```
> restart:
> ##Lemma 4
> V0:=(x,N)->add((j+1-(j^2+j+1)*x)*x^(j+1)/(j+1)!/product((1-i*x),i=1..j+1),j=1..N)/add((j+1-j^2*x)*x^j/(j+1)!/product((1-i*x),i=1..j),j=1..N);
```

$$V0 := (x, N) \rightarrow \frac{\operatorname{add}\left(\frac{(j+1-(j^2+j+1)\,x)\,x^{(j+1)}}{(j+1)!\left(\prod_{i=1}^{j+1}(1-i\,x)\right)}, j=1\,..\,N\right)}{\operatorname{add}\left(\frac{(j+1-j^2\,x)\,x^{j}}{(j+1)!\left(\prod_{i=1}^{j}(1-i\,x)\right)}, j=1\,..\,N\right)}$$

```
> #Lemma 4
> Vp:=(x,p,N)->add((-1)^j*(-1+p-j*p^2*x+(2*j+1)*p*x-(j^2+j+1)*p^2*x^2)*p^(2*j+1)*x^(2*j+1)/(1-p+p*x)/product((1-i*p*x),i=1..j+1)/product((1-p-i*p*x),i=1..j+1),j=0..N)+V0(x,N)*add((-1)^j*((1-j*p*x)^2-p+(j-1)*p^2*x)*p^(2*j)*x^(2*j)/(1-p+p*x)/product((1-i*p*x),i=1..j)/product((1-p-i*p*x),i=1..j+1),j=0..N);
```

$$Vp := (x, p, N) \rightarrow \operatorname{add}\left(\frac{(-1)^j\,(-1+p-j\,p^2\,x+(2\,j+1)\,p\,x-(j^2+j+1)\,p^2\,x^2)\,p^{(2\,j+1)}\,x^{(2\,j+1)}}{(1-p+p\,x)\left(\prod_{i=1}^{j+1}(1-i\,p\,x)\right)\left(\prod_{i=1}^{j+1}(1-p-i\,p\,x)\right)}, j=0\,..\,N\right) + \mathrm{V0}(x, N)\,\operatorname{add}\left(\frac{(-1)^j\,((1-j\,p\,x)^2-p+(j-1)\,p^2\,x)\,p^{(2\,j)}\,x^{(2\,j)}}{(1-p+p\,x)\left(\prod_{i=1}^{j}(1-i\,p\,x)\right)\left(\prod_{i=1}^{j+1}(1-p-i\,p\,x)\right)}, j=0\,..\,N\right)$$

```
> ##Lemma 6
> C11:=(x,N)->(1-x)*V0(x,N)-x;
```

$$C11 := (x, N) \rightarrow (1-x)\,\mathrm{V0}(x, N) - x$$

```
> ##Equation 19
> C1u:=(x,u,N)->(1-u*x)*x/(1-x)/(1-u+u*x)*C11(x,N)+(1-u)*u*x^4/(1-u+u*x)/(1-u-2*u*x)*Vp(x,1,N)-(1-u)*u^2*x^4/(1-u+u*x)/(1-u-2*u*x)/(1-2*u*x)*Vp(x,u/(1-2*u*x),N)+(1-u)*(1-u*x-u*x^2)*x^3/(1-x)/(1-u+u*x)/(1-2*u*x);
```

$$C1u := (x, u, N) \rightarrow \frac{(1-u\,x)\,x\,\mathrm{C11}(x, N)}{(1-x)\,(1-u+u\,x)} + \frac{(1-u)\,u\,x^4\,\mathrm{Vp}(x, 1, N)}{(1-u+u\,x)\,(1-u-2\,u\,x)}$$

$$-\frac{(1-u)\,u^2\,x^4\,\mathrm{Vp}\left(x,\frac{u}{1-2\,u\,x},N\right)}{(1-u+u\,x)\,(1-u-2\,u\,x)\,(1-2\,u\,x)}+\frac{(1-u)\,(1-u\,x-u\,x^2)\,x^3}{(1-x)\,(1-u+u\,x)\,(1-2\,u\,x)}$$

```
> ##Lemma 5
> Cp:=(x,v,u,N)->u*x^4/(1-u-2*u*x)*(Vp(x,1,N)-u/(1-2*u*x)*Vp(x,u/(1-
  2*u*x),N))+u*x^4/(1-2*u*x)+u*v*x/(1-u)*(C1u(x,v,N)-C1u(x,u*v,N))+v
  *x/(1-v*x)*C1u(x,v,N)+v*x^3/(1-v*x);
```

$$Cp := (x, v, u, N) \rightarrow \frac{u\,x^4\left(\mathrm{Vp}(x,1,N)-\frac{u\,\mathrm{Vp}\left(x,\frac{u}{1-2\,u\,x},N\right)}{1-2\,u\,x}\right)}{1-u-2\,u\,x}+\frac{u\,x^4}{1-2\,u\,x}+\frac{u\,v\,x\,(\mathrm{C1u}(x,v,N)-\mathrm{C1u}(x,u\,v,N))}{1-u}+\frac{v\,x\,\mathrm{C1u}(x,v,N)}{1-v\,x}+\frac{v\,x^3}{1-v\,x}$$

```
> #Lemma 8
> B11:=(x,N)->-(C11(x,N)/(1-x)*add(j^2*x^(j+1)/(j+1)!/product((1-i*x
  ),i=1..j),j=1..N)+add(j*x^(j+1)/(j+1)!/product((1-i*x),i=1..j+2)*C
  1u(x,1/(1-(j+1)*x),N),j=1..N)+1/(1-x)*add((1-(j+1)*x)^2*x^(j+2)/(j
  -1)!/product((1-i*x),i=1..j+2),j=1..N))/add((j^2*x-j-1)*x^j/(j+1)!
  /product((1-i*x),i=1..j),j=1..N);
```

$$B11 := (x, N) \rightarrow -\left(\frac{\mathrm{C11}(x,N)\,\mathrm{add}\left(\frac{j^2\,x^{(j+1)}}{(j+1)!\left(\prod_{i=1}^{j}(1-i\,x)\right)},j=1\,..\,N\right)}{1-x}+\mathrm{add}\left(\frac{j\,x^{(j+1)}\,\mathrm{C1u}\left(x,\frac{1}{1-(j+1)\,x},N\right)}{(j+1)!\left(\prod_{i=1}^{j+2}(1-i\,x)\right)},j=1\,..\,N\right)+\frac{\mathrm{add}\left(\frac{(1-(j+1)\,x)^2\,x^{(j+2)}}{(j-1)!\left(\prod_{i=1}^{j+2}(1-i\,x)\right)},j=1\,..\,N\right)}{1-x}\right)\Big/\mathrm{add}\left(\frac{(j^2\,x-j-1)\,x^j}{(j+1)!\left(\prod_{i=1}^{j}(1-i\,x)\right)},j=1\,..\,N\right)$$

```
> ##Theorem 9
```

```
> A:=(x,N)->x+x*V0(x,N)+x*B11(x,N);
```

$$A := (x, N) \to x + x\,\mathrm{V0}(x, N) + x\,\mathrm{B11}(x, N)$$

```
>
> st:=time(): nn:=10: simplify(taylor(A(x,nn),x,nn)); print("running time =",time()-st);
```

$$x + x^2 + 2\,x^3 + 5\,x^4 + 15\,x^5 + 50\,x^6 + 180\,x^7 + 690\,x^8 + \mathrm{O}(x^9)$$

"running time =", 0.156

```
> st:=time(): nn:=20: simplify(taylor(A(x,nn),x,nn)); print("running time =",time()-st);
```

$$x + x^2 + 2\,x^3 + 5\,x^4 + 15\,x^5 + 50\,x^6 + 180\,x^7 + 690\,x^8 + 2792\,x^9 + 11857\,x^{10} + 52633\,x^{11} + 243455\,x^{12} + 1170525\,x^{13} + \mathrm{O}(x^{14})$$

"running time =", 0.688

```
> st:=time(): nn:=30: simplify(taylor(A(x,nn),x,nn)); print("running time =",time()-st);
```

$$x + x^2 + 2\,x^3 + 5\,x^4 + 15\,x^5 + 50\,x^6 + 180\,x^7 + 690\,x^8 + 2792\,x^9 + 11857\,x^{10} + 52633\,x^{11} + 243455\,x^{12} + 1170525\,x^{13} + 5837934\,x^{14} + 30151474\,x^{15} + 161021581\,x^{16} + 888001485\,x^{17} + 5051014786\,x^{18} + \mathrm{O}(x^{19})$$

"running time =", 4.203

```
> st:=time(): nn:=40: simplify(taylor(A(x,nn),x,nn)); print("running time =",time()-st);
```

$$x + x^2 + 2\,x^3 + 5\,x^4 + 15\,x^5 + 50\,x^6 + 180\,x^7 + 690\,x^8 + 2792\,x^9 + 11857\,x^{10} + 52633\,x^{11} + 243455\,x^{12} + 1170525\,x^{13} + 5837934\,x^{14} + 30151474\,x^{15} + 161021581\,x^{16} + 888001485\,x^{17} + 5051014786\,x^{18} + 29600662480\,x^{19} + 178541105770\,x^{20} + 1107321666920\,x^{21} + 7055339825171\,x^{22} + 46142654894331\,x^{23} + \mathrm{O}(x^{24})$$

"running time =", 78.734

```
> st:=time(): nn:=50: simplify(taylor(A(x,nn),x,nn)); print("running time =",time()-st);
```

$$x + x^2 + 2\,x^3 + 5\,x^4 + 15\,x^5 + 50\,x^6 + 180\,x^7 + 690\,x^8 + 2792\,x^9 + 11857\,x^{10} + 52633\,x^{11} + 243455\,x^{12} + 1170525\,x^{13} + 5837934\,x^{14} + 30151474\,x^{15} + 161021581\,x^{16} + 888001485\,x^{17} + 5051014786\,x^{18} + 29600662480\,x^{19} + 178541105770\,x^{20} + 1107321666920\,x^{21} + 7055339825171\,x^{22} + 46142654894331\,x^{23} + 309513540865544\,x^{24} + 2127744119042216\,x^{25} + 14979904453920111\,x^{26} + 107932371558460341\,x^{27} + 795363217306369817\,x^{28} + 5990768203554158167\,x^{29} + \mathrm{O}(x^{30})$$

"running time =", 354.188

```
> st:=time(): nn:=60: simplify(taylor(A(x,nn),x,nn)); print("running time =",time()-st);
```

$$x + x^2 + 2\,x^3 + 5\,x^4 + 15\,x^5 + 50\,x^6 + 180\,x^7 + 690\,x^8 + 2792\,x^9 + 11857\,x^{10} + 52633\,x^{11} + 243455\,x^{12} + 1170525\,x^{13} + 5837934\,x^{14} + 30151474\,x^{15} + 161021581\,x^{16} + 888001485\,x^{17} + 5051014786\,x^{18} + 29600662480\,x^{19} + 178541105770\,x^{20} + 1107321666920\,x^{21} + 7055339825171\,x^{22} + 46142654894331\,x^{23} + 309513540865544\,x^{24} + 2127744119042216\,x^{25} + 14979904453920111\,x^{26} + 107932371558460341\,x^{27} + 795363217306369817\,x^{28} + 5990768203554158167\,x^{29} + 46094392105916344968\,x^{30} + 362092868720288824992\,x^{31} + 2902468778263996723996\,x^{32} + 23728550137026791789338\,x^{33} + 197752305704274251817235\,x^{34} + O(x^{35})$$

"running time =", 1139.687

>