2 x^2 + x^2 - 2 * x + 1)^{(1/2)}) / 2 ;$$

$$\frac{x}{2} + \frac{1}{2} + \frac{\sqrt{4 q^2 x^2 - 4 q x^2 - 4 q^2 x^2 - 4 q^2 x^2 + x^2 - 2 x + 1}}{2}, \frac{x}{2} + \frac{1}{2} - \frac{\sqrt{4 q^2 x^2 - 4 q x^2 - 4 q^2 x^2 - 4 q^2 x^2 + x^2 - 2 x + 1}}{2}$$

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$$v0 := \frac{x}{2} + \frac{1}{2} + \frac{\sqrt{4q^2x^2 - 4qx^2 - 4qx + x^2 - 2x + 1}}{2}$$


> ##formula of A(x,1,1) in the paper
> FINAx11:=simplify(simplify(rationalize(subs(v=v0,solve(subs(FAA(x/v,v,1)=0,eq26)=0,FAA(x,1,1))))+1))-1;

$$FINAx11 := \frac{-2q^2x^2 + 2qx^2 + 2qx + \sqrt{4q^2x^2 - 4qx^2 - 4qx + x^2 - 2x + 1} + x - 1}{2q(qx - x - 1)}$$


> factor(subs(x=v*x,solve(subs(FAA(x,1,1)=FINAx11,eq26)=0,FAA(x/v,v,1)));

$$FINAv1 := (x, v) \rightarrow 1/2 * v * x * (-2*q^3*v^2*x^3 + 4*q^2*v^2*x^3 + 2*q^2*v^2*x^2 - 2*q*v^2*x^3 + 2*q^2*v*x^2 - 3*q*x^2*v^2 + \sqrt{4q^2*v^2*x^2 - 4q*v^2*x^2 + v^2*x^2 - 4q*vx - 2*vx + 1} * q*vx + 2*v^2*x^2 - 3*q*x*vx - v^2*x - 3*v*x^2 - \sqrt{4q^2*v^2*x^2 - 4q*v^2*x^2 + v^2*x^2 - 4q*vx - 2*vx + 1} * v - \sqrt{4q^2*v^2*x^2 - 4q*v^2*x^2 + v^2*x^2 - 4q*vx - 2*vx + 1} * x + 2*vx + v - x) / (2(q*vx - vx - 1) * (q^2*vx^2 - q*vx^2 - qx + vx - v - x + 1))$$



$$FINAv1 := (x, v) \rightarrow \frac{1}{2} v x (-2q^3v^2x^3 + 4q^2v^2x^3 + 2q^2v^2x^2 - 2qv^2x^3 + 2q^2vx^2 - 3qx^2v^2 + \sqrt{4q^2v^2x^2 - 4qvx^2 + v^2x^2 - 4qvx - 2vx + 1} * qvx + 2v^2x^2 - 3qvx - v^2x - 3vx^2 - \sqrt{4q^2v^2x^2 - 4qvx^2 + v^2x^2 - 4qvx - 2vx + 1} * v - \sqrt{4q^2v^2x^2 - 4qvx^2 + v^2x^2 - 4qvx - 2vx + 1} * x + 2vx + v - x) / ((qvx - vx - 1) * (q^2vx^2 - qvx^2 - qx + vx - v - x + 1))$$


> #presentation of theorem 11 , formula A(x,v,1)
> factor(coeff(FINAv1(x,v), (4*q^2*v^2*x^2 - 4*q*v^2*x^2 + v^2*x^2 - 4*q*v*x^2 - 2*v*x + 1)^{(1/2)}, 1));

$$\frac{vx(qvx - v - x)}{2(qvx - vx - 1)(q^2vx^2 - qvx^2 - qx + vx - v - x + 1)}$$


> s:=factor(coeff(FINAv1(x,v), (4*q^2*v^2*x^2 - 4*q*v^2*x^2 + v^2*x^2 - 4*q*v*x^2 - 2*v*x + 1)^{(1/2)}, 0) - v^2*q*x^2 / (1 - v*q*x));

$$factor(taylor(numer(s)/v/x, v, 10));$$


s := vx(2q^3v^3x^3 - q^2v^3x^3 - 2q^2v^3x^2 - 2q^2v^2x^3 - 3q^2v^2x^2 + qv^3x^2 + qv^2x^3 + qv^2x^2 + 3qv^2x + qvx^2 - 2vx^2 + qvx + v^2x + 3vx^2 - 2vx - v + x) / (2(qvx - vx - 1))

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$$(q^2 v x^2 - q v x^2 - q x + v x - v - x + 1) (q v x - 1))$$


$$x + (q x^2 + q x + 3 x^2 - 2 x - 1) v - x (2 q^2 x^2 + 3 q^2 x - q x^2 - q x - 3 q + 2 x - 1) v^2 +$$


$$q x^2 (2 q - 1) (q x - 1) v^3$$

> #formula for A^-=FAAN=AAAN from eq1;
> eq1;
AAAN:=(x,v,w)->q*v^2*w*x^2+q*x/(1-v)*(v*FINAv1(x,v*w)-v^2*FINAv1(v*x,w));

$$-FAAN(x, v, w) + q v^2 w x^2 + \frac{q x (v \text{FAA}(x, v w, 1) - v^2 \text{FAA}(v x, w, 1))}{1 - v}$$


$$AAAN := (x, v, w) \rightarrow q v^2 w x^2 + \frac{q x (v \text{FINAv1}(x, v w) - v^2 \text{FINAv1}(v x, w))}{1 - v}$$

> eq2; FINCC:=(x,v)->+v*x^2+x*FINAv1(x,v);

$$-\text{FCAA}(x, v) + v x^2 + x \text{FAA}(x, v, 1)$$


$$FINCC := (x, v) \rightarrow v x^2 + x \text{FINAv1}(x, v)$$

> eq3;
FINDD:=(x,v)->1/(1-q*x)*(-q^2*v^2*x^4+q*x^2/(1-v)*(FINAv1(x,v)-FINAv1(v*x,1))+q*x^2*(FINAv1(v*x,1)-q*v*x*FINAv1(v*x,1))+q*v*x^2*(FINAv1(v*x,1)+v*x+1/q));

$$-\text{FDAA}(x, v) + q x \text{FDAA}(x, v) - q^2 v^2 x^4 + \frac{q x^2 (\text{FAA}(x, v, 1) - \text{FAA}(v x, 1, 1))}{1 - v}$$


$$+ q x^2 (\text{FAA}(v x, 1, 1) - q v x \text{FAA}(v x, 1, 1)) + q v x^2 \left( \text{FAA}(v x, 1, 1) + v x + \frac{1}{q} \right)$$


$$FINDD := (x, v) \rightarrow \left( -q^2 v^2 x^4 + \frac{q x^2 (\text{FINAv1}(x, v) - \text{FINAv1}(v x, 1))}{1 - v} \right.$$


$$\left. + q x^2 (\text{FINAv1}(v x, 1) - q v x \text{FINAv1}(v x, 1)) + q v x^2 \left( \text{FINAv1}(v x, 1) + v x + \frac{1}{q} \right) \right) / (1 - q x)$$

> eq4;
AAAP:=(x,v,w)->w*FINCC(x,v*w)+FINDD(w*x,v)-q*x^2*v*w^2*FINAv1(v*w*x,1,1)-v*w^2*x^2

$$-q v^2 w^3 x^3$$

AAAP:=(x,v,w)-

$$w \text{FINCC}(x, v w) + \text{FINDD}(w x, v) - q x^2 v w^2 \text{FINAv1}(v w x, 1) - v w^2 x^2 - q v^2 w^3 x^3$$

> #checking all equations
> eq1;
simplify(-AAAN(x,v,w)+q*v^2*w*x^2+q*x/(1-v)*(v*FINAv1(x,v*w)-v^2*FINAv1(v*x,w)));

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$$-\text{FAAN}(x, v, w) + q v^2 w x^2 + \frac{q x (v \text{FAA}(x, v w, 1) - v^2 \text{FAA}(v x, w, 1))}{1 - v}$$


$$0$$


$$> \text{eq2}; \text{simplify}(-\text{FINCC}(x, v) + v*x^2 + x*\text{FINAv1}(x, v));$$


$$-\text{FCAA}(x, v) + v x^2 + x \text{FAA}(x, v, 1)$$


$$0$$


$$> \text{eq3};$$


$$\text{simplify}(-\text{FINDD}(x, v) + q*x*\text{FINDD}(x, v) - q^2 v^2 x^4 + q*x^2 / (1-v) * (\text{FINAv1}(x, v) - \text{FINAv1}(v*x, 1)) + q*x^2 * (\text{FINAv1}(v*x, 1) - q*v*x*\text{FINAv1}(v*x, 1)) + q*x^2 * (\text{FINAv1}(v*x, 1) + v*x + 1/q));$$


$$-\text{FDAA}(x, v) + q x \text{FDAA}(x, v) - q^2 v^2 x^4 + \frac{q x^2 (\text{FAA}(x, v, 1) - \text{FAA}(v x, 1, 1))}{1 - v}$$


$$+ q x^2 (\text{FAA}(v x, 1, 1) - q v x \text{FAA}(v x, 1, 1)) + q v x^2 \left( \text{FAA}(v x, 1, 1) + v x + \frac{1}{q} \right)$$


$$0$$


$$> \text{eq4};$$


$$\text{simplify}(-\text{AAP}(x, v, w) + w*\text{FINCC}(x, v*w) + \text{FINDD}(w*x, v) - q*x^2 * v * w^2 * \text{FINAv1}(v*w*x, 1) - v*w^2 x^2 - q*v^2 w^3 x^3);$$


$$-\text{FAAP}(x, v, w) + w \text{FCAA}(x, v w) + \text{FDAA}(w x, v) - q x^2 v w^2 \text{FAA}(v w x, 1, 1) - v w^2 x^2$$


$$- q v^2 w^3 x^3$$


$$0$$


$$>$$


$$>$$


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