

INVERSION SEQUENCES AVOIDING 021 AND A 5-LETTER PATTERN

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ABSTRACT. We study the enumeration of inversion sequences that avoid the pattern 021 and another pattern of length five that avoids 021. We determine the generating trees and the corresponding generating functions for all possible pattern pairs.

1. MAIN RESULTS

An integer sequence $e = e_0e_1 \cdots e_n$ of length n is called an *inversion sequence* if $0 \leq e_i \leq i$ for each $0 \leq i \leq n$. We use \mathbf{I}_n to denote the set of length n inversion sequences. A pattern τ is any word of length k over the alphabet $[k] := \{0, 1, \dots, k-1\}$. An inversion sequence $e \in \mathbf{I}_n$ is said to contain the length- k pattern τ if there is a subsequence of length k in e that has the same relative order with τ ; otherwise, we say that e avoids the pattern τ . The first systematic study of pattern-avoidance for inversion sequences was initiated by Mansour and Shattuck [1] and Corteel et al. [2] for the patterns of length three. For a summary of the existing results and main references, see the introduction of [7]. This note extends the results of [8] to the pairs 021 and another pattern of length five, whereas the earlier paper studied the pattern pairs 021 and another pattern of length four. We use the same algorithm developed in [7], and the five-step procedure detailed in [8], based on generating trees and kernel method.

There are 106 patterns τ of length five that avoid 021. For each pair $\{021, \tau\}$, we characterize the corresponding generating trees and obtain analytic expressions for the generating functions

$$F_{\{021, \tau\}}(x) = \sum_{n \geq 0} |I_n(\{021, \tau\})| x^{n+1}.$$

For the summary of the results, see Table 1. The conjecture of [8] states that for any 021-avoiding pattern τ , the generating tree $\mathcal{T}'(\{021, \tau\})$ is d -regular for some d . The results of this note justify this conjecture for patterns of length five.

For the following four cases $\tau = 00000, 00001, 0011, 0012$, we present the details of the system of the equations that yield the exact expressions for the corresponding generating functions. As a summary, we note that

- for guessing the generating tree rules, we used our algorithm in [7] for each pattern of length five. The output of the algorithm can be found in [6].
- the generating tree $\mathcal{T}'(\{021, 00000\})$ has 71 succession rules. We translate these rules into a system of equations involving the corresponding generating functions. The Maple file [3] contains the output of this long, hard work of computations.
- the generating tree $\mathcal{T}'(\{021, 00001\})$ has more than 20 rules. The Maple file [4] contains most of the steps of the computations.

2010 *Mathematics Subject Classification.* 05A05, 05A15, 05A16.

Key words and phrases. Pattern-avoiding inversion sequences, generating functions, generating trees, kernel method.

G. Yıldırım was partially supported by Tubitak-Ardeb-120F352.

- for all other cases of length-five patterns, we present the system of the equations and the solutions in [5].

1.1. **Case $B = \{021, 00000\}$.** By applying our algorithm, we found that the generating tree $\mathcal{T}'(\{021, 00000\})$ which has the root a_0 , and the following set of succession rules

$$\begin{aligned}
0 &\rightsquigarrow 0^2, b_{11}, & 0^2 &\rightsquigarrow 0^3, b_{21}, b_{22}, \\
b_{11} &\rightsquigarrow b_{21}, 01^2, b_{11}, & a_{000} &\rightsquigarrow 0^4, b_{31}, b_{32}, b_{33}, \\
b_{21} &\rightsquigarrow a_{31}, 0^2 1^2, b_{21} b_{22}, & b_{22} &\rightsquigarrow b_{32}, 0^2 2^2, b_{22}, \\
01^2 &\rightsquigarrow 0^2 1^2, 01^3, a_{121}(2), b_{11}, & 0^4 &\rightsquigarrow b_{41}, b_{42}, b_{43}, b_{44}, \\
b_{31} &\rightsquigarrow b_{41}, 0^3 1^2, b_{31}, b_{32}, b_{33}, & b_{32} &\rightsquigarrow b_{42}, 0^3 2^2, b_{32}, b_{33}, \\
b_{33} &\rightsquigarrow b_{43}, 0^3 3^2, b_{33}, & 0^2 1^2 &\rightsquigarrow 0^3 1^2, 0^2 1^3, a_{221}(2), b_{21}, b_{22}, \\
0^2 2^2 &\rightsquigarrow 0^3 2^2, 0^2 2^3, b_{21}, b_{22}, & 01^3 &\rightsquigarrow 0^2 1^3, 01^4, a_{131}(2), a_{121}(2), b_{11}, \\
b_{41} &\rightsquigarrow 0^4 1^2, b_{41}, b_{42}, b_{43}, b_{44}, & b_{42} &\rightsquigarrow 0^4 2^2, b_{42}, b_{43}, b_{44}, \\
b_{43} &\rightsquigarrow 0^4 3^2, b_{43}, b_{44}, & b_{44} &\rightsquigarrow 0^4 4^2, b_{44}, \\
0^3 1^2 &\rightsquigarrow 0^4 1^2, 0^3 1^3, a_{321}(2), b_{31}, b_{32}, b_{33}, & 0^3 2^2 &\rightsquigarrow 0^4 2^2, 0^3 2^3, b_{31}, b_{32}, b_{33}, \\
0^3 3^2 &\rightsquigarrow 0^4 3^2, 0^3 3^3, b_{32}, b_{33}, & 0^2 1^3 &\rightsquigarrow 0^3 1^3, 0^2 1^4, a_{231}(2), a_{221}(2), b_{21}, b_{22}, \\
0^2 2^3 &\rightsquigarrow 0^3 2^3, 0^2 2^4, a_{221}(2), b_{21}, b_{22}, & 01^4 &\rightsquigarrow 0^2 1^4, a_{141}(2), a_{131}(2), a_{121}(2), b_{11}, \\
0^4 1^2 &\rightsquigarrow 0^4 1^3, a_{421}(2), b_{41}, b_{42}, b_{43}, b_{44}, & 0^4 2^2 &\rightsquigarrow 0^4 2^3, b_{41}, b_{42}, b_{43}, b_{44}, \\
0^4 3^2 &\rightsquigarrow 0^4 3^3, b_{42}, b_{43}, b_{44}, & 0^4 4^2 &\rightsquigarrow 0^4 4^3, b_{43}, b_{44}, \\
0^3 1^3 &\rightsquigarrow 0^4 1^3, 0^3 1^4, a_{331}(2), a_{321}(2), b_{31}, b_{32}, b_{33}, & 0^3 2^3 &\rightsquigarrow 0^4 2^3, 0^3 2^4, a_{321}(2), b_{31}, b_{32}, b_{33}, \\
0^3 3^3 &\rightsquigarrow 0^4 3^3, 0^3 3^4, b_{31}, b_{32}, b_{33}, & 0^2 1^4 &\rightsquigarrow 0^3 1^4, a_{241}(2), a_{231}(2), a_{221}(2), b_{21}, b_{22}, \\
0^2 2^4 &\rightsquigarrow 0^3 2^4, a_{231}(2), a_{221}(2), b_{21}, b_{22}, & 0^4 1^3 &\rightsquigarrow 0^4 1^4, a_{431}(2), a_{421}(2), b_{41}, b_{42}, b_{43}, b_{44}, \\
0^4 2^3 &\rightsquigarrow 0^4 2^4, a_{421}(2), b_{41}, b_{42}, b_{43}, b_{44}, & 0^4 3^3 &\rightsquigarrow 0^4 3^4, b_{41}, b_{42}, b_{43}, b_{44}, \\
0^4 4^3 &\rightsquigarrow 0^4, b_{42}, b_{43}, b_{44}, & 0^3 1^4 &\rightsquigarrow 0^4 1^4, a_{341}(2), a_{331}(2), a_{321}(2), b_{31}, b_{32}, b_{33}, \\
0^3 2^4 &\rightsquigarrow 0^4 2^4, a_{331}(2), a_{321}(2), b_{31}, b_{32}, b_{33}, & 0^3 3^4 &\rightsquigarrow 0^4 4^4, a_{321}(2), b_{31}, b_{32}, b_{33}, \\
0^4 1^4 &\rightsquigarrow a_{441}(2), a_{431}(2), a_{421}(2), b_{41}, b_{42}, b_{43}, b_{44}, & 0^4 2^4 &\rightsquigarrow a_{431}(2), a_{421}(2), b_{41}, b_{42}, b_{43}, b_{44}, \\
0^4 3^4 &\rightsquigarrow a_{421}(2), b_{41}, b_{42}, b_{43}, b_{44},
\end{aligned}$$

$$\begin{aligned}
a_{121}(m) &\rightsquigarrow a_{221}(m)a_{122}(m)a_{121}(m)\cdots a_{121}(2)a_{131}(m-1)\cdots a_{131}(2)a_{141}(m-1)\cdots a_{141}(2)b_{11}, \\
a_{122}(m) &\rightsquigarrow a_{222}(m)a_{123}(m)a_{121}(m)\cdots a_{121}(2)a_{131}(m)\cdots a_{131}(2)a_{141}(m-1)\cdots a_{141}(2)b_{11}, \\
a_{123}(m) &\rightsquigarrow a_{223}(m)a_{124}(m)a_{121}(m)\cdots a_{121}(2)a_{131}(m)\cdots a_{131}(2)a_{141}(m)\cdots a_{141}(2)b_{11}, \\
a_{124}(m) &\rightsquigarrow a_{224}(m)a_{121}(m+1)\cdots a_{121}(2)a_{131}(m)\cdots a_{131}(2)a_{141}(m)\cdots a_{141}(2)b_{11}, \\
a_{131}(m) &\rightsquigarrow a_{231}(m)a_{132}(m)a_{121}(m)\cdots a_{121}(2)a_{131}(m)\cdots a_{131}(2)a_{141}(m-1)\cdots a_{141}(2)b_{11}, \\
a_{132}(m) &\rightsquigarrow a_{232}(m)a_{133}(m)a_{121}(m)\cdots a_{121}(2)a_{131}(m)\cdots a_{131}(2)a_{141}(m)\cdots a_{141}(2)b_{11}, \\
a_{133}(m) &\rightsquigarrow a_{233}(m)a_{134}(m)a_{121}(m+1)\cdots a_{121}(2)a_{131}(m)\cdots a_{131}(2)a_{141}(m)\cdots a_{141}(2)b_{11}, \\
a_{134}(m) &\rightsquigarrow a_{234}(m)a_{121}(m+1)\cdots a_{121}(2)a_{131}(m+1)\cdots a_{131}(2)a_{141}(m)\cdots a_{141}(2)b_{11}, \\
a_{141}(m) &\rightsquigarrow a_{241}(m)a_{142}(m)a_{121}(m)\cdots a_{121}(2)a_{131}(m)\cdots a_{131}(2)a_{141}(m)\cdots a_{141}(2)b_{11}, \\
a_{142}(m) &\rightsquigarrow a_{242}(m)a_{143}(m)a_{121}(m+1)\cdots a_{121}(2)a_{131}(m)\cdots a_{131}(2)a_{141}(m)\cdots a_{141}(2)b_{11}, \\
a_{143}(m) &\rightsquigarrow a_{243}(m)a_{144}(m)a_{121}(m+1)\cdots a_{121}(2)a_{131}(m+1)\cdots a_{131}(2)a_{141}(m)\cdots a_{141}(2)b_{11}, \\
a_{144}(m) &\rightsquigarrow a_{244}(m)a_{121}(m+1)\cdots a_{121}(2)a_{131}(m+1)\cdots a_{131}(2)a_{141}(m+1)\cdots a_{141}(2)b_{11}, \\
a_{221}(m) &\rightsquigarrow a_{321}(m)a_{222}(m)a_{221}(m)\cdots a_{221}(2)a_{231}(m-1)\cdots a_{231}(2)a_{241}(m-1)\cdots a_{241}(2)b_{21}b_{22}, \\
a_{222}(m) &\rightsquigarrow a_{322}(m)a_{223}(m)a_{221}(m)\cdots a_{221}(2)a_{231}(m)\cdots a_{231}(2)a_{241}(m-1)\cdots a_{241}(2)b_{21}b_{22},
\end{aligned}$$

$$\begin{aligned}
a_{223}(m) &\rightsquigarrow a_{323}(m)a_{224}(m)a_{221}(m)\cdots a_{221}(2)a_{231}(m)\cdots a_{231}(2)a_{241}(m)\cdots a_{241}(2)b_{21}b_{22}, \\
a_{224}(m) &\rightsquigarrow a_{324}(m)a_{221}(m+1)\cdots a_{221}(2)a_{231}(m)\cdots a_{231}(2)a_{241}(m)\cdots a_{241}(2)b_{21}b_{22}, \\
a_{231}(m) &\rightsquigarrow a_{331}(m)a_{232}(m)a_{221}(m)\cdots a_{221}(2)a_{231}(m)\cdots a_{231}(2)a_{241}(m-1)\cdots a_{241}(2)b_{21}b_{22}, \\
a_{232}(m) &\rightsquigarrow a_{332}(m)a_{233}(m)a_{221}(m)\cdots a_{221}(2)a_{231}(m)\cdots a_{231}(2)a_{241}(m)\cdots a_{241}(2)b_{21}b_{22}, \\
a_{233}(m) &\rightsquigarrow a_{333}(m)a_{234}(m)a_{221}(m+1)\cdots a_{221}(2)a_{231}(m)\cdots a_{231}(2)a_{241}(m)\cdots a_{241}(2)b_{21}b_{22}, \\
a_{234}(m) &\rightsquigarrow a_{334}(m)a_{221}(m+1)\cdots a_{221}(2)a_{231}(m+1)\cdots a_{231}(2)a_{241}(m)\cdots a_{241}(2)b_{21}b_{22}, \\
a_{241}(m) &\rightsquigarrow a_{341}(m)a_{242}(m)a_{221}(m)\cdots a_{221}(2)a_{231}(m)\cdots a_{231}(2)a_{241}(m)\cdots a_{241}(2)b_{21}b_{22}, \\
a_{242}(m) &\rightsquigarrow a_{342}(m)a_{243}(m)a_{221}(m+1)\cdots a_{221}(2)a_{231}(m)\cdots a_{231}(2)a_{241}(m)\cdots a_{241}(2)b_{21}b_{22}, \\
a_{243}(m) &\rightsquigarrow a_{343}(m)a_{244}(m)a_{221}(m+1)\cdots a_{221}(2)a_{231}(m+1)\cdots a_{231}(2)a_{241}(m)\cdots a_{241}(2)b_{21}b_{22}, \\
a_{244}(m) &\rightsquigarrow a_{344}(m)a_{221}(m+1)\cdots a_{221}(2)a_{231}(m+1)\cdots a_{231}(2)a_{241}(m+1)\cdots a_{241}(2)b_{21}b_{22}, \\
a_{321}(m) &\rightsquigarrow a_{421}(m)a_{322}(m)a_{321}(m)\cdots a_{321}(2)a_{331}(m-1)\cdots a_{331}(2)a_{341}(m-1)\cdots a_{341}(2)b_{31}b_{32}b_{33}, \\
a_{322}(m) &\rightsquigarrow a_{422}(m)a_{323}(m)a_{321}(m)\cdots a_{321}(2)a_{331}(m)\cdots a_{331}(2)a_{341}(m-1)\cdots a_{341}(2)b_{31}b_{32}b_{33}, \\
a_{323}(m) &\rightsquigarrow a_{423}(m)a_{324}(m)a_{321}(m)\cdots a_{321}(2)a_{331}(m)\cdots a_{331}(2)a_{341}(m)\cdots a_{341}(2)b_{31}b_{32}b_{33}, \\
a_{324}(m) &\rightsquigarrow a_{424}(m)a_{321}(m+1)\cdots a_{321}(2)a_{331}(m)\cdots a_{331}(2)a_{341}(m)\cdots a_{341}(2)b_{31}b_{32}b_{33}, \\
a_{331}(m) &\rightsquigarrow a_{431}(m)a_{332}(m)a_{321}(m)\cdots a_{321}(2)a_{331}(m)\cdots a_{331}(2)a_{341}(m-1)\cdots a_{341}(2)b_{31}b_{32}b_{33}, \\
a_{332}(m) &\rightsquigarrow a_{432}(m)a_{333}(m)a_{321}(m)\cdots a_{321}(2)a_{331}(m)\cdots a_{331}(2)a_{341}(m)\cdots a_{341}(2)b_{31}b_{32}b_{33}, \\
a_{333}(m) &\rightsquigarrow a_{433}(m)a_{334}(m)a_{321}(m+1)\cdots a_{321}(2)a_{331}(m)\cdots a_{331}(2)a_{341}(m)\cdots a_{341}(2)b_{31}b_{32}b_{33}, \\
a_{334}(m) &\rightsquigarrow a_{434}(m)a_{321}(m+1)\cdots a_{321}(2)a_{331}(m+1)\cdots a_{331}(2)a_{341}(m)\cdots a_{341}(2)b_{31}b_{32}b_{33}, \\
a_{341}(m) &\rightsquigarrow a_{441}(m)a_{342}(m)a_{321}(m)\cdots a_{321}(2)a_{331}(m)\cdots a_{331}(2)a_{341}(m)\cdots a_{341}(2)b_{31}b_{32}b_{33}, \\
a_{342}(m) &\rightsquigarrow a_{442}(m)a_{343}(m)a_{321}(m+1)\cdots a_{321}(2)a_{331}(m)\cdots a_{331}(2)a_{341}(m)\cdots a_{341}(2)b_{31}b_{32}b_{33}, \\
a_{343}(m) &\rightsquigarrow a_{443}(m)a_{344}(m)a_{321}(m+1)\cdots a_{321}(2)a_{331}(m+1)\cdots a_{331}(2)a_{341}(m)\cdots a_{341}(2)b_{31}b_{32}b_{33}, \\
a_{344}(m) &\rightsquigarrow a_{444}(m)a_{321}(m+1)\cdots a_{321}(2)a_{331}(m+1)\cdots a_{331}(2)a_{341}(m+1)\cdots a_{341}(2)b_{31}b_{32}b_{33}, \\
a_{421}(m) &\rightsquigarrow a_{422}(m)a_{421}(m)\cdots a_{421}(2)a_{431}(m-1)\cdots a_{431}(2)a_{441}(m-1)\cdots a_{441}(2)b_{41}b_{42}b_{43}b_{44}, \\
a_{422}(m) &\rightsquigarrow a_{423}(m)a_{421}(m)\cdots a_{421}(2)a_{431}(m)\cdots a_{431}(2)a_{441}(m-1)\cdots a_{441}(2)b_{41}b_{42}b_{43}b_{44}, \\
a_{423}(m) &\rightsquigarrow a_{424}(m)a_{421}(m)\cdots a_{421}(2)a_{431}(m)\cdots a_{431}(2)a_{441}(m)\cdots a_{441}(2)b_{41}b_{42}b_{43}b_{44}, \\
a_{424}(m) &\rightsquigarrow a_{421}(m+1)\cdots a_{421}(2)a_{431}(m)\cdots a_{431}(2)a_{441}(m)\cdots a_{441}(2)b_{41}b_{42}b_{43}b_{44}, \\
a_{431}(m) &\rightsquigarrow a_{432}(m)a_{421}(m)\cdots a_{421}(2)a_{431}(m)\cdots a_{431}(2)a_{441}(m-1)\cdots a_{441}(2)b_{41}b_{42}b_{43}b_{44}, \\
a_{432}(m) &\rightsquigarrow a_{433}(m)a_{421}(m)\cdots a_{421}(2)a_{431}(m)\cdots a_{431}(2)a_{441}(m)\cdots a_{441}(2)b_{41}b_{42}b_{43}b_{44}, \\
a_{433}(m) &\rightsquigarrow a_{434}(m)a_{421}(m+1)\cdots a_{421}(2)a_{431}(m)\cdots a_{431}(2)a_{441}(m)\cdots a_{441}(2)b_{41}b_{42}b_{43}b_{44}, \\
a_{434}(m) &\rightsquigarrow a_{421}(m+1)\cdots a_{421}(2)a_{431}(m+1)\cdots a_{431}(2)a_{441}(m)\cdots a_{441}(2)b_{41}b_{42}b_{43}b_{44}, \\
a_{441}(m) &\rightsquigarrow a_{442}(m)a_{421}(m)\cdots a_{421}(2)a_{431}(m)\cdots a_{431}(2)a_{441}(m)\cdots a_{441}(2)b_{41}b_{42}b_{43}b_{44}, \\
a_{442}(m) &\rightsquigarrow a_{443}(m)a_{421}(m+1)\cdots a_{421}(2)a_{431}(m)\cdots a_{431}(2)a_{441}(m)\cdots a_{441}(2)b_{41}b_{42}b_{43}b_{44}, \\
a_{443}(m) &\rightsquigarrow a_{444}(m)a_{421}(m+1)\cdots a_{421}(2)a_{431}(m+1)\cdots a_{431}(2)a_{441}(m)\cdots a_{441}(2)b_{41}b_{42}b_{43}b_{44}, \\
a_{444}(m) &\rightsquigarrow a_{421}(m+1)\cdots a_{421}(2)a_{431}(m+1)\cdots a_{431}(2)a_{441}(m+1)\cdots a_{441}(2)b_{41}b_{42}b_{43}b_{44},
\end{aligned}$$

where $b_{ij} = 0^i j$ and $a_{ijk}(m) = 0^i 1^4 \cdots (m-2)^4 (m-1)^j m^k$ for all $m \geq 2$, $1 \leq i, k \leq 4$ and $j = 2, 3, 4$.

Define

$$\begin{aligned}
A_r &= A_r(x) = \sum_{n \geq 0} (\text{number nodes at level } n \text{ in } \mathcal{T}'(B; r))x^{n+1}, \\
B_{ij} &= B_{ij}(x) = \sum_{n \geq 0} (\text{number nodes at level } n \text{ in } \mathcal{T}'(B; b_{ij}))x^{n+1}, \\
A_{ijk}(v) &= A_{ijk}(x; v) = \sum_{n \geq 0} \sum_{m \geq 2} (\text{number nodes at level } n \text{ in } \mathcal{T}'(B; a_{ijk}(m)))x^{n+2}v^{m-2}.
\end{aligned}$$

Then, by translating each rule of the generating tree $T'(B)$ into an equation for the generating functions, we obtain the following equations (we distinguish between three sets of equations):

System S1:

$$\begin{aligned}
A_0 &= x + xA_{02} + xB_{11}, \\
A_{02} &= x + xA_{03} + xB_{21} + B_{22}, \\
B_{11} &= x + xB_{21} + 01^2 + xB_{11}, \\
A_{000} &= x + xA_{04} + xB_{31} + xB_{32} + xB_{33}, \\
B_{21} &= x + xA_{31} + xA_{0212} + xB_{21} + xB_{22}, \\
B_{22} &= x + xB_{32} + xA_{0222} + xB_{22}, \\
A_{012} &= x + xA_{0212} + xA_{013} + xA_{121}(2) + xB_{11}, \\
A_{04} &= x + xB_{41} + xB_{42} + xB_{43} + xB_{44}, \\
B_{31} &= x + xB_{41} + xA_{0312} + xB_{31} + xB_{32} + xB_{33}, \\
B_{32} &= x + xB_{42} + xA_{0322} + xB_{32} + xB_{33}, \\
B_{33} &= x + xB_{43} + xA_{0332} + xB_{33}, \\
A_{0212} &= x + xA_{0312} + xA_{0213} + xA_{221}(2) + xB_{21} + xB_{22}, \\
A_{0222} &= x + xA_{0322} + xA_{0223} + xB_{21} + xB_{22}, \\
A_{013} &= x + xA_{0213} + xA_{014} + xA_{131}(2) + xA_{121}(2) + xB_{11}, \\
B_{41} &= x + xA_{0412} + xB_{41} + xB_{42} + xB_{43} + xB_{44}, \\
B_{42} &= x + xA_{0422} + xB_{42} + xB_{43} + xB_{44}, \\
B_{43} &= x + xA_{0432} + xB_{43} + xB_{44}, \\
B_{44} &= x + xA_{0442} + xB_{44}, \\
A_{0312} &= x + xA_{0412} + xA_{0313} + xA_{321}(2) + xB_{31} + xB_{32} + xB_{33}, \\
A_{0322} &= x + xA_{0422} + xA_{0323} + xB_{31} + xB_{32} + xB_{33}, \\
A_{0332} &= x + xA_{0432} + xA_{0333} + xB_{32} + xB_{33}, \\
A_{0213} &= x + xA_{0313} + xA_{0214} + xA_{231}(2) + xA_{221}(2) + xB_{21} + xB_{22}, \\
A_{0223} &= x + xA_{0323} + xA_{0224} + xA_{221}(2) + xB_{21} + xB_{22}, \\
A_{014} &= x + xA_{0214} + xA_{141}(2) + xA_{131}(2) + xA_{121}(2) + xB_{11}, \\
A_{0412} &= x + xA_{0413} + xA_{421}(2) + xB_{41} + xB_{42} + xB_{43} + xB_{44}, \\
A_{0422} &= x + xA_{0423} + xB_{41} + xB_{42} + xB_{43} + xB_{44}, \\
A_{0432} &= x + xA_{0433} + xB_{42} + xB_{43} + xB_{44}, \\
A_{0442} &= x + xA_{0443} + xB_{43} + xB_{44}, \\
A_{0313} &= x + xA_{0413} + xA_{0314} + xA_{331}(2) + xA_{321}(2) + xB_{31} + xB_{32} + xB_{33}, \\
A_{0323} &= x + xA_{0423} + xA_{0324} + xA_{321}(2) + xB_{31} + xB_{32} + xB_{33}, \\
A_{0333} &= x + xA_{0433} + xA_{0334} + xB_{31} + xB_{32} + xB_{33}, \\
A_{0214} &= x + xA_{0314} + xA_{241}(2) + xA_{231}(2) + xA_{221}(2) + xB_{21} + xB_{22}, \\
A_{0224} &= x + xA_{0324} + xA_{231}(2) + xA_{221}(2) + xB_{21} + xB_{22}, \\
A_{0413} &= x + xA_{0414} + xA_{431}(2) + xA_{421}(2) + xB_{41} + xB_{42} + xB_{43} + xB_{44}, \\
A_{0423} &= x + xA_{0424} + xA_{421}(2) + xB_{41} + xB_{42} + xB_{43} + xB_{44}, \\
A_{0433} &= x + xA_{0434} + xB_{41} + xB_{42} + xB_{43} + xB_{44}, \\
A_{0443} &= x + xA_{044} + xB_{42} + xB_{43} + xB_{44}, \\
A_{0314} &= x + xA_{0414} + xA_{341}(2) + xA_{331}(2) + xA_{321}(2) + xB_{31} + xB_{32} + xB_{33}, \\
A_{0324} &= x + xA_{0424} + xA_{331}(2) + xA_{321}(2) + xB_{31} + xB_{32} + xB_{33}, \\
A_{0334} &= x + xA_{0444} + xA_{321}(2) + xB_{31} + xB_{32} + xB_{33}, \\
A_{0414} &= x + xA_{441}(2) + xA_{431}(2) + xA_{421}(2) + xB_{41} + xB_{42} + xB_{43} + xB_{44},
\end{aligned}$$

$$\begin{aligned} A_{0^4 2^4} &= x + xA_{431}(2) + xA_{421}(2) + xB_{41} + xB_{42} + xB_{43} + xB_{44}, \\ A_{0^4 3^4} &= x + xA_{421}(2) + xB_{41} + xB_{42} + xB_{43} + xB_{44} \end{aligned}$$

and System S2:

$$\begin{aligned} A_{121}(v) &= \frac{x}{1-v} + xA_{221}(v) + xA_{122}(v) + \frac{x}{1-v}(A_{121}(v) + vA_{131}(v) + vA_{141}(v) + B_{11}), \\ A_{122}(v) &= \frac{x}{1-v} + xA_{222}(v) + xA_{123}(v) + \frac{x}{1-v}(A_{121}(v) + A_{131}(v) + vA_{141}(v) + B_{11}), \\ A_{123}(v) &= \frac{x}{1-v} + xA_{223}(v) + xA_{124}(v) + \frac{x}{1-v}(A_{121}(v) + A_{131}(v) + A_{141}(v) + B_{11}), \\ A_{124}(v) &= \frac{x}{1-v} + xA_{224}(v) + \frac{x}{v}(A_{121}(v) - A_{121}(0)) + \frac{x}{1-v}(A_{121}(v) + A_{131}(v) + A_{141}(v) + B_{11}), \\ A_{131}(v) &= \frac{x}{1-v} + xA_{231}(v) + xA_{132}(v) + \frac{x}{1-v}(A_{121}(v) + A_{131}(v) + vA_{141}(v) + B_{11}), \\ A_{132}(v) &= \frac{x}{1-v} + xA_{232}(v) + xA_{133}(v) + \frac{x}{1-v}(A_{121}(v) + A_{131}(v) + A_{141}(v) + B_{11}), \\ A_{133}(v) &= \frac{x}{1-v} + xA_{233}(v) + xA_{134}(v) + \frac{x}{v}(A_{121}(v) - A_{121}(0)) + \frac{x}{1-v}(A_{121}(v) + A_{131}(v) + A_{141}(v) + B_{11}), \\ A_{134}(v) &= \frac{x}{1-v} + xA_{234}(v) + \frac{x}{v}(A_{121}(v) + A_{131}(v) - A_{121}(0) - A_{131}(0)) + \frac{x}{1-v}(A_{121}(v) + A_{131}(v) + A_{141}(v) + B_{11}), \\ A_{141}(v) &= \frac{x}{1-v} + xA_{241}(v) + xA_{142}(v) + \frac{x}{1-v}(A_{121}(v) + A_{131}(v) + A_{141}(v) + B_{11}), \\ A_{142}(v) &= \frac{x}{1-v} + xA_{242}(v) + xA_{143}(v) + \frac{x}{v}(A_{121}(v) - A_{121}(0)) + \frac{x}{1-v}(A_{121}(v) + A_{131}(v) + A_{141}(v) + B_{11}), \\ A_{143}(v) &= \frac{x}{1-v} + xA_{243}(v) + xA_{144}(v) + \frac{x}{v}(A_{121}(v) + A_{131}(v) - A_{121}(0) - A_{131}(0)) + \frac{x}{1-v}(A_{121}(v) + A_{131}(v) + A_{141}(v) + B_{11}), \\ A_{144}(v) &= \frac{x}{1-v} + xA_{244}(v) + \frac{x}{v}(A_{121}(v) + A_{131}(v) + A_{141}(v) - A_{121}(0) - A_{131}(0) - A_{141}(0)) + \frac{x}{1-v}(A_{121}(v) + A_{131}(v) + A_{141}(v) + B_{11}), \\ A_{221}(v) &= \frac{x}{1-v} + xA_{321}(v) + xA_{222}(v) + \frac{x}{1-v}(A_{221}(v) + vA_{231}(v) + vA_{241}(v) + B_{21} + B_{22}), \\ A_{222}(v) &= \frac{x}{1-v} + xA_{322}(v) + xA_{223}(v) + \frac{x}{1-v}(A_{221}(v) + A_{231}(v) + vA_{241}(v) + B_{21} + B_{22}), \\ A_{223}(v) &= \frac{x}{1-v} + xA_{323}(v) + xA_{224}(v) + \frac{x}{1-v}(A_{221}(v) + A_{231}(v) + A_{241}(v) + B_{21} + B_{22}), \\ A_{224}(v) &= \frac{x}{1-v} + xA_{324}(v) + \frac{x}{v}(A_{221}(v) - A_{221}(0)) + \frac{x}{1-v}(A_{221}(v) + A_{231}(v) + A_{241}(v) + B_{21} + B_{22}), \\ A_{231}(v) &= \frac{x}{1-v} + xA_{331}(v) + xA_{232}(v) + \frac{x}{1-v}(A_{221}(v) + A_{231}(v) + vA_{241}(v) + B_{21} + B_{22}), \\ A_{232}(v) &= \frac{x}{1-v} + xA_{332}(v) + xA_{233}(v) + \frac{x}{1-v}(A_{221}(v) + A_{231}(v) + A_{241}(v) + B_{21} + B_{22}), \\ A_{233}(v) &= \frac{x}{1-v} + xA_{333}(v) + xA_{234}(v) + \frac{x}{v}(A_{221}(v) - A_{221}(0)) + \frac{x}{1-v}(A_{221}(v) + A_{231}(v) + A_{241}(v) + B_{21} + B_{22}), \\ A_{234}(v) &= \frac{x}{1-v} + xA_{334}(v) + \frac{x}{v}(A_{221}(v) + A_{231}(v) - A_{221}(0) - A_{231}(0)) + \frac{x}{1-v}(A_{221}(v) + A_{231}(v) + A_{241}(v) + B_{21} + B_{22}), \\ A_{241}(v) &= \frac{x}{1-v} + xA_{341}(v) + xA_{242}(v) + \frac{x}{1-v}(A_{221}(v) + A_{231}(v) + A_{241}(v) + B_{21} + B_{22}), \\ A_{242}(v) &= \frac{x}{1-v} + xA_{342}(v) + xA_{243}(v) + \frac{x}{v}(A_{221}(v) - A_{221}(0)) + \frac{x}{1-v}(A_{221}(v) + A_{231}(v) + A_{241}(v) + B_{21} + B_{22}), \\ A_{243}(v) &= \frac{x}{1-v} + xA_{343}(v) + xA_{244}(v) + \frac{x}{v}(A_{221}(v) + A_{231}(v) - A_{221}(0) - A_{231}(0)) + \frac{x}{1-v}(A_{221}(v) + A_{231}(v) + A_{241}(v) + B_{21} + B_{22}), \\ A_{244}(v) &= \frac{x}{1-v} + xA_{344}(v) + \frac{x}{v}(A_{221}(v) + A_{231}(v) + A_{241}(v) - A_{221}(0) - A_{231}(0) - A_{241}(0)) + \frac{x}{1-v}(A_{221}(v) + A_{231}(v) + A_{241}(v) + B_{21} + B_{22}), \\ A_{321}(v) &= \frac{x}{1-v} + xA_{421}(v) + xA_{322}(v) + \frac{x}{1-v}(A_{321}(v) + vA_{331}(v) + vA_{341}(v) + B_{31} + B_{32} + B_{33}), \\ A_{322}(v) &= \frac{x}{1-v} + xA_{422}(v) + xA_{323}(v) + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + cA_{341}(v) + B_{31} + B_{32} + B_{33}), \\ A_{323}(v) &= \frac{x}{1-v} + xA_{423}(v) + xA_{324}(v) + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{341}(v) + B_{31} + B_{32} + B_{33}), \\ A_{324}(v) &= \frac{x}{1-v} + xA_{424}(v) + \frac{x}{v}(A_{321}(v) - A_{321}(0)) + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{341}(v) + B_{31} + B_{32} + B_{33}), \\ A_{331}(v) &= \frac{x}{1-v} + xA_{431}(v) + xA_{332}(v) + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + vA_{341}(v) + B_{31} + B_{32} + B_{33}), \\ A_{332}(v) &= \frac{x}{1-v} + xA_{432}(v) + xA_{333}(v) + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{341}(v) + B_{31} + B_{32} + B_{33}), \\ A_{333}(v) &= \frac{x}{1-v} + xA_{433}(v) + xA_{334}(v) + \frac{x}{v}(A_{321}(v) - A_{321}(0)) + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{341}(v) + B_{31} + B_{32} + B_{33}), \\ A_{334}(v) &= \frac{x}{1-v} + xA_{434}(v) + \frac{x}{v}(A_{321}(v) + A_{331}(v) - A_{321}(0) - A_{331}(0)) + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{341}(v) + B_{31} + B_{32} + B_{33}), \\ A_{341}(v) &= \frac{x}{1-v} + xA_{441}(v) + xA_{342}(v) + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{341} * v) + B_{31} + B_{32} + B_{33}, \end{aligned}$$

$$\begin{aligned}
A_{342}(v) &= \frac{x}{1-v} + xA_{442}(v) + xA_{343}(v) + \frac{x}{v}(A_{321}(v) - A_{321}(0)) + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{341}(v) + B_{31} + B_{32} + B_{33}), \\
A_{343}(v) &= \frac{x}{1-v} + xA_{443}(v) + xA_{344}(v) + \frac{x}{v}(A_{321}(v) + A_{331}(v) - A_{321}(0) - A_{331}(0)) + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{341}(v) + B_{31} + B_{32} + B_{33}), \\
A_{344}(v) &= \frac{x}{1-v} + xA_{444}(v) + \frac{x}{v}(A_{321}(v) + A_{331}(v) + A_{341}(v) - A_{321}(0) - A_{331}(0) - A_{341}(0)) + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{341}(v) + B_{31} + B_{32} + B_{33}), \\
A_{421}(v) &= \frac{x}{1-v} + xA_{422}(v) + \frac{x}{1-v}(A_{421}(v) + vA_{431}(v) + vA_{441}(v) + B_{41} + B_{42} + B_{43} + B_{44}), \\
A_{422}(v) &= \frac{x}{1-v} + xA_{423}(v) + \frac{x}{1-v}(A_{421}(v) + A_{431}(v) + vA_{441}(v) + B_{41} + B_{42} + B_{43} + B_{44}), \\
A_{423}(v) &= \frac{x}{1-v} + xA_{424}(v) + \frac{x}{1-v}(A_{421}(v) + A_{431}(v) + A_{441}(v) + B_{41} + B_{42} + B_{43} + B_{44}), \\
A_{424}(v) &= \frac{x}{1-v} + \frac{x}{v}(A_{421}(v) - A_{421}(0)) + \frac{x}{1-v}(A_{421}(v) + A_{431}(v) + A_{441}(v) + B_{41} + B_{42} + B_{43} + B_{44}), \\
A_{431}(v) &= \frac{x}{1-v} + xA_{432}(v) + \frac{x}{1-v}(A_{421}(v) + A_{431}(v) + vA_{441}(v) + B_{41} + B_{42} + B_{43} + B_{44}), \\
A_{432}(v) &= \frac{x}{1-v} + xA_{433}(v) + \frac{x}{1-v}(A_{421}(v) + A_{431}(v) + A_{441}(v) + B_{41} + B_{42} + B_{43} + B_{44}), \\
A_{433}(v) &= \frac{x}{1-v} + xA_{434}(v) + \frac{x}{v}(A_{421}(v) - A_{421}(0)) + \frac{x}{1-v}(A_{421}(v) + A_{431}(v) + A_{441}(v) + B_{41} + B_{42} + B_{43} + B_{44}), \\
A_{434}(v) &= \frac{x}{1-v} + \frac{x}{v}(A_{421}(v) + A_{431}(v) - A_{421}(0) - A_{431}(0)) + \frac{x}{1-v}(A_{421}(v) + A_{431}(v) + A_{441}(v) + B_{41} + B_{42} + B_{43} + B_{44}), \\
A_{441}(v) &= \frac{x}{1-v} + xA_{442}(v) + \frac{x}{1-v}(A_{421}(v) + A_{431}(v) + A_{441}(v) + B_{41} + B_{42} + B_{43} + B_{44}), \\
A_{442}(v) &= \frac{x}{1-v} + xA_{443}(v) + \frac{x}{v}(A_{421}(v) - A_{421}(0)) + \frac{x}{1-v}(A_{421}(v) + A_{431}(v) + A_{441}(v) + B_{41} + B_{42} + B_{43} + B_{44}), \\
A_{443}(v) &= \frac{x}{1-v} + xA_{444}(v) + \frac{x}{v}(A_{421}(v) + A_{431}(v) - A_{421}(0) - A_{431}(0)) + \frac{x}{1-v}(A_{421}(v) + A_{431}(v) + A_{441}(v) + B_{41} + B_{42} + B_{43} + B_{44}), \\
A_{444}(v) &= \frac{x}{1-v} + \frac{x}{v}(A_{421}(v) + A_{431}(v) + A_{441}(v) - A_{421}(0) - A_{431}(0) - A_{441}(0)) + \frac{x}{1-v}(A_{421}(v) + A_{431}(v) + A_{441}(v) + B_{41} + B_{42} + B_{43} + B_{44}).
\end{aligned}$$

By solving S1+S2, we obtain the following result.

Theorem 1. *The generating function $x^{10}(1-x^2-x^3)(125x^4+50x^3+15x^2+94x-27)^3 F_{\{021,00000\}}(x)$ is given by*

$$\begin{aligned}
&x^8(3515625x^{16} - 3125000x^{15} - 9546875x^{14} + 5000000x^{13} - 11446875x^{12} - 12779875x^{11} \\
&\quad + 14491750x^{10} - 9406595x^9 - 1595415x^8 + 13143527x^7 - 7338375x^6 + 1788936x^5 \\
&\quad + 2704378x^4 - 2839572x^3 + 1009781x^2 - 154080x + 8505) + x^4(8203125x^{17} + 468750x^{16} \\
&\quad - 3203125x^{15} + 25915625x^{14} - 10040625x^{13} - 4703250x^{12} + 33813225x^{11} - 28404825x^{10} \\
&\quad + 3265507x^9 + 18069436x^8 - 25346008x^7 + 10525772x^6 + 1761959x^5 - 6890115x^4 \\
&\quad + 5088705x^3 - 1721699x^2 + 276030x - 17010)\rho + (5078125x^{18} - 78125x^{17} + 2921875x^{16} \\
&\quad + 22912500x^{15} - 11662500x^{14} + 5755875x^{13} + 26732525x^{12} - 37191910x^{11} + 13951143x^{10} \\
&\quad + 10453444x^9 - 30982743x^8 + 21158423x^7 - 3731245x^6 - 7215579x^5 + 8638210x^4 \\
&\quad - 4420744x^3 + 1173711x^2 - 157752x + 8505)\rho^2 + (-1953125x^{16} + 390625x^{15} + 4609375x^{14} \\
&\quad - 3546875x^{13} + 3059375x^{12} + 7703750x^{11} - 8943875x^{10} + 3558775x^9 + 3133860x^8 \\
&\quad - 8574039x^7 + 4400941x^6 - 253023x^5 - 2270761x^4 + 2081239x^3 - 777000x^2 + 132237x \\
&\quad - 8505)\rho^3,
\end{aligned}$$

where $x^{12} + x^8(x-3)\rho + x^4(x^2-3x+3)\rho^2 + (x^3+3x-1)\rho^3 + \rho^4 = 0$.

1.2. Case $\{021, 00001\}$. Let $a_{ijk}(m) = 0^i 1^4 \cdots (m-2)^4 (m-1)^j m^k$ for all $m \geq 2$, $1 \leq i, k \leq 4$ and $j = 2, 3$. The set of rules of the generating tree $\mathcal{T}'(\{021, 00001\})$ with root 0 at level 0 is given by

$$\begin{aligned}
0 &\rightsquigarrow 0^2, 01, & 0^2 &\rightsquigarrow 0^3, 0^2 1, 0^2 2, \\
01 &\rightsquigarrow 0^2 1, 01^2, 01, & 0^3 &\rightsquigarrow 0^4, 0^3 1, 0^3 2, 0^3 3,
\end{aligned}$$

$$\begin{aligned}
0^21 &\rightsquigarrow 0^31, 0^21^2, 0^21, 0^22, & 0^22 &\rightsquigarrow 0^32, 0^22, 0^22, \\
01^2 &\rightsquigarrow 0^21^2, 01^3, a_{121}(2), 01, & 0^4 &\rightsquigarrow 0^4, \\
0^31 &\rightsquigarrow 0^4, 0^31^2, 0^31, 0^32, 0^33, & 0^32 &\rightsquigarrow 0^4, 0^32^2, 0^32, 0^33, \\
0^33 &\rightsquigarrow 0^4, 0^33^2, 0^33, & 0^21^2 &\rightsquigarrow 0^31^2, 0^21^3, a_{221}(2), 0^21, 0^22, \\
0^20^2 &\rightsquigarrow 0^32^2, 0^22^3, 0^21, 0^22, & 0^31^2 &\rightsquigarrow 0^4, 0^31^3, a_{321}(2), 0^31, 0^32, 0^33, \\
01^3 &\rightsquigarrow 0^21^3, 01^4, a_{131}(2), a_{121}(2), 01, & 0^32^2 &\rightsquigarrow 0^4, 0^33^2, 0^32, 0^33, \\
0^32^2 &\rightsquigarrow 0^4, 0^32^3, 0^31, 0^32, 0^33, & 0^22^3 &\rightsquigarrow 0^32^3, 0^21^4, a_{221}(2), 0^21, 0^22, \\
0^21^3 &\rightsquigarrow 0^31^3, 0^21^4, a_{231}(2), a_{221}(2), 0^21, 0^22, & 0^31^3 &\rightsquigarrow 0^4, 0^31^4, a_{331}(2), a_{321}(2), 0^31, 0^32, 0^33, \\
01^4 &\rightsquigarrow 0^21^4, 01^4, & 0^33^2 &\rightsquigarrow 0^4, 0^31^4, 0^31, 0^32, 0^33, \\
0^32^3 &\rightsquigarrow 0^4, 0^31^4, a_{321}(2), 0^31, 0^32, 0^33, & 0^31^4 &\rightsquigarrow 0^4, 0^31^4, \\
0^21^4 &\rightsquigarrow 0^31^4, 0^21^4, &
\end{aligned}$$

$$\begin{aligned}
a_{121}(m) &\rightsquigarrow a_{221}(m), a_{122}(m), a_{121}(m), \dots, a_{121}(2), a_{131}(m-1), \dots, a_{131}(2), 01, \\
a_{122}(m) &\rightsquigarrow a_{222}(m), a_{123}(m), a_{121}(m), \dots, a_{121}(2), a_{131}(m), \dots, a_{131}(2), 01, \\
a_{123}(m) &\rightsquigarrow a_{223}(m), 01^4, a_{121}(m+1), \dots, a_{121}(2), a_{131}(m-1), \dots, a_{131}(2), 01, \\
a_{131}(m) &\rightsquigarrow a_{231}(m), a_{132}(m), a_{121}(m), \dots, a_{121}(2), a_{131}(m), \dots, a_{131}(2), 01, \\
a_{132}(m) &\rightsquigarrow a_{232}(m), a_{133}(m), a_{121}(m+1), \dots, a_{121}(2), a_{131}(m), \dots, a_{131}(2), 01, \\
a_{133}(m) &\rightsquigarrow a_{233}(m), 01^4, a_{121}(m+1), \dots, a_{121}(2), a_{131}(m+1), \dots, a_{131}(2), 01, \\
a_{221}(m) &\rightsquigarrow a_{321}(m), a_{222}(m), a_{221}(m), \dots, a_{221}(2), a_{231}(m-1), \dots, a_{231}(2), 0^21, 0^22, \\
a_{222}(m) &\rightsquigarrow a_{322}(m), a_{223}(m), a_{221}(m), \dots, a_{221}(2), a_{231}(m), \dots, a_{231}(2), 0^21, 0^22, \\
a_{223}(m) &\rightsquigarrow a_{323}(m), 0^21^4, a_{221}(m+1), \dots, a_{221}(2), a_{231}(m), \dots, a_{231}(2), 0^21, 0^22, \\
a_{231}(m) &\rightsquigarrow a_{331}(m), a_{232}(m), a_{221}(m), \dots, a_{221}(2), a_{231}(m), \dots, a_{231}(2), 0^21, 0^22, \\
a_{232}(m) &\rightsquigarrow a_{332}(m), a_{233}(m), a_{221}(m+1), \dots, a_{221}(2), a_{231}(m), \dots, a_{231}(2), 0^21, 0^22, \\
a_{233}(m) &\rightsquigarrow a_{333}(m), 0^21^4, a_{221}(m+1), \dots, a_{221}(2), a_{231}(m+1), \dots, a_{231}(2), 0^21, 0^22, \\
a_{321}(m) &\rightsquigarrow 0^4, a_{322}(m), a_{321}(m), \dots, a_{321}(2), a_{331}(m-1), \dots, a_{331}(2), 0^31, 0^32, 0^33, \\
a_{322}(m) &\rightsquigarrow 0^4, a_{323}(m), a_{321}(m), \dots, a_{321}(2), a_{331}(m), \dots, a_{331}(2), 0^31, 0^32, 0^33, \\
a_{323}(m) &\rightsquigarrow 0^4, 0^31^4, a_{321}(m+1), \dots, a_{321}(2), a_{331}(m), \dots, a_{331}(2), 0^31, 0^32, 0^33, \\
a_{331}(m) &\rightsquigarrow 0^4, a_{332}(m), a_{321}(m), \dots, a_{321}(2), a_{331}(m), \dots, a_{331}(2), 0^31, 0^32, 0^33, \\
a_{332}(m) &\rightsquigarrow 0^4, a_{333}(m), a_{321}(m+1), \dots, a_{321}(2), a_{331}(m), \dots, a_{331}(2), 0^31, 0^32, 0^33, \\
a_{333}(m) &\rightsquigarrow 0^4, 0^31^4, a_{321}(m+1), \dots, a_{321}(2), a_{331}(m+1), \dots, a_{331}(2), 0^31, 0^32, 0^33.
\end{aligned}$$

Based on the rules of the generating tree $T'(B)$, we obtain the following equations (we separate them into three sets of equations):

System S1: Define

$$A_r = A_r(x) = \sum_{n \geq 0} (\text{number nodes at level } n \text{ in } T'(B; r)) x^{n+1},$$

$$A_{ijk}(v) = A_{ijk}(x; v) = \sum_{n \geq 0} \sum_{m \geq 2} (\text{number nodes at level } n \text{ in } T'(B; a_{ijk}(m))) x^{n+2} v^{m-2}.$$

By rewriting the above rules in terms of A_r and $A_{ijk}(v)$, we obtain the following equations:

$$A_0 = x + xA_{02} + xA_{01},$$

$$A_{02} = x + xA_{03} + xA_{021} + xA_{022},$$

$$\begin{aligned}
A_{01} &= x + xA_{0^21} + xA_{01^2} + xA_{01}, \\
A_{0^3} &= x + xA_{0^4} + xA_{0^31} + xA_{0^32} + xA_{0^33}, \\
A_{0^21} &= x + xA_{0^31} + xA_{0^21^2} + xA_{0^21} + xA_{0^22}, \\
A_{0^22} &= x + xA_{0^32} + xA_{0^22^2} + xA_{0^22}, \\
A_{01^2} &= x + xA_{0^21^2} + xA_{01^3} + xA_{121}(0) + xA_{01}, \\
A_{0^4} &= x + xA_{0^4}, \\
A_{0^31} &= x + xA_{0^4} + xA_{0^31^2} + xA_{0^31} + xA_{0^32} + xA_{0^33}, \\
A_{0^32} &= x + xA_{0^4} + xA_{0^32^2} + xA_{0^32} + xA_{0^33}, \\
A_{0^33} &= x + xA_{0^4} + xA_{0^33^2} + xA_{0^33}, \\
A_{0^21^2} &= x + xA_{0^31^2} + xA_{0^21^3} + xA_{221}(0) + xA_{0^21} + xA_{0^22}, \\
A_{0^22^2} &= x + xA_{0^32^2} + xA_{0^22^2} + xA_{0^21} + xA_{0^22}, \\
A_{01^3} &= x + xA_{0^21^3} + xA_{01^4} + xA_{131}(0) + xA_{121}(0) + xA_{01}, \\
A_{0^31^2} &= x + xA_{0^4} + xA_{0^31^3} + xA_{321}(0) + xA_{0^31} + xA_{0^32} + xA_{0^33}, \\
A_{0^32^2} &= x + xA_{0^4} + xA_{0^32^2} + xA_{0^31} + xA_{0^32} + xA_{0^33}, \\
A_{0^33^2} &= x + xA_{0^4} + xA_{0^33^2} + xA_{0^32} + xA_{0^33}, \\
A_{0^21^3} &= x + xA_{0^31^3} + xA_{0^21^4} + xA_{231}(0) + xA_{221}(0) + xA_{0^21} + xA_{0^22}, \\
A_{0^22^3} &= x + xA_{0^32^3} + xA_{0^21^4} + xA_{221}(0) + xA_{0^21} + xA_{0^22}, \\
A_{01^4} &= x + xA_{0^21^4} + xA_{01^4}, \\
A_{0^31^3} &= x + xA_{0^4} + xA_{0^31^4} + xA_{331}(0) + xA_{321}(0) + xA_{0^31} + xA_{0^32} + xA_{0^33}, \\
A_{0^32^3} &= x + xA_{0^4} + xA_{0^31^4} + xA_{321}(0) + xA_{0^31} + xA_{0^32} + xA_{0^33}, \\
A_{0^33^3} &= x + xA_{0^4} + xA_{0^31^4} + xA_{0^31} + xA_{0^32} + xA_{0^33}, \\
A_{0^21^4} &= x + xA_{0^31^4} + xA_{0^21^4}, \\
A_{0^31^4} &= x + xA_{0^4} + xA_{0^31^4},
\end{aligned}$$

System S2:

$$\begin{aligned}
A_{121}(v) &= \frac{x}{1-v} + xA_{221}(v) + xA_{122}(v) + \frac{x}{1-v}(A_{121}(v) + vA_{131}(v) + A_{01}), \\
A_{122}(v) &= \frac{x}{1-v} + xA_{222}(v) + xA_{123}(v) + \frac{x}{1-v}(A_{131}(v) + A_{121}(v) + A_{01}), \\
A_{123}(v) &= \frac{x}{1-v}(1 + A_{01^4}) + xA_{223}(v) + \frac{x}{v}(A_{121}(v) - A_{121}(0)) \\
&\quad + \frac{x}{1-v}(A_{121}(v) + A_{131}(v) + A_{01}), \\
A_{131}(v) &= \frac{x}{1-v} + xA_{231}(v) + xA_{132}(v) + \frac{x}{1-v}(A_{131}(v) + A_{121}(v) + A_{01}), \\
A_{132}(v) &= \frac{x}{1-v} + xA_{232}(v) + xA_{133}(v) + \frac{x}{v}(A_{121}(v) - A_{121}(0)) \\
&\quad + \frac{x}{1-v}(A_{121}(v) + A_{131}(v) + A_{01}), \\
A_{133}(v) &= \frac{x}{1-v}(1 + A_{01^4}) + xA_{233}(v) + \frac{x}{v}(A_{131}(v) + A_{121}(v) - A_{131}(0) - A_{121}(0))
\end{aligned}$$

$$\begin{aligned}
& + \frac{x}{1-v}(A_{131}(v) + A_{121}(v) + A_{01}), \\
A_{221}(v) &= \frac{x}{1-v} + xA_{321}(v) + xA_{222}(v) + \frac{x}{1-v}(A_{221}(v) + vA_{231}(v) + A_{0^21} + A_{0^22}), \\
A_{222}(v) &= \frac{x}{1-v} + xA_{322}(v) + xA_{223}(v) + \frac{x}{1-v}(A_{231}(v) + A_{221}(v) + A_{0^21} + A_{0^22}), \\
A_{223}(v) &= \frac{x}{1-v}(1 + A_{0^214}) + xA_{323}(v) + \frac{x}{v}(A_{221}(v) - A_{221}(0)) \\
& + \frac{x}{1-v}(A_{221}(v) + A_{231}(v) + A_{0^21} + A_{0^22}), \\
A_{231}(v) &= \frac{x}{1-v} + xA_{331}(v) + xA_{232}(v) + \frac{x}{1-v}(A_{231}(v) + A_{221}(v) + A_{0^21} + A_{0^22}), \\
A_{232}(v) &= \frac{x}{1-v} + xA_{332}(v) + xA_{233}(v) + \frac{x}{v}(A_{221}(v) - A_{221}(0)) \\
& + \frac{x}{1-v}(A_{221}(v) + A_{231}(v) + A_{0^21} + A_{0^22}), \\
A_{233}(v) &= \frac{x}{1-v}(1 + A_{0^214}) + xA_{333}(v) + \frac{x}{v}(A_{231}(v) + A_{221}(v) - A_{231}(0) - A_{221}(0)) \\
& + \frac{x}{1-v}(A_{231}(v) + A_{221}(v) + A_{0^21} + A_{0^22}), \\
A_{321}(v) &= \frac{x}{1-v}(1 + A_{0^4}) + xA_{322}(v) + \frac{x}{1-v}(A_{321}(v) + vA_{331}(v) + A_{0^31} + A_{0^32} + A_{0^33}), \\
A_{322}(v) &= \frac{x}{1-v}(1 + A_{0^4}) + xA_{323}(v) + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{0^31} + A_{0^32} + A_{0^33}), \\
A_{323}(v) &= \frac{x}{1-v}(1 + A_{0^4} + A_{0^314}) + \frac{x}{v}(A_{321}(v) - A_{321}(0)) \\
& + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{0^31} + A_{0^32} + A_{0^33}), \\
A_{331}(v) &= \frac{x}{1-v}(1 + A_{0^4}) + xA_{332}(v) + \frac{x}{1-v}(A_{321}(v) + vA_{331}(v) + A_{0^31} + A_{0^32} + A_{0^33}), \\
A_{332}(v) &= \frac{x}{1-v}(1 + A_{0^4}) + xA_{333}(v) + \frac{x}{v}(A_{321}(v) - A_{321}(0)) \\
& + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{0^31} + A_{0^32} + A_{0^33}), \\
A_{333}(v) &= \frac{x}{1-v}(1 + A_{0^4} + A_{0^314}) + \frac{x}{v}(A_{331}(v) + A_{321}(v) - A_{331}(0) - A_{321}(0)) \\
& + \frac{x}{1-v}(A_{321}(v) + A_{331}(v) + A_{0^31} + A_{0^32} + A_{0^33}).
\end{aligned}$$

By solving S1+S2, we obtain an explicit formula for the generating function $F_{\{021,00001\}}(x)$.

Theorem 2. *The generating function $2x^5(1-x)^4(1+x+x^2)^3(16x^3+8x^2+11x-4)^2F_{\{021,00001\}}(x)$ is given by*

$$\begin{aligned}
& -2x^3(256x^{18} + 2560x^{17} - 608x^{16} + 3024x^{15} - 2039x^{14} + 6073x^{13} + 139x^{12} \\
& + 6412x^{11} - 839x^{10} + 4003x^9 - 2417x^8 + 1124x^7 - 1857x^6 + 1252x^5 - 13x^4 \\
& + 392x^3 - 143x^2 - 31x + 10) + (3072x^{17} - 8192x^{16} + 4480x^{15} - 9792x^{14} \\
& + 11772x^{13} - 16376x^{12} + 1204x^{11} - 13516x^{10} + 12020x^9 - 1260x^8 + 5094x^7
\end{aligned}$$

$$\begin{aligned}
& -6354x^6 + 462x^5 - 776x^4 + 972x^3 - 4x^2 - 124x + 20)\rho + 2(x-1)(1536x^{15} \\
& + 512x^{14} + 1984x^{13} - 32x^{12} + 3286x^{11} + 1532x^{10} + 2314x^9 - 1290x^8 - 265x^7 \\
& - 584x^6 + 975x^5 + 291x^4 + 194x^3 - 108x^2 - 32x + 10)\rho^2,
\end{aligned}$$

where $\rho^3 = x^6 + x^3(x-2)\rho + (1-2x-x^2)\rho^2 = 0$.

1.3. Case $\{021, 00011\}$.

Theorem 3. *We have*

$$\begin{aligned}
F_{\{021, 00011\}}(x) = & -\frac{(2-2x+3x^2)\sqrt{1-4x}+2x^3-8x^2+7x-2}{2x^3} \\
& + \frac{(1-x-3x^2)\sqrt{1-4x}-9x^3-3x^2+4x-1}{2x^3\sqrt{(1+x)(1-3x)}}.
\end{aligned}$$

Proof. Let $a_m = 0^m$, $b_m = 01^m$, $c_m = 001^m$, $d_m = 0^21^2..m^2$, $e_m = 0^21^2..(m-1)^2m$, $f_m = 01^22^2..m^2$, and $g_m = 01^22^2..(m-1)^2m$, for all $m \geq 1$. Then, with help of the algorithm we guess and then we prove that the generating tree $\mathcal{T}[\{021, 00011\}]$ has a root a_1 and satisfies the following rules

$$\begin{aligned}
a_0 &\rightsquigarrow a_{00}g_1, & a_{00} &\rightsquigarrow a_3e_1a_{002}, \\
a_{002} &\rightsquigarrow a_3a_{0022}a_{002}, & a_{0022} &\rightsquigarrow a_4a_{00222}e_1a_{002}, \\
a_{00222} &\rightsquigarrow a_5c_3a_3a_{0002}a_{0003}, & & \\
a_m &\rightsquigarrow a_{m+1}\cdots a_3a_{0002}a_{0003}, & b_m &\rightsquigarrow b_{m+1}c_ma_m\cdots a_3a_{0002}a_{0003}, \quad m \geq 3 \\
c_m &\rightsquigarrow a_{m+3}c_{m+1}a_{m+1}\cdots a_3a_{0002}a_{0003}, & d_m &\rightsquigarrow a_{m+4}c_{m+2}e_{m+1}\cdots e_1a_{002}, \quad m \geq 1 \\
e_m &\rightsquigarrow a_{m+3}d_me_m\cdots e_1a_{002}, & f_m &\rightsquigarrow d_mb_{m+2}g_{m+1}\cdots g_1, \quad m \geq 1 \\
g_m &\rightsquigarrow e_mf_mg_m\cdots g_1, \quad m \geq 1. & &
\end{aligned}$$

By translating these rules to generating functions as we did before, we obtain an explicit formula for the generating function $A_0(x)$ for the number of nodes in $\mathcal{T}[\{021, 00011\}]$ at level n , where the root stay at level 0. \square

1.4. Case $\{021, 00012\}$.

Theorem 4. *We have*

$$\begin{aligned}
F_{\{021, 00012\}}(x) = & \frac{4x^9 - 8x^8 - 8x^7 + 10x^6 - 18x^5 + 6x^4 + 8x^3 - 9x^2 + 4x - 1}{2x^2(1+x)^2(1-x)^4(1-2x)} \\
& + \frac{18x^7 - 24x^6 + 24x^5 + 8x^4 - 26x^3 + 17x^2 - 6x + 1}{2x^2(1+x)(1-x)^4(1-2x)\sqrt{(1+x)(1-3x)}}.
\end{aligned}$$

Proof. Let $a_m = 0^m$, $b_m = 01^m$, $c_m = 001^m$, $d_m = 0^21^2..m^2$, $e_m = 0^21^2..(m-1)^2m$, $f_m = 01^22^2..m^2$, and $g_m = 01^22^2..(m-1)^2m$, for all $m \geq 1$. The generating tree $\mathcal{T}[\{021, 00012\}]$ has the root a_1 and satisfies the following rules

$$\begin{aligned}
a_0 &\rightsquigarrow a_{00}g_1, & a_{00} &\rightsquigarrow a_3e_1a_{002}, \\
a_{002} &\rightsquigarrow a_3a_{0022}a_{002}, & a_{0022} &\rightsquigarrow a_4e_1a_{002}a_{0022}, \\
a_{00222} &\rightsquigarrow a_5c_3a_{0001}^3, & a_{0001} &\rightsquigarrow a_{0001}^2, \\
a_m &\rightsquigarrow a_{m+1}a_{0001}^m, & b_m &\rightsquigarrow b_{m+1}c_ma_{0001}^m,
\end{aligned}$$

$$\begin{aligned}
c_m &\rightsquigarrow a_{m+3}c_{m+1}a_{0001}^{m+1}, & d_m &\rightsquigarrow a_{m+4}c_{m+2}e_{m+1}\cdots e_1a_{002}, \\
e_m &\rightsquigarrow a_{m+3}d_m e_m \cdots e_1 a_{002}, & f_m &\rightsquigarrow d_m b_{m+2}g_{m+1}\cdots g_1, \\
g_m &\rightsquigarrow e_m f_m g_m \cdots g_1.
\end{aligned}$$

By translating these rules to generating functions, we obtain an explicit formula for the generating function $A_0(x)$ for the number of nodes in $\mathcal{T}[\{021, 00012\}]$ at level n , where the root stays at the level 0. \square

1.5. Case $B = \{021, \tau\}$, where $\tau \neq 0^5, 0^41, 0^311, 0^312$. Table 1 contains all the generating trees $\mathcal{T}'(\{021, \tau\})$ and the corresponding generating function $F_{\{021, \tau\}}(x)$, whenever τ is a 5-letter pattern avoids 021. In this table, we used the following notation: for any sequence b_m , we define \bar{b}_m to be $b_m b_{m-1} \cdots b_1$, for all $m \geq 1$.

Table 1: Generating functions $F_B(x)$ for pattern set $B = \{021, \tau\}$, where τ is any pattern of length five that avoids 021.

Beginning of Table 1			
τ	d	$F_{\{021, \tau\}}(x)$	Reference
00000	12		Thm. 1
00001	6		Thm. 2
00010	3	$a_1 \rightsquigarrow a\bar{2}c_1, a_2 \rightsquigarrow a_3a_2b_1, c_1 \rightsquigarrow b_1c_2c_1, b_1 \rightsquigarrow a_4b_2b_1a_2,$ $c_2 \rightsquigarrow b_2c_3c_2c_1, b_2 \rightsquigarrow a_5b_3b_2b_1a_2, a_m \rightsquigarrow a_{m+1}\cdots a_3de,$ $b_m \rightsquigarrow a_{m+3}\bar{b}_{m+1}a_2, c_m \rightsquigarrow b_m\bar{c}_{m+1}, d \rightsquigarrow a_3de, e \rightsquigarrow de,$ $a_m = 0^m, b_m = 001^m, c_m = 01^m, d = 0002, e = 0003$	Thm. 5(1)
00011	5		Thm. 3
00012	5		Thm. 4
00100	4	$a_1 \rightsquigarrow a2b_1, b_1 \rightsquigarrow a_3b_1d_2, a_m \rightsquigarrow a_{m+1}\bar{b}_m e, b_m \rightsquigarrow \bar{b}_{m+1}c_m e,$ $c_m \rightsquigarrow \bar{c}_{m+1}fg, d_m \rightsquigarrow a_{m+2}\bar{d}_{m+1}b_1, e \rightsquigarrow b_2^2eg, f \rightsquigarrow fg,$ $g \rightsquigarrow c_2fg, a_m = 0^m, b_m = 0^m1, c_m = 0^m10, d_m = 01^m,$ $e = 002, f = 00104, g = 0020$	Thm. 5(2)
00110	4	$a_1 \rightsquigarrow a2b_1, b_1 \rightsquigarrow a_3b_1d_2, a_m \rightsquigarrow a_{m+1}\bar{b}_m e, b_m \rightsquigarrow \bar{b}_{m+1}c_m e,$ $c_m \rightsquigarrow \bar{c}_{m+1}fg, d_m \rightsquigarrow a_{m+2}\bar{d}_{m+1}b_1, e \rightsquigarrow b_2^2eg, f \rightsquigarrow fg,$ $g \rightsquigarrow c_2fg, a_m = 0^m, b_m = 0^m1, c_m = 0^m11, d_m = 01^m,$ $e = 002, f = 00114, g = 0022$	
00101	3	$a_1 \rightsquigarrow a2b_1, b_1 \rightsquigarrow a_3b_1c_2, a_m \rightsquigarrow a_{m+1}\bar{b}_m e,$ $b_m \rightsquigarrow \bar{b}_{m+1}a_{m+1}e, c_m \rightsquigarrow \bar{c}_{m+1}a_{m+2}b_1, e \rightsquigarrow a_2b_2e,$ $a_m = 0^m, b_m = 0^m1, c_m = 01^m, e = 002$	Thm. 5(3)
00102	4	$a_1 \rightsquigarrow a2b_1, b_1 \rightsquigarrow a_3b_1d_2, a_m \rightsquigarrow a_{m+1}\bar{b}_m e,$ $b_m \rightsquigarrow b_{m+1}\bar{c}_m fh, c_m \rightsquigarrow \bar{c}_{m+1}fg, d_m \rightsquigarrow a_{m+2}\bar{d}_{m+1}b_1,$ $e \rightsquigarrow b_2fh, f \rightsquigarrow c_2fg, g \rightsquigarrow g, a_m = 0^m, b_m = 0^m1,$ $c_m = 0^m12, d_m = 01^m, e = 002, f = 0013, g = 00120,$ $h = 0010$	Thm. 5(4)
00111	3	$a_1 \rightsquigarrow a2b_1, b_1 \rightsquigarrow a_3b_1c_2, a_m \rightsquigarrow a_{m+1}\bar{b}_m e,$ $b_m \rightsquigarrow \bar{b}_{m+1}a_{m+1}e, c_m \rightsquigarrow c_{m+1}a_{m+2}\bar{b}_m e, e \rightsquigarrow a_2b_2e,$ $a_m = 0^m, b_m = 0^m1, c_m = 01^m, e = 002$	Thm. 5(5)
00112	3	$a_1 \rightsquigarrow a2b_1, b_1 \rightsquigarrow a_3b_1c_2, a_m \rightsquigarrow a_{m+1}\bar{b}_m e, b_m \rightsquigarrow \bar{b}_{m+1}ef,$ $c_m \rightsquigarrow c_{m+1}a_{m+2}\bar{b}_m e, e \rightsquigarrow b_2ef, f \rightsquigarrow f^2, a_m = 0^m,$ $b_m = 0^m1, c_m = 01^m, e = 002, f = 0011$	Thm. 5(6)
00120	4	$a_1 \rightsquigarrow a2b_1, b_1 \rightsquigarrow a_3b_1d_2, a_m \rightsquigarrow a_{m+1}\bar{b}_m e, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m f,$ $c_m \rightsquigarrow \bar{c}_{m+1}f, d_m \rightsquigarrow a_{m+2}\bar{d}_{m+1}b_1, e \rightsquigarrow b_2^2f, f \rightsquigarrow c_2f,$ $a_m = 0^m, b_m = 0^m1, c_m = 0^m12, d_m = 01^m, e = 002,$ $f = 0013$	Thm. 5(7)

Continuation of Table 1

τ	d	$F_{\{021,\tau\}}(x)$	Reference
00122	4	$a_1 \rightsquigarrow a_2 b_1, b_1 \rightsquigarrow a_3 b_1 d_2, a_m \rightsquigarrow a_{m+1} \bar{b}_m f, b_m \rightsquigarrow b_{m+1}^2 \bar{c}_m e,$ $c_m \rightsquigarrow \bar{c}_{m+1} e, d_m \rightsquigarrow a_{m+2} d_{m+1} \bar{b}_m f, e \rightsquigarrow c_2 e, f \rightsquigarrow b_2^2 e,$ $a_m = 0^m, b_m = 0^m 1, c_m = 0^m 12, d_m = 01^m, e = 0013, f = 002$	Thm. 5(8)
00123	3	$a_1 \rightsquigarrow a_2 b_1, b_1 \rightsquigarrow a_3 b_1 c_2, a_m \rightsquigarrow a_{m+1} \bar{b}_m d, b_m \rightsquigarrow b_{m+1}^2 e^m,$ $c_m \rightsquigarrow c_{m+1} a_{m+2} \bar{b}_m d, d \rightsquigarrow b_2 e, e \rightsquigarrow e^2, a_m = 0^m,$ $b_m = 0^m 1, c_m = 01^m, d = 002, e = 0012$	Thm. 5(9)
01000	4	$a_m \rightsquigarrow a_{m+1} \bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1} c_m, c_m \rightsquigarrow \bar{c}_{m+1} d_m e,$ $d_m \rightsquigarrow \bar{d}_{m+1} f g, e \rightsquigarrow c_1 e f, f \rightsquigarrow d_1 f g, g \rightsquigarrow f g, a_m = 0^m,$ $b_m = 0^m 1, c_m = 0^m 10, d_m = 0^m 100, e = 0103, f = 01003, g = 01004$	Thm. 5(10)
01100	4	$a_m \rightsquigarrow a_{m+1} \bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1} c_m, c_m \rightsquigarrow \bar{c}_{m+1} d_m e,$ $d_m \rightsquigarrow \bar{d}_{m+1} f g, e \rightsquigarrow c_1 e f, f \rightsquigarrow d_1 f g, g \rightsquigarrow f g, a_m = 0^m,$ $b_m = 0^m 1, c_m = 0^m 11, d_m = 0^m 110, e = 0113, f = 01103, g = 01104$	
01001	3		
01011	3	$a_m \rightsquigarrow a_{m+1} \bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1} c_m, c_m \rightsquigarrow c_{m+1} \bar{b}_{m+1} a_{m+2},$ $d \rightsquigarrow d^2, a_m = 0^m, b_m = 0^m 1, c_m = 0^m 10$	
01101	3		
01111	3	$a_m \rightsquigarrow a_{m+1} \bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1} c_m, c_m \rightsquigarrow c_{m+1} \bar{b}_{m+1} a_{m+2},$ $d \rightsquigarrow d^2, a_m = 0^m, b_m = 0^m 1, c_m = 0^m 11$	Thm. 5(11)
01002	5	$a_m \rightsquigarrow a_{m+1} \bar{b}_m, b_m \rightsquigarrow b_{m+1} c_m \bar{d}_m, c_m \rightsquigarrow c_{m+1} \bar{e}_m f g,$ $d_m \rightsquigarrow \bar{d}_{m+1} e_m, e_m \rightsquigarrow \bar{e}_{m+1} f h, f \rightsquigarrow e_1 f h, g \rightsquigarrow g^2, h \rightsquigarrow h,$ $a_m = 0^m, b_m = 0^m 1, c_m = 0^m 10, d_m = 0^m 12,$ $e_m = 0^m 102, f = 0103, g = 0100, h = 01020$	Thm. 5(12)
01010	4	$a_m \rightsquigarrow a_{m+1} \bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1} c_m, c_m \rightsquigarrow c_{m+1} \bar{b}_{m+1} d_m,$ $d_m \rightsquigarrow \bar{d}_{m+1} e f, e \rightsquigarrow d_1 e f, f \rightsquigarrow e f, a_m = 0^m, b_m = 0^m 1,$ $c_m = 0^m 10, d_m = 0^m 101, e = 01013, f = 01014$	Thm. 5(13)
01110	4	$a_m \rightsquigarrow a_{m+1} \bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1} c_m, c_m \rightsquigarrow c_{m+1} \bar{b}_{m+1} d_m,$ $d_m \rightsquigarrow \bar{d}_{m+1} e f, e \rightsquigarrow d_1 e f, f \rightsquigarrow e f, a_m = 0^m, b_m = 0^m 1,$ $c_m = 0^m 10, d_m = 0^m 101, e = 01013, f = 01014$	
01012	3	$a_m \rightsquigarrow a_{m+1} \bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1} c_m, c_m \rightsquigarrow c_{m+1} \bar{b}_{m+1} d, d \rightsquigarrow d^2,$ $a_m = 0^m, b_m = 0^m 1, c_m = 0^m 10, d = 0101$	
01112	3	$a_m \rightsquigarrow a_{m+1} \bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1} c_m, c_m \rightsquigarrow c_{m+1} \bar{b}_{m+1} d, d \rightsquigarrow d^2,$ $a_m = 0^m, b_m = 0^m 1, c_m = 0^m 11, d = 0111$	Thm. 5(14)
01020	5		
01022	5	$a_m \rightsquigarrow a_{m+1} \bar{b}_m, b_m \rightsquigarrow b_{m+1} c_m \bar{d}_m, c_m \rightsquigarrow c_{m+1}^2 \bar{e}_{m-1} f g,$ $d_m \rightsquigarrow \bar{d}_{m+1} e_m, e_m \rightsquigarrow \bar{e}_{m+1} f g, f \rightsquigarrow e_1 f g, g \rightsquigarrow f g,$ $a_m = 0^m, b_m = 0^m 1, c_m = 0^m 10, d_m = 0^m 12,$ $e_m = 0^m 120, f = 0102, g = 0103$	Thm. 5(15)
01023	5	$a_m \rightsquigarrow a_{m+1} \bar{b}_m, b_m \rightsquigarrow b_{m+1} c_m \bar{d}_m, c_m \rightsquigarrow c_{m+1}^2 f^{m+1},$ $d_m \rightsquigarrow \bar{d}_{m+1} e_m, e_m \rightsquigarrow e_{m+1} f^{m+2}, f \rightsquigarrow f^2, a_m = 0^m,$ $b_m = 0^m 1, c_m = 0^m 10, d_m = 0^m 12, e_m = 0^m 120, f = 0102$	Thm. 5(16)
01102	4	$a_m \rightsquigarrow a_{m+1} \bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1} c_m, c_m \rightsquigarrow c_{m+1} \bar{d}_m e f,$ $d_m \rightsquigarrow \bar{d}_{m+1} e g, e \rightsquigarrow d_1 e g, f \rightsquigarrow f^2, g \rightsquigarrow g, a_m = 0^m,$ $b_m = 0^m 1, c_m = 0^m 11, d_m = 0^m 112, e = 0113, f = 0110,$ $g = 01120$	Thm. 5(17)
01120	4		

Continuation of Table 1

τ	d	$F_{\{021,\tau\}}(x)$	Reference
01122	4	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}^2\bar{d}_m e,$ $d_m \rightsquigarrow \bar{d}_{m+1}e, e \rightsquigarrow d_1e, a_m = 0^m, b_m = 0^m1, c_m = 0^m12,$ $d_m = 0^m112, e = 0113$	Thm. 5(18)
01123	3	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}d^{m+1}, d \rightsquigarrow d^2,$ $a_m = 0^m, b_m = 0^m1, c_m = 0^m11, d = 0112$	Thm. 5(19)
01200	4	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow \bar{c}_{m+1}d_m,$ $d_m \rightsquigarrow \bar{d}_{m+1}e, e \rightsquigarrow d_1e, a_m = 0^m, b_m = 0^m1, c_m = 0^m12,$ $d_m = 0^m120, e = 01204$	Thm. 5(20)
	4	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow \bar{c}_{m+1}d_m,$ $d_m \rightsquigarrow \bar{d}_{m+1}e, e \rightsquigarrow d_1e, a_m = 0^m, b_m = 0^m1, c_m = 0^m12,$ $d_m = 0^m122, e = 01224$	
01202	4	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow c_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}\bar{c}_{m+1}, a_m = 0^m, b_m = 0^m1, c_m = 0^m12,$ $d = 0^m120$	Thm. 5(21)
	4	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow c_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}\bar{c}_{m+1}, a_m = 0^m, b_m = 0^m1, c_m = 0^m12,$ $d = 0^m122$	
01203	4	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow c_{m+1}\bar{d}_m e,$ $d_m \rightsquigarrow \bar{d}_{m+1}f, e \rightsquigarrow e^2, f \rightsquigarrow f, a_m = 0^m, b_m = 0^m1,$ $c_m = 0^m12, d_m = 0^m123, e = 0120, f = 01230$	Thm. 5(22)
01223	3	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow c_{m+1}d, d \rightsquigarrow d^2,$ $a_m = 0^m, b_m = 0^m1, c_m = 0^m12, d = 0122$	Thm. 5(23)
01230	4	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow c_{m+1}^2d^m,$ $d_m \rightsquigarrow \bar{d}_{m+1}, a_m = 0^m, b_m = 0^m1, c_m = 0^m12,$ $d_m = 0^m123$	Thm. 5(24)
01233			
01234	3	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow c_{m+1}^2d^m, d \rightsquigarrow d^2,$ $a_m = 0^m, b_m = 0^m1, c_m = 0^m12, d = 0123$	Thm. 5(25)
10000	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow \bar{c}_{m+1}d_m f,$ $d_m \rightsquigarrow \bar{d}_{m+1}e_m g h, e_m \rightsquigarrow \bar{e}_{m+1}j k l, f \rightsquigarrow c_1 f g, g \rightsquigarrow d_1 g h j,$ $h \rightsquigarrow g h k, j \rightsquigarrow e_1 j k l, k \rightsquigarrow j k l, l \rightsquigarrow k l, a_m = 0^m, b_m = 0^m1,$ $c_m = 0^m11, d_m = 0^m100, e_m = 0^m1000, f = 0103,$ $g = 01003, h = 01004, j = 010003, l = 010004, k = 010005$	Thm. 5(26)
	5		
11000	5	$a_m \rightsquigarrow a_{m+1}b_m, b_m \rightsquigarrow b_{m+1}c_m, c_m \rightsquigarrow \bar{c}_{m+1}d_m f,$ $d_m \rightsquigarrow \bar{d}_{m+1}e_m g h, e_m \rightsquigarrow \bar{e}_{m+1}j k l, f \rightsquigarrow c_1 f g, g \rightsquigarrow d_1 g h j,$ $h \rightsquigarrow g h k, j \rightsquigarrow e_1 j k l, k \rightsquigarrow j k l, l \rightsquigarrow k l, a_m = 0^m, b_m = 0^m1,$ $c_m = 0^m11, d_m = 0^m110, e_m = 0^m1100, f = 0113,$ $g = 01103, h = 01104, j = 011003, l = 011004, k = 011005$	Thm. 5(26)
10001	4	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}a_{m+3}\bar{b}_{m+1}, e_m \rightsquigarrow \bar{e}_{m+1}g h j, f \rightsquigarrow f^2,$ $g \rightsquigarrow e_1 g h j, h \rightsquigarrow g h j, j \rightsquigarrow j, a_m = 0^m, b_m = 0^m1,$ $c_m = 0^m10, d_m = 0^m100$	
10011	4		
10101	4		
10111	4	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}a_{m+3}\bar{b}_{m+1}, e_m \rightsquigarrow \bar{e}_{m+1}g h j, f \rightsquigarrow f^2,$ $g \rightsquigarrow e_1 g h j, h \rightsquigarrow g h j, j \rightsquigarrow j, a_m = 0^m, b_m = 0^m1,$ $c_m = 0^m10, d_m = 0^m101$	
11001	4		

Continuation of Table 1

τ	d	$F_{\{021,\tau\}}(x)$	Reference
11011	4	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}a_{m+3}\bar{b}_{m+1}, e_m \rightsquigarrow \bar{e}_{m+1}ghj, f \rightsquigarrow f^2,$ $g \rightsquigarrow e_1ghj, h \rightsquigarrow ghj, j \rightsquigarrow j, a_m = 0^m, b_m = 0^m1,$ $c_m = 0^m11, d_m = 0^m110$	Thm. 5(27)
11101	4	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}a_{m+3}\bar{b}_{m+1}, e_m \rightsquigarrow \bar{e}_{m+1}ghj, f \rightsquigarrow f^2,$ $g \rightsquigarrow e_1ghj, h \rightsquigarrow ghj, j \rightsquigarrow j, a_m = 0^m, b_m = 0^m1,$ $c_m = 0^m11, d_m = 0^m111$	
10002	7	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}c_m\bar{d}_m, c_m \rightsquigarrow c_{m+1}e_m\bar{f}_mh,$ $d_m \rightsquigarrow \bar{d}_{m+1}f_m, e_m \rightsquigarrow e_{m+1}\bar{g}_{m+jkl}, f_m \rightsquigarrow \bar{f}_{m+1}g_mh,$ $g_m \rightsquigarrow g_{m+1}jkp, h \rightsquigarrow f_1jh, j \rightsquigarrow g_1jkp, k \rightsquigarrow jkp, l \rightsquigarrow l^2,$ $p \rightsquigarrow p, a_m = 0^m, b_m = 0^m1, c_m = 0^m10, d_m = 0^m12,$ $e_m = 0^m100, f_m = 0^m102, g_m = 0^m1002, h = 0103,$ $j = 01003, k = 01004, l = 01000, p = 010020$	Thm. 5(28)
10010	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}e_m\bar{b}_{m+2}, e_m \rightsquigarrow \bar{e}_{m+1}fgh, f \rightsquigarrow e_1fgh,$ $g \rightsquigarrow fgh, h \rightsquigarrow gh, a_m = 0^m, b_m = 0^m1, c_m = 0^m10,$ $d_m = 0^m100, e_m = 0^m1001, f = 010013, g = 010014,$ $h = 010015$	Thm. 5(29)
10110	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}e_m\bar{b}_{m+2}, e_m \rightsquigarrow \bar{e}_{m+1}fgh, f \rightsquigarrow e_1fgh,$ $g \rightsquigarrow fgh, h \rightsquigarrow gh, a_m = 0^m, b_m = 0^m1, c_m = 0^m10,$ $d_m = 0^m101, e_m = 0^m1011, f = 010113, g = 010114,$ $h = 010115$	
11010	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}e_m\bar{b}_{m+2}, e_m \rightsquigarrow \bar{e}_{m+1}fgh, f \rightsquigarrow e_1fgh,$ $g \rightsquigarrow fgh, h \rightsquigarrow gh, a_m = 0^m, b_m = 0^m1, c_m = 0^m11,$ $d_m = 0^m110, e_m = 0^m1101, f = 011013, g = 011014,$ $h = 011015$	
11110	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}e_m\bar{b}_{m+2}, e_m \rightsquigarrow \bar{e}_{m+1}fgh, f \rightsquigarrow e_1fgh,$ $g \rightsquigarrow fgh, h \rightsquigarrow gh, a_m = 0^m, b_m = 0^m1, c_m = 0^m11,$ $d_m = 0^m111, e_m = 0^m1111, f = 011113, g = 011114,$ $h = 011115$	
10012	4	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}\bar{b}_{m+2}e, e \rightsquigarrow e^2, a_m = 0^m, b_m = 0^m1,$ $c_m = 0^m10, d_m = 0^m100, e = 01001$	Thm. 5(30)
10112	4	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}\bar{b}_{m+2}e, e \rightsquigarrow e^2, a_m = 0^m, b_m = 0^m1,$ $c_m = 0^m10, d_m = 0^m101, e = 01011$	
11012	4	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}\bar{b}_{m+2}e, e \rightsquigarrow e^2, a_m = 0^m, b_m = 0^m1,$ $c_m = 0^m10, d_m = 0^m110, e = 01101$	
10020 10022	7	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}c_m\bar{d}_m, c_m \rightsquigarrow c_{m+1}e_m\bar{f}_mh,$ $d_m \rightsquigarrow \bar{d}_{m+1}f_m, e_m \rightsquigarrow e_{m+1}^2\bar{g}_{m-1}jkl, f_m \rightsquigarrow \bar{f}_{m+1}g_mh,$ $g_m \rightsquigarrow g_{m+1}jkl, h \rightsquigarrow f_1jh, j \rightsquigarrow g_1jkl, k \rightsquigarrow jkl, l \rightsquigarrow kl,$ $a_m = 0^m, b_m = 0^m1, c_m = 0^m10, d_m = 0^m12,$ $e_m = 0^m100, f_m = 0^m102, g_m = 0^m1020, h = 0103,$ $j = 01002, k = 01003, l = 01004$	Thm. 5(31)

Continuation of Table 1

τ	d	$F_{\{021,\tau\}}(x)$	Reference
10023	7	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}c_m\bar{d}_m, c_m \rightsquigarrow c_{m+1}e_m\bar{f}_m h,$ $d_m \rightsquigarrow \bar{d}_{m+1}f_m, e_m \rightsquigarrow e_{m+1}^2 j^{m+2}, f_m \rightsquigarrow \bar{f}_{m+1}g_m h,$ $g_m \rightsquigarrow g_{m+1}j^{m+3}, h \rightsquigarrow f_1 h k, j \rightsquigarrow j^2, k \rightsquigarrow g_1 j^3, a_m = 0^m,$ $b_m = 0^m 1, c_m = 0^m 10, d_m = 0^m 12, e_m = 0^m 100,$ $f_m = 0^m 102, g_m = 0^m 1020, h = 0103, j = 01002,$ $k = 01030$	Thm. 5(32)
10100	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow \bar{d}_{m+1}e_m f g, e_m \rightsquigarrow \bar{e}_{m+1}h j k, f \rightsquigarrow d_1 f g h, g \rightsquigarrow f g j,$ $h \rightsquigarrow e_1 h j k, j \rightsquigarrow h j k, k \rightsquigarrow j k, a_m = 0^m, b_m = 0^m 1,$ $c_m = 0^m 10, d_m = 0^m 101, e_m = 0^m 1010, f = 01013,$ $g = 01014, h = 010103, j = 010104, k = 010105$	Thm. 5(33)
11100	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow \bar{d}_{m+1}e_m f g, e_m \rightsquigarrow \bar{e}_{m+1}h j k, f \rightsquigarrow d_1 f g h, g \rightsquigarrow f g j,$ $h \rightsquigarrow e_1 h j k, j \rightsquigarrow h j k, k \rightsquigarrow j k, a_m = 0^m, b_m = 0^m 1,$ $c_m = 0^m 11, d_m = 0^m 111, e_m = 0^m 1110, f = 01113,$ $g = 01114, h = 011103, j = 011104, k = 011105$	Thm. 5(33)
10102	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}\bar{e}_m f g h, e_m \rightsquigarrow \bar{e}_{m+1}g h j, f \rightsquigarrow f^2, g \rightsquigarrow e_1 g h j,$ $h \rightsquigarrow g h j, j \rightsquigarrow j, a_m = 0^m, b_m = 0^m 1, c_m = 0^m 10,$ $d_m = 0^m 101, e_m = 0^m 1012, f = 01010, g = 01013,$ $h = 01014, j = 010120$	Thm. 5(34)
11102	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}\bar{e}_m f g h, e_m \rightsquigarrow \bar{e}_{m+1}g h j, f \rightsquigarrow f^2, g \rightsquigarrow e_1 g h j,$ $h \rightsquigarrow g h j, j \rightsquigarrow j, a_m = 0^m, b_m = 0^m 1, c_m = 0^m 11,$ $d_m = 0^m 111, e_m = 0^m 1112, f = 01110, g = 01113,$ $h = 01114, j = 011120$	Thm. 5(34)
10120	5		
10122	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}^2 \bar{e}_m f g, e_m \rightsquigarrow \bar{e}_{m+1}f g, f \rightsquigarrow e_1 f g, g \rightsquigarrow g h,$ $a_m = 0^m, b_m = 0^m 1, c_m = 0^m 10, d_m = 0^m 101,$ $e_m = 0^m 1012, f = 01013, g = 01014$	Thm. 5(35)
11120	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}^2 \bar{e}_m f g, e_m \rightsquigarrow \bar{e}_{m+1}f g, f \rightsquigarrow e_1 f g, g \rightsquigarrow g h,$ $a_m = 0^m, b_m = 0^m 1, c_m = 0^m 11, d_m = 0^m 111,$ $e_m = 0^m 1112, f = 01113, g = 01114$	Thm. 5(35)
10123	4	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}\bar{b}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}^2 e^{m+2}, e \rightsquigarrow e^2, a_m = 0^m, b_m = 0^m 1,$ $c_m = 0^m 10, d_m = 0^m 101, e = 01012$	Thm. 5(36)
10200	7	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}c_m\bar{d}_m, c_m \rightsquigarrow c_{m+1}^2 \bar{e}_m h,$ $d_m \rightsquigarrow \bar{d}_{m+1}f_m, e_m \rightsquigarrow \bar{e}_{m+1}g_m h, f_m \rightsquigarrow f_{m+1}\bar{e}_{m+1}h,$ $g_m \rightsquigarrow \bar{g}_{m+1}j k, h \rightsquigarrow e_1 j h, j \rightsquigarrow g_1 j k, k \rightsquigarrow j k, a_m = 0^m,$ $b_m = 0^m 1, c_m = 0^m 10, d_m = 0^m 12, e_m = 0^m 102,$ $f_m = 0^m 120, g_m = 0^m 1022, h = 0103, j = 01030,$ $k = 010205$	Thm. 5(37)
10220	7	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}c_m\bar{d}_m, c_m \rightsquigarrow c_{m+1}^2 \bar{e}_m h,$ $d_m \rightsquigarrow \bar{d}_{m+1}f_m, e_m \rightsquigarrow \bar{e}_{m+1}g_m h, f_m \rightsquigarrow f_{m+1}\bar{e}_{m+1}h,$ $g_m \rightsquigarrow \bar{g}_{m+1}j k, h \rightsquigarrow e_1 j h, j \rightsquigarrow g_1 j k, k \rightsquigarrow j k, a_m = 0^m,$ $b_m = 0^m 1, c_m = 0^m 10, d_m = 0^m 12, e_m = 0^m 102,$ $f_m = 0^m 120, g_m = 0^m 1022, h = 0103, j = 01033,$ $k = 010225$	Thm. 5(37)
10202	6		

Continuation of Table 1

τ	d	$F_{\{021,\tau\}}(x)$	Reference
10222	6	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}c_m\bar{d}_m, c_m \rightsquigarrow c_{m+1}^2\bar{e}_mg,$ $d_m \rightsquigarrow \bar{d}_{m+1}f_m, e_m \rightsquigarrow \bar{e}_{m+1}f_mg, f_m \rightsquigarrow f_{m+1}\bar{e}_{m+1}g,$ $g \rightsquigarrow e_1gh, h \rightsquigarrow e_1f_1g, a_m = 0^m, b_m = 0^m1, c_m = 0^m10,$ $d_m = 0^m12, e_m = 0^m102, f_m = 0^m120, g = 0103,$ $h = 01033$	Thm. 5(38)
10203	7	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}c_m\bar{d}_m, c_m \rightsquigarrow c_{m+1}^2\bar{e}_mh,$ $d_m \rightsquigarrow \bar{d}_{m+1}f_m, e_m \rightsquigarrow e_{m+1}g_mj_l, f_m \rightsquigarrow f_{m+1}\bar{e}_{m+1}h,$ $g_m \rightsquigarrow \bar{g}_{m+1}jk, h \rightsquigarrow e_1jl, j \rightsquigarrow g_1jk, k \rightsquigarrow k, l \rightsquigarrow l^2,$ $a_m = 0^m, b_m = 0^m1, c_m = 0^m10, d_m = 0^m12,$ $e_m = 0^m102, f_m = 0^m120, g_m = 0^m1023, h = 0103,$ $j = 01024, k = 010230, l = 01020$	Thm. 5(39)
10223	6	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}c_m\bar{d}_m, c_m \rightsquigarrow c_{m+1}^2\bar{e}_mg,$ $d_m \rightsquigarrow \bar{d}_{m+1}f_m, e_m \rightsquigarrow \bar{e}_{m+1}gh, f_m \rightsquigarrow f_{m+1}\bar{e}_{m+1}g, h \rightsquigarrow h^2,$ $j \rightsquigarrow e_1gh, a_m = 0^m, b_m = 0^m1, c_m = 0^m11, d_m = 0^m12,$ $e_m = 0^m102, f_m = 0^m120, h = 01022, g = 0103$	Thm. 5(40)
10230 10233	7	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}c_m\bar{d}_m, c_m \rightsquigarrow c_{m+1}^2\bar{e}_mj,$ $d_m \rightsquigarrow \bar{d}_{m+1}f_m, e_m \rightsquigarrow e_{m+1}^2\bar{g}_mh, f_m \rightsquigarrow f_{m+1}\bar{e}_{m+1}j,$ $g_m \rightsquigarrow \bar{g}_{m+1}h, h \rightsquigarrow g_1h, j \rightsquigarrow e_1^2h, a_m = 0^m, b_m = 0^m1,$ $c_m = 0^m10, d_m = 0^m12, e_m = 0^m102, f_m = 0120,$ $g_m = 0^m1023, h = 01024, j = 0103$	Thm. 5(41)
10234	6	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}c_m\bar{d}_m, c_m \rightsquigarrow c_{m+1}^2\bar{e}_mg,$ $d_m \rightsquigarrow \bar{d}_{m+1}f_m, e_m \rightsquigarrow e_{m+1}^2h^{m+1}, f_m \rightsquigarrow f_{m+1}\bar{e}_{m+1}g,$ $g \rightsquigarrow e_1^2h, h \rightsquigarrow h^2, a_m = 0^m, b_m = 0^m1, c_m = 0^m10,$ $d_m = 0^m12, e_m = 0^m102, f_m = 0^m120, g = 0103,$ $h = 01023$	Thm. 5(42)
11002	6	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}d_m\bar{e}_mg,$ $d_m \rightsquigarrow d_{m+1}f_mhjl, e_m \rightsquigarrow \bar{e}_{m+1}f_mg, f_m \rightsquigarrow \bar{f}_{m+1}hjk,$ $g \rightsquigarrow e_1gh, h \rightsquigarrow f_1hjk, j \rightsquigarrow hjk, k \rightsquigarrow k, l \rightsquigarrow l^2, a_m = 0^m,$ $b_m = 0^m1, c_m = 0^m11, d_m = 0^m110, e_m = 0^m112,$ $f_m = 0^m1102, g = 0113, h = 01103, j = 01104,$ $k = 011020, l = 01100$	Thm. 5(43)
11020 11022	6	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}d_m\bar{e}_mg,$ $d_m \rightsquigarrow d_{m+1}^2\bar{f}_{m-1}hjk, e_m \rightsquigarrow \bar{e}_{m+1}f_mg, f_m \rightsquigarrow \bar{f}_{m+1}hjk,$ $g \rightsquigarrow e_1gh, h \rightsquigarrow f_1hjk, j \rightsquigarrow hjk, k \rightsquigarrow jk, a_m = 0^m,$ $b_m = 0^m1, c_m = 0^m11, d_m = 0^m110, e_m = 0^m112,$ $f_m = 0^m1120, g = 0113, h = 01102, j = 01103, k = 01104$	Thm. 5(44)
11023	6	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}d_m\bar{e}_mg,$ $d_m \rightsquigarrow d_{m+1}^2h^{m+2}, e_m \rightsquigarrow \bar{e}_{m+1}f_mg, f_m \rightsquigarrow f_{m+1}h^{m+3},$ $g \rightsquigarrow e_1gj, h \rightsquigarrow h^2, j \rightsquigarrow f_1h^3, a_m = 0^m, b_m = 0^m1,$ $c_m = 0^m11, d_m = 0^m110, e_m = 0^m112, f_m = 0^m1120,$ $g = 0113, h = 01102, j = 01130$	Thm. 5(45)
11200	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}^2\bar{d}_mf,$ $d_m \rightsquigarrow \bar{d}_{m+1}e_mf, e_m \rightsquigarrow \bar{e}_{m+1}gh, f \rightsquigarrow d_1fg, g \rightsquigarrow e_1gh,$ $h \rightsquigarrow gh, a_m = 0^m, b_m = 0^m1, c_m = 0^m11, d_m = 0^m112,$ $e_m = 0^m1120, f = 0113, g = 01130, h = 011205$	Thm. 5(46)
11220	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}^2\bar{d}_mf,$ $d_m \rightsquigarrow d_{m+1}\bar{e}_mfe_m, e_m \rightsquigarrow \bar{e}_{m+1}gh, f \rightsquigarrow d_1fg, g \rightsquigarrow e_1gh,$ $h \rightsquigarrow gh, a_m = 0^m, b_m = 0^m1, c_m = 0^m11, d_m = 0^m112,$ $e_m = 0^m1122, f = 0113, g = 01133, h = 011225$	

Continuation of Table 1

τ	d	$F_{\{021,\tau\}}(x)$	Reference
11202	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}^2\bar{d}_m f,$ $d_m \rightsquigarrow \bar{d}_{m+1}e_m f, e_m \rightsquigarrow e_{m+1}\bar{d}_{m+1}f, f \rightsquigarrow d_1fg,$ $g \rightsquigarrow d_1e_1f, a_m = 0^m, b_m = 0^m1, c_m = 0^m11, d_m = 0^m112,$ $e_m = 0^m1120, f = 0113, g = 01130$	Thm. 5(47)
11203	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}^2\bar{d}_m f,$ $d_m \rightsquigarrow d_{m+1}\bar{e}_mgh, e_m \rightsquigarrow \bar{e}_{m+1}hj, f \rightsquigarrow d_1gh, g \rightsquigarrow g^2,$ $h \rightsquigarrow e_1hj, j \rightsquigarrow j, a_m = 0^m, b_m = 0^m1, c_m = 0^m11,$ $d_m = 0^m112, e_m = 0^m1123, f = 0113, g = 01120,$ $h = 01124, j = 011230$	Thm. 5(48)
11230	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow \bar{b}_{m+1}c_m, c_m \rightsquigarrow c_{m+1}^2\bar{d}_m f,$ $d_m \rightsquigarrow d_{m+1}^2\bar{e}_mg, e_m \rightsquigarrow \bar{e}_{m+1}g, f \rightsquigarrow d_1^2g, g \rightsquigarrow e_1g,$ $a_m = 0^m, b_m = 0^m1, c_m = 0^m11, d_m = 0^m112,$ $e_m = 0^m1123, f = 0113, g = 01124$	Thm. 5(49)
12000	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow \bar{c}_{m+1}d_m,$ $d_m \rightsquigarrow \bar{d}_{m+1}e_m f, e_m \rightsquigarrow \bar{e}_{m+1}gh, f \rightsquigarrow d_1fg, g \rightsquigarrow e_1gh,$ $h \rightsquigarrow gh, a_m = 0^m, b_m = 0^m1, c_m = 0^m12, d_m = 0^m120,$ $e_m = 0^m1200, f = 01204, g = 012004, h = 012005$	Thm. 5(50)
12200	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow \bar{c}_{m+1}d_m,$ $d_m \rightsquigarrow \bar{d}_{m+1}e_m f, e_m \rightsquigarrow \bar{e}_{m+1}gh, f \rightsquigarrow d_1fg, g \rightsquigarrow e_1gh,$ $h \rightsquigarrow gh, a_m = 0^m, b_m = 0^m1, c_m = 0^m12, d_m = 0^m122,$ $e_m = 0^m1220, f = 01224, g = 012204, h = 012205$	
12002	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow \bar{c}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}\bar{c}_{m+1}e_m, e_m \rightsquigarrow e_{m+1}\bar{c}_{m+2}, a_m = 0^m,$ $b_m = 0^m1, c_m = 0^m12, d_m = 0^m120, e_m = 0^m1200$	Thm. 5(51)
12022	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow \bar{c}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}\bar{c}_{m+1}e_m, e_m \rightsquigarrow e_{m+1}\bar{c}_{m+2}, a_m = 0^m,$ $b_m = 0^m1, c_m = 0^m12, d_m = 0^m120, e_m = 0^m1202$	
12202	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{ooc}_m, c_m \rightsquigarrow \bar{c}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}\bar{c}_{m+1}e_m, e_m \rightsquigarrow e_{m+1}\bar{c}_{m+2}, a_m = 0^m,$ $b_m = 0^m1, c_m = 0^m12, d_m = 0^m122, e_m = 0^m1220$	
12003	6	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow d_m c_{m+1}\bar{e}_m,$ $d_m \rightsquigarrow d_{m+1}\bar{f}_mgh, e_m \rightsquigarrow \bar{e}_{m+1}f_m, f_m \rightsquigarrow \bar{f}_{m+1}hj, g \rightsquigarrow g^2,$ $h \rightsquigarrow f_1jh, j \rightsquigarrow j, a_m = 0^m, b_m = 0^m1, c_m = 0^m12,$ $d_m = 0^m120, e_m = 0^m123, f_m = 0^m1203, g = 01200,$ $h = 01204, j = 012030$	Thm. 5(52)
12020	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow \bar{c}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}\bar{c}_{m+1}e_m, e_m \rightsquigarrow \bar{e}_{m+1}fg, f \rightsquigarrow e_1fg, g \rightsquigarrow fg,$ $a_m = 0^m, b_m = 0^m1, c_m = 0^m12, d_m = 0^m120,$ $e_m = 0^m1202, f = 012024, g = 012025$	Thm. 5(53)
12220	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow \bar{c}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}\bar{c}_{m+1}e_m, e_m \rightsquigarrow \bar{e}_{m+1}fg, f \rightsquigarrow e_1fg, g \rightsquigarrow fg,$ $a_m = 0^m, b_m = 0^m1, c_m = 0^m12, d_m = 0^m122,$ $e_m = 0^m1222, f = 012224, g = 012225$	
12023	4	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow d_m \bar{c}_{m+1},$ $d_m \rightsquigarrow d_{m+1}\bar{c}_{m+1}e, e \rightsquigarrow e^2, a_m = 0^m, b_m = 0^m1,$ $c_m = 0^m12, d_m = 0^m120, e = 1202$	Thm. 5(54)
12030			

Continuation of Table 1

τ	d	$F_{\{021,\tau\}}(x)$	Reference
12033	6	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow d_m c_{m+1}\bar{e}_m,$ $d_m \rightsquigarrow d_{m+1}^2\bar{f}_{m-1}gh, e_m \rightsquigarrow f_m\bar{e}_{m+1}, f_m \rightsquigarrow \bar{f}_{m+1}gh,$ $g \rightsquigarrow f_1gh, h \rightsquigarrow gh, a_m = 0^m, b_m = 0^m 1, c_m = 0^m 12,$ $d_m = 0^m 120, e_m = 0^m 123, f_m = 0^m 1230, g = 01203,$ $h = 01204$	Thm. 5(55)
12034	6	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow d_m c_{m+1}\bar{e}_m,$ $d_m \rightsquigarrow d_{m+1}g^{m+1}, e_m \rightsquigarrow f_m\bar{e}_{m+1}, f_m \rightsquigarrow f_{m+1}g^{m+2},$ $g \rightsquigarrow g^2, a_m = 0^m, b_m = 0^m 1, c_m = 0^m 12, d_m = 0^m 120,$ $e_m = 0^m 123, f_m = 0^m 1230, g = 01203, h = 012230$	Thm. 5(56)
12203	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow \bar{c}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}\bar{e}_mfg, e_m \rightsquigarrow \bar{e}_{m+1}gh, f \rightsquigarrow f^2, g \rightsquigarrow e_1gh,$ $h \rightsquigarrow h, a_m = 0^m, b_m = 0^m 1, c_m = 0^m 12, d_m = 0^m 122,$ $e_m = 0^m 1223, f = 01220, g = 01224, h = 012230$	Thm. 5(57)
12230	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow \bar{c}_{m+1}d_m,$ $d_m \rightsquigarrow d_{m+1}^2\bar{e}_mf, e_m \rightsquigarrow \bar{e}_{m+1}f, f \rightsquigarrow e_1f, a_m = 0^m,$ $b_m = 0^m 1, c_m = 0^m 12, d_m = 0^m 122, e_m = 0^m 1223,$ $f = 01224$	Thm. 5(58)
12300	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow c_{m+1}^2\bar{d}_m,$ $d_m \rightsquigarrow e_m\bar{d}_{m+1}, e_m \rightsquigarrow \bar{e}_{m+1}f, f \rightsquigarrow e_1f, a_m = 0^m,$ $b_m = 0^m 1, c_m = 0^m 12, d_m = 0^m 123, e_m = 0^m 1233,$ $f = 012335$	Thm. 5(59)
12330	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow c_{m+1}^2\bar{d}_m,$ $d_m \rightsquigarrow e_m\bar{d}_{m+1}, e_m \rightsquigarrow \bar{e}_{m+1}f, f \rightsquigarrow e_1f, a_m = 0^m,$ $b_m = 0^m 1, c_m = 0^m 12, d_m = 0^m 123, e_m = 0^m 1230,$ $f = 012305$	
12303	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow c_{m+1}^2\bar{d}_m,$ $d_m \rightsquigarrow e_m\bar{d}_{m+1}, e_m \rightsquigarrow e_{m+1}\bar{d}_{m+1}, a_m = 0^m, b_m = 0^m 1,$ $c_m = 0^m 12, d_m = 0^m 123, e_m = 0^m 1230$	Thm. 5(60)
12304	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow c_{m+1}^2\bar{d}_m,$ $d_m \rightsquigarrow d_{m+1}\bar{e}_mf, e_m \rightsquigarrow \bar{e}_{m+1}g, f \rightsquigarrow f^2, g \rightsquigarrow g, a_m = 0^m,$ $b_m = 0^m 1, c_m = 0^m 12, d_m = 0^m 123, e_m = 0^m 1234,$ $f = 1230, g = 012340$	Thm. 5(61)
12340	5	$a_m \rightsquigarrow a_{m+1}\bar{b}_m, b_m \rightsquigarrow b_{m+1}^2\bar{c}_m, c_m \rightsquigarrow c_{m+1}^2\bar{d}_m,$ $d_m \rightsquigarrow d_{m+1}^2\bar{e}_m, e_m \rightsquigarrow \bar{e}_{m+1}, a_m = 0^m, b_m = 0^m 1,$ $c_m = 0^m 12, d_m = 0^m 123, e_m = 0^m 1234$	Thm. 5(62)
End of Table 1			

Theorem 5. Let $G_\tau(x) = F_{\{021,\tau\}}(x)$. We have

- (1) $G_{00010} = -\frac{(19x^2-9x+1)\sqrt{1-4x}+12x^3-23x^2+9x-1}{2(1-4x)^2};$
- (2) $G_{00100} = G_{00110} = \frac{(4x-1)(x^3+4x^2-5x+1)+(-7x^3-10x^2+7x-1)\sqrt{1-4x}}{2x(1-4x)^2};$
- (3) $G_{00101} = \frac{\text{series}((1+\sqrt{1-4x})(2x-(x+1)\rho+\rho^2)}{2x^2}, \text{ where } \rho^3 = (x\rho + \rho - x)(\rho - x);$
- (4) $G_{00102} = \frac{x(4x-1)(2x^4-4x^2+4x-1)+x(12x^4-32x^3+38x^2-18x+3)\sqrt{1-4x}}{2(1-x)^4(1-2x)(1-4x)};$
- (5) $G_{00111} = \frac{x(x^2+x+3)-(x+2)\rho+2\rho^2}{x^2(1-x)}, \text{ where } \rho^3 = (x\rho + \rho - x)(\rho - x);$
- (6) $G_{00112} = \frac{-4x^6+6x^5+2x^4-17x^3+17x^2-7x+1-(2x-1)(x-1)^3\sqrt{1-4x}}{2x^2(1-x)^3(1-2x)};$

$$(7) \quad G_{00120} = \frac{-x^2 - 3x + 1 - (x^2 - 3x + 1)\sqrt{1-4x}}{2(1-x)(1-4x)};$$

$$(8) \quad G_{00122} = \frac{(4x-1)(2x^4 - 2x^3 - x^2 + 3x - 1) + (-4x^3 + 7x^2 - 5x + 1)\sqrt{1-4x}}{2x(1-x)^3(1-4x)};$$

$$(9) \quad G_{00123} = -\frac{x(8x^7 - 36x^6 + 86x^5 - 105x^4 + 78x^3 - 36x^2 + 9x - 1)}{(1-x)^5(1-2x)^3};$$

$$(10) \quad G_{01000} = G_{01100} = \frac{(1-x)(1-4x)^2(1-2x)^2 + (2x^5 - 56x^4 + 78x^3 - 44x^2 + 11x - 1)\sqrt{1-4x}}{2x^3(1-4x)^2};$$

$$(11) \quad G_{01001} = G_{01011} = G_{01101} = G_{01111} = \frac{x}{\rho-x}, \text{ where } \rho^4 = (\rho - x)(2x\rho^2 + (\rho - x)^2);$$

$$(12) \quad G_{01002} = \frac{(4x-1)(2x^2 - 2x + 1)(2x^6 - 12x^5 + 25x^4 - 34x^3 + 24x^2 - 8x + 1) + (2x^2 - 4x + 1)(2x-1)^2(x-1)^4\sqrt{1-4x}}{2x^2(1-x)^5(1-2x)(1-4x)};$$

$$(13) \quad G_{01010} = G_{01110} = \frac{x((x-2)\sqrt{1-4x} + 2x^2 - 3x + 2) + (\sqrt{1-4x} - 1)(1-\rho)\rho}{2x^3}, \text{ where } \rho^3 = (x\rho + \rho - x)(\rho - x);$$

$$(14) \quad G_{01012} = G_{01112} = \frac{2x^2(x-1)(2x^3 - 2x + 1) + 2x(x-1)^4(1-\rho)\rho}{2x^3(1-x)^2(1-2x)}, \text{ where } \rho^3 = (x\rho + \rho - x)(\rho - x);$$

$$(15) \quad G_{01020} = G_{01022} = \frac{(4x-1)(2x^4 + 4x^3 - 16x^2 + 11x - 2) + (-22x^3 + 34x^2 - 15x + 2)\sqrt{1-4x}}{2x^3(1-x)(1-4x)};$$

$$(16) \quad G_{01023} = \frac{-16x^9 + 76x^8 - 212x^7 + 342x^6 - 374x^5 + 286x^4 - 151x^3 + 53x^2 - 11x + 1 + (2x^2 - 2x + 1)(2x-1)^2(x-1)^3\sqrt{1-4x}}{2x(1-x)^5(1-2x)^3};$$

$$(17) \quad G_{01102} = \frac{(4x-1)(4x^6 - 15x^5 + 35x^4 - 40x^3 + 25x^2 - 8x + 1) + (x-1)(4x^5 - 31x^4 + 46x^3 - 30x^2 + 9x - 1)\sqrt{1-4x}}{2x^2(1-x)^3(1-2x)(1-4x)};$$

$$(18) \quad G_{01120} = G_{01122} = \frac{(3x-1)\sqrt{1-4x} + 5x^2 - 5x + 1}{x(1-4x)};$$

$$(19) \quad G_{01123} = \frac{16x^6 - 42x^5 + 60x^4 - 52x^3 + 27x^2 - 8x + 1 + (2x^3 - 6x^2 + 4x - 1)(1-x)^2\sqrt{1-4x}}{2x(1-x)^2(1-2x)^3};$$

$$(20) \quad G_{01200} = G_{01220} = \frac{(1-x)(1-4x)(2x-1)(5x-1) + (22x^3 - 29x^2 + 10x - 1)\sqrt{1-4x}}{2x(1-x)(1-4x)^2};$$

$$(21) \quad G_{01202} = G_{01222} = \frac{x}{1-x} - \frac{5x^2 - x + (1-4x-2x^2)\rho - (1-4x)\rho^2}{x(1-x)(1-6x)}, \text{ where } \rho^3 = (x\rho + \rho - x)(\rho - x);$$

$$(22) \quad G_{01203} = \frac{(4x-1)(16x^8 - 76x^7 + 188x^6 - 270x^5 + 246x^4 - 145x^3 + 53x^2 - 11x + 1) + (1-x)^3(1-2x)^4\sqrt{1-4x}}{2(1-x)^5(1-2x)^3(1-4x)};$$

$$(23) \quad G_{01223} = \frac{x}{1-x} - \frac{x((2x^3 + 4x^2 - 4x + 1)\sqrt{1-4x} - 8x^4 + 26x^3 - 22x^2 + 8x - 1)}{(1-x)^2(16x^3 - 20x^2 + 8x - 1)(1 + \sqrt{1-4x})};$$

$$(24) \quad G_{01230} = G_{01233} = \frac{x}{1-x} - \frac{2x^2(x\sqrt{1-4x} - (1-x)(1-2x))}{(1-x)^2(1-2x)(1-4x)(1 + \sqrt{1-4x})};$$

$$(25) \quad G_{01234} = \frac{x^2(28x^6 - 78x^5 + 102x^4 - 80x^3 + 37x^2 - 9x + 1)}{(1-2x)^5(1-x)^4};$$

$$(26) \quad G_{10000} = G_{11000} = \frac{(1-x)(5x^3 - 8x^2 + 5x - 1)(1-4x)^3 + (-14x^7 + 316x^6 - 650x^5 + 635x^4 - 343x^3 + 103x^2 - 16x + 1)\sqrt{1-4x}}{2x^4(1-4x)^3};$$

$$(27) \quad G_{10001} = G_{10011} = G_{10101} = G_{10111} = G_{11001} = G_{11011} = G_{11101} = \frac{x}{\rho-x}, \text{ where } \rho^5 = (\rho - x)(3x\rho^3 - x^2\rho^2 + x^3\rho + (\rho - x)^3);$$

$$(28) \quad G_{10002} = \frac{6x^{10} - 47x^9 + 181x^8 - 399x^7 + 582x^6 - 587x^5 + 414x^4 - 201x^3 + 64x^2 - 12x + 1}{2x^3(1-x)^6(1-2x)} - \frac{(38x^5 - 89x^4 + 88x^3 - 45x^2 + 11x - 1)(x-1)^5\sqrt{1-4x}}{2x^3(1-x)^6(1-4x)^2};$$

$$(29) \quad G_{10010} = G_{10110} = G_{11010} = G_{11110} = \frac{x^3(3-4x-\sqrt{1-4x}) + x(x+2)(1-x)(\sqrt{1-4x}-1)\rho + (x-1)(x+1)(\sqrt{1-4x}-1)\rho^2 + (1-x)(\sqrt{1-4x}-1)\rho^3}{2x^4}, \text{ where } \rho^4 = (\rho - x)(2x\rho^2 + (\rho - x)^2);$$

$$(30) \quad G_{10012} = G_{10112} = G_{11012} = \frac{x^2(1-2x)(x^4 - 2x^3 + 4x^2 - 3x + 1) - x(1-x)^4(x+2)\rho + (x-1)^4(x+1)\rho^2 - (x-1)^4\rho^3}{x^3(1-x)(1-2x)(1-x+x^2)}, \text{ where } \rho^4 = (\rho - x)(2x\rho^2 + (\rho - x)^2);$$

$$(31) \quad G_{10020} = G_{10022} = \frac{(4x-1)(11x^5 - 38x^4 + 69x^3 - 55x^2 + 18x - 2) + (35x^5 - 116x^4 + 151x^3 - 87x^2 + 22x - 2)\sqrt{1-4x}}{2x^3(1-x)(1-4x)^2};$$

$$(32) \quad G_{10023} = \frac{16x^{11}-128x^{10}+452x^9-1079x^8+1732x^7-1937x^6+1538x^5-866x^4+340x^3-89x^2+14x-1}{2x^2(1-x)^6(1-2x)^3} - \frac{(4x^5-27x^4+40x^3-25x^2+8x-1)\sqrt{1-4x}}{2x^2(1-x)^2(1-2x)(1-4x)};$$

$$(33) \quad G_{10100} = G_{11100} = \frac{13x^3-25x^2+9x-1)\sqrt{1-4x}+12x^4-47x^3+39x^2-11x+1+((1-4x)(1-x)+(5x-1)\sqrt{1-4x})(1-\rho)\rho}{2x^3(1-4x)}, \text{ where } \rho^3 = (x\rho + \rho - x)(\rho - x);$$

$$(34) \quad G_{10102} = G_{11102} = \frac{x(-4x^6+8x^5-14x^4+21x^3-22x^2+11x-2+(2x^4-9x^3+14x^2-9x+2)\sqrt{1-4x})}{2x^3(1-x)^3(1-2x)} - \frac{(1-x)(-2x^5+4x^4-6x^3+3x-1+(2x^2-3x+1)\sqrt{1-4x})(1-\rho)\rho}{2x^3(1-x)^3(1-2x)}, \text{ where } \rho^3 = (x\rho + \rho - x)(\rho - x);$$

$$(35) \quad G_{10120} = G_{10122} = G_{11120} = \frac{2x(4x^2-x-(1-x)\sqrt{1-4x})+(\sqrt{1-4x}-4x+1)(1-\rho)\rho}{2x^2(1-4x)}, \text{ where } \rho^3 = (x\rho + \rho - x)(\rho - x);$$

$$(36) \quad G_{10123} = \frac{x(8x^6-12x^5-3x^4+21x^3-19x^2+7x-1)-(x-1)^3(2x^4-5x^3+9x^2-5x+1)(1-\rho)\rho}{x^2(1-x)^2(1-2x)^3}, \text{ where } \rho^3 = (x\rho + \rho - x)(\rho - x);$$

$$(37) \quad G_{10200} = G_{10220} = \frac{(4x-1)(7x^6+63x^5-296x^4+394x^3-221x^2+55x-5)+(-23x^6+343x^5-812x^4+746x^3-321x^2+65x-5)\sqrt{1-4x}}{2x^5(1-x)(1-4x)^2};$$

$$(38) \quad G_{10202} = G_{10222} = \frac{(1-6x)(1-x)\sqrt{1-4x}-6x^4+2x^3-36x^2+12x-1}{x^3(1-x)(1-6x)} + \frac{(x^2-1)(1-6x)\sqrt{1-4x}-8x^4+28x^3+31x^2-12x+1)\rho+((1-6x)(1-x)\sqrt{1-4x}+4x^3-42x^2+13x-1)\rho^2}{2x^4(1-x)(1-6x)}, \text{ where } \rho^3 = (x\rho + \rho - x)(\rho - x);$$

$$(39) \quad G_{10203} = -\frac{16x^13-108x^12+364x^11-510x^10-304x^9+2688x^8-5540x^7+6646x^6-5258x^5+2829x^4-1027x^3+241x^2-33x+2}{2x^4(1-x)^7(1-2x)^3} - \frac{(8x^7+26x^6-158x^5+280x^4-236x^3+104x^2-23x+2)(2x-1)^2(x-1)^3\sqrt{1-4x}}{2x^4(1-x)^7(1-2x)^3(1-4x)};$$

$$(40) \quad G_{10223} = \frac{(4x-1)(4x^3-7x^2+5x-1)(2x^6-2x^5-9x^4+31x^3-31x^2+13x-2)+(9x^3-23x^2+13x-2)(2x-1)^2(x-1)^3\sqrt{1-4x}}{2x^3(1-x)^4(1-2x)^2(1-4x)};$$

$$(41) \quad G_{10230} = G_{10233} = \frac{-16x^8+40x^7+x^6-149x^5+260x^4-215x^3+96x^2-22x+2+(-15x^6+63x^5-118x^4+117x^3-64x^2+18x-2)\sqrt{1-4x}}{2x^2(1-x)^4(1-2x)(1-4x)};$$

$$(42) \quad G_{10234} = \frac{64x^{12}-424x^{11}+1500x^{10}-3168x^9+4638x^8-5025x^7+4132x^6-2604x^5+1242x^4-432x^3+103x^2-15x+1}{2x(1-x)^6(1-2x)^5} + \frac{(6x^5-15x^4+21x^3-16x^2+6x-1)\sqrt{1-4x}}{2x(1-x)^3(1-2x)^3};$$

$$(43) \quad G_{11002} = \frac{1-\sqrt{1-4x}}{((x-1)\sqrt{1-4x}+3x-1)(x-1)};$$

$$(44) \quad G_{11020} = G_{11022} = \frac{(1-4x)(3x-1)(3x^4+7x^3-22x^2+13x-2)-(7x^5+98x^4-165x^3+95x^2-23x+2)\sqrt{1-4x}}{2x^4(1-4x)^2};$$

$$(45) \quad G_{11023} = \frac{16x^9-82x^8+260x^7-464x^6+530x^5-401x^4+200x^3-64x^2+12x-1}{2x^2(1-x)^4(1-2x)^3} + \frac{(8x^8-138x^7+392x^6-554x^5+458x^4-233x^3+73x^2-13x+1)\sqrt{1-4x}}{2x^2(1-x)^3(1-2x)^3(1-4x)};$$

$$(46) \quad G_{11200} = G_{11220} = \frac{(1-x)(53x^3-45x^2+12x-1)}{2x^2(1-4x)^2} + \frac{(x^4-40x^3+37x^2-11x+1)\sqrt{1-4x}}{2x^2(1-4x)^2};$$

$$(47) \quad G_{11202} = \frac{(\rho^2-5x\rho+3x^2)(1-4x-\sqrt{1-4x})}{2(1-6x)(\rho-x)^2}, \text{ where } \rho^3 = (x\rho + \rho - x)(\rho - x);$$

$$(48) \quad G_{11203} = -\frac{64x^9-328x^8+946x^7-1660x^6+1858x^5-1348x^4+627x^3-180x^2+29x-2}{2x(1-x)^4(1-2x)^3(1-4x)} - \frac{(8x^6-90x^5+194x^4-184x^3+88x^2-21x+2)\sqrt{1-4x}}{2x(1-x)^2(1-2x)^3(1-4x)};$$

$$(49) \quad G_{11230} = \frac{-9x^3+12x^2-7x+1}{2x(1-x)(1-4x)} - \frac{(30x^4-57x^3+40x^2-11x+1)\sqrt{1-4x}}{2x(1-x)(1-2x)(1-4x)^2};$$

$$(50) \quad G_{12000} = G_{12200} = \frac{(-4x^5+154x^4-174x^3+74x^2-14x+1)\sqrt{1-4x}}{2x^2(1-4x)^3} - \frac{52x^4-92x^3+52x^2-12x+1}{2x^2(1-4x)^2};$$

$$(51) \quad G_{12002} = G_{12022} = G_{12202} = \frac{x^2(8x^3-4x^2+6x-1)-x(12x^3-x^2+10x-2)\rho+(8x^3+x^2+3x-1)\rho^2-(2x^2+3x-1)\rho^3}{(1-x)(1-8x)(\rho^3-2x\rho^2+x(1+2x)\rho-x^3)}, \text{ where } \rho^4 = (\rho - x)(2x\rho^2 + (\rho - x)^2);$$

- $$(52) \quad G_{12003} = -\frac{(2x^2-4x+1)(4x^2-6x+1)\sqrt{1-4x}}{2x(1-x)^2(1-4x)^2} - \frac{64x^{11}-496x^{10}+1668x^9-3512x^8+5026x^7-5054x^6+3614x^5-1821x^4+628x^3-140x^2+18x-1}{2x(1-x)^6(1-2x)^3(1-4x)};$$
- $$(53) \quad G_{12020} = G_{12220} = -\frac{48x^4-68x^3+66x^2-21x+2+(12x^3-46x^2+19x-2)\sqrt{1-4x}}{2x(1-x)(1-6x)(1-4x)} + \frac{(2x^2+4x-1)(1-\sqrt{1-4x})}{2x^2(1-x)(1-6x)}\rho + \frac{(1-4x)(1-\sqrt{1-4x})}{2x^2(1-x)(1-6x)}\rho^2, \\ \text{where } \rho^3 = (x\rho + \rho - x)(\rho - x);$$
- $$(54) \quad G_{12023} = \frac{x(20x^5-16x^4-14x^3+23x^2-9x+1)+(2x^2+4x-1)(1-x)^3(1-2x)\rho+(1-x)^3(1-2x)(1-4x)\rho^2}{x(1-x)^2(1-2x)^2(1-6x)}, \text{ where } \rho^3 = (x\rho + \rho - x)(\rho - x);$$
- $$(55) \quad G_{12030} = G_{12033} = \frac{(118x^5-284x^4+264x^3-118x^2+25x-2)\sqrt{1-4x}}{2x^2(1-x)^2(1-2x)(1-4x)^2} - \frac{16x^6-2x^5-92x^4+136x^3-80x^2+21x-2}{2x^2(1-x)^2(1-2x)(1-4x)};$$
- $$(56) \quad G_{12034} = \frac{(2x^2-2x+1)\sqrt{1-4x}}{2(1-x)^3(1-4x)} + \frac{64x^{11}-424x^{10}+1444x^9-2940x^8+4008x^7-3848x^6+2640x^5-1296x^4+449x^3-105x^2+15x-1}{(1-x)^6(1-2x)^5};$$
- $$(57) \quad G_{12203} = \frac{(32x^8-144x^7+352x^6-497x^5+437x^4-240x^3+79x^2-14x+1)}{2x(1-x)^4(1-2x)^2(1-4x)} + \frac{(4x^5-35x^4+58x^3-40x^2+11x-1)\sqrt{1-4x}}{2x(1-x)^3(1-4x)^2};$$
- $$(58) \quad G_{12230} = \frac{(3x-1)(6x^2-6x+1)}{(1-x)(1-4x)^2} + \frac{(7x^2-6x+1)\sqrt{1-4x}}{(1-x)/(1-4x)^2};$$
- $$(59) \quad G_{12330} = G_{12304} = -\frac{(72x^4-118x^3+68x^2-15x+1)}{2(1-x)(1-4x)^2(1-2x)} + \frac{(26x^4-54x^3+41x^2-12x+1)\sqrt{1-4x}}{2(1-2x)(1-4x)^2(1-x)^2};$$
- $$(60) \quad G_{12303} = \frac{-x^4(72x^3-78x^2+50x-9)+x^3(72x^3-102x^2+68x-13)\rho-x(72x^4-102x^3+64x^2-15x+1)\rho^2}{(1-2x)(1-6x)^2(1-x)^2(\rho^2-x\rho+x^2)}, \text{ where } \rho^3 = (x\rho + \rho - x)(\rho - x);$$
- $$(61) \quad G_{12304} = -\frac{x(256x^{11}-1760x^{10}+5976x^9-12292x^8+16836x^7-16104x^6+10942x^5-5268x^4+1761x^3-389x^2+51x-3)}{2(1-4x)(1-2x)^5(1-x)^6} - \frac{x(2x^2-2x+1)\sqrt{1-4x}}{2(1-x)^3(1-2x)(1-4x)};$$
- $$(62) \quad G_{12340}(x) = \frac{x(64x^6-208x^5+306x^4-242x^3+104x^2-23x+2)}{2(1-x)^3(1-2x)^3(1-4x)} - \frac{x^2(16x^5-32x^4+38x^3-26x^2+8x-1)\sqrt{1-4x}}{2(1-4x)^2(1-2x)^3(1-x)^3};$$

There are 106 cases of length-five patterns; determining the generating trees and corresponding generating functions for each case required huge and hard computational work. In case you find any typos, please consider the following source files [3–6] and feel free to contact the authors.

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