```
> #------------Avoiding 021 and another pattern tau----------------
> #-----------------------------------------------------------------
> #Case tau=0001
> restart:
> eqA:=-A+x/(1-v)+x*E/(1-v)+x*B+x/(1-v)*A+x/(1-v)*F;
  eqB:=-B+x/(1-v)+x*E/(1-v)+x/(1-v)*H+x/v*(A-A1)+x/(1-v)*A+x/(1-v)*F
  ;                          eqC:=-C+x/(1-v)+x*A+x*D1+x/(1-v)*C;
  eqD:=-D1+x/(1-v)+x*B+x/(1-v)*I1+x/v*(C-C1)+x/(1-v)*C;
```

$$eqA := -A + \frac{x}{1-v} + \frac{x\,E}{1-v} + x\,B + \frac{x\,A}{1-v} + \frac{x\,F}{1-v}$$

$$eqB := -B + \frac{x}{1-v} + \frac{x\,E}{1-v} + \frac{x\,H}{1-v} + \frac{x\,(A-A1)}{v} + \frac{x\,A}{1-v} + \frac{x\,F}{1-v}$$

$$eqC := -C + \frac{x}{1-v} + x\,A + x\,\mathrm{D1} + \frac{x\,C}{1-v}$$

$$eqD := -\mathrm{D1} + \frac{x}{1-v} + x\,B + \frac{x\,I1}{1-v} + \frac{x\,(C-C1)}{v} + \frac{x\,C}{1-v}$$

```
> eqB1:=subs(A=solve(eqA=0,A),eqB); coeff(eqB1,B);
  kb:=solve(%=0,v)[2];
  eqB11:=subs(v=kb,subs(B=0,eqB1));
```

$$eqB1 := -B + \frac{x}{1-v} + \frac{x\,E}{1-v} + \frac{x\,H}{1-v} + \frac{x\left(\frac{x\,(B\,v-B-E-F-1)}{-1+v+x} - A1\right)}{v} + \frac{x^2\,(B\,v-B-E-F-1)}{(1-v)\,(-1+v+x)} + \frac{x\,F}{1-v}$$

$$-1 + \frac{x^2\,(-1+v)}{v\,(-1+v+x)} + \frac{x^2\,(-1+v)}{(1-v)\,(-1+v+x)}$$

$$kb := -\frac{x}{2} + \frac{1}{2} - \frac{\sqrt{-3\,x^2-2\,x+1}}{2}$$

$$eqB11 := \frac{x}{\frac{1}{2}+\frac{x}{2}+\frac{\sqrt{-3\,x^2-2\,x+1}}{2}} + \frac{x\,E}{\frac{1}{2}+\frac{x}{2}+\frac{\sqrt{-3\,x^2-2\,x+1}}{2}} + \frac{x\,H}{\frac{1}{2}+\frac{x}{2}+\frac{\sqrt{-3\,x^2-2\,x+1}}{2}} + \frac{x\left(\frac{x\,(-E-F-1)}{-\frac{1}{2}+\frac{x}{2}-\frac{\sqrt{-3\,x^2-2\,x+1}}{2}} - A1\right)}{-\frac{x}{2}+\frac{1}{2}-\frac{\sqrt{-3\,x^2-2\,x+1}}{2}}$$

$$+\frac{x^2(-E-F-1)}{\left(\frac{1}{2}+\frac{x}{2}+\frac{\sqrt{-3x^2-2x+1}}{2}\right)\left(-\frac{1}{2}+\frac{x}{2}-\frac{\sqrt{-3x^2-2x+1}}{2}\right)}+\frac{xF}{\frac{1}{2}+\frac{x}{2}+\frac{\sqrt{-3x^2-2x+1}}{2}}$$

```
> #solving with all other rules
> solve({eqB11=0,A0=x+x*B0+x*C1,B0=x+x*E+x*A1+x*F,E=x+x*E,F=x+x*E+x*
G+x*F,G=x+x*E+x*H+x*A1+x*F,H=x+x*E+x*H,I1=x+x*H+x*I1},{A1,B0,C1,E,
F,G,H,I1});
```

$$\{A1=\frac{\sqrt{-3x^2-2x+1}\,x^2+x^3+\sqrt{-3x^2-2x+1}\,x-2x^2-\sqrt{-3x^2-2x+1}-2x+1}{2x^4(x+1)(x-1)^2},$$

$$B0=-\frac{2x^6-\sqrt{-3x^2-2x+1}\,x+x^2+\sqrt{-3x^2-2x+1}+2x-1}{2(x-1)^2(x+1)x^3},\ C1=(2A0x^5-2A0x^4+2x^5$$

$$-2A0x^3+2x^4+2A0x^2-2x^3-\sqrt{-3x^2-2x+1}\,x+x^2+\sqrt{-3x^2-2x+1}+2x-1)\Big/(2$$

$$(x-1)^2(x+1)x^3),\ E=-\frac{x}{x-1},\ F=-\frac{-1+2x^2+\sqrt{-3x^2-2x+1}+x}{2x^2(x-1)^2(x+1)},$$

$$G=\frac{2x^4+2x^3+2x^2+\sqrt{-3x^2-2x+1}+x-1}{2x^3(x+1)(x-1)},\ H=\frac{x}{(x-1)^2},\ I1=-\frac{x(x^2-x+1)}{(x^2-2x+1)(x-1)}\}$$

```
> #copying the solutions with suitable variables
> AA1:=1/2*((-3*x^2-2*x+1)^(1/2)*x^2+x^3+(-3*x^2-2*x+1)^(1/2)*x-2*x^
2-(-3*x^2-2*x+1)^(1/2)-2*x+1)/x^4/(x+1)/(x-1)^2:
BB0:=-1/2*(2*x^6-(-3*x^2-2*x+1)^(1/2)*x+x^2+(-3*x^2-2*x+1)^(1/2)+2
*x-1)/(x-1)^2/x^3/(x+1):
CC1:=1/2/(x-1)^2/x^3/(x+1)*(2*A0*x^5-2*A0*x^4+2*x^5-2*A0*x^3+2*x^4
+2*A0*x^2-2*x^3-(-3*x^2-2*x+1)^(1/2)*x+x^2+(-3*x^2-2*x+1)^(1/2)+2*
x-1): EE:=-x/(x-1):
FF:=-1/2*(-1+2*x^2+(-3*x^2-2*x+1)^(1/2)+x)/x^2/(x-1)^2/(x+1):
GG:=1/2*(2*x^4+2*x^3+2*x^2+(-3*x^2-2*x+1)^(1/2)+x-1)/x^3/(x+1)/(x-
1): HH:=x/(x-1)^2:    II1:=-x*(x^2-x+1)/(x^2-2*x+1)/(x-1):
> #solving eqA and eqB
> solve({subs(E=EE,F=FF,A1=AA1,eqA),subs(E=EE,F=FF,A1=AA1,H=HH,eqB)}
,{A,B});
```

$$\{A=(-1+2x^2+\sqrt{-3x^2-2x+1}+x)\Big/((-\sqrt{-3x^2-2x+1}\,x^3+2vx^3+x^4$$

$$+\sqrt{-3x^2-2x+1}\,x^2-2x^2v-2x^3+\sqrt{-3x^2-2x+1}\,x-2vx-\sqrt{-3x^2-2x+1}+2v+2x$$

$$-1)x^2),\ B=-(2x^4+2x^3+2x^2+\sqrt{-3x^2-2x+1}+x-1)(-\sqrt{-3x^2-2x+1}\,x^4+2vx^4+x^5$$

$$-x^4-\sqrt{-3x^2-2x+1}\,x^2+2x^2v+x^3-2vx-3x^2-2v+2x+2)\Big/(2($$

$$-v\sqrt{-3x^2-2x+1}\,x^3+2v^2x^3+vx^4+\sqrt{-3x^2-2x+1}\,vx^2+\sqrt{-3x^2-2x+1}\,x^3-2v^2x^2$$

$$-4\,v\,x^3 - x^4 + \sqrt{-3\,x^2 - 2\,x + 1}\;v\,x - \sqrt{-3\,x^2 - 2\,x + 1}\;x^2 - 2\,v^2\,x + 2\,x^2\,v + 2\,x^3$$
$$-\sqrt{-3\,x^2 - 2\,x + 1}\;v - \sqrt{-3\,x^2 - 2\,x + 1}\;x + 2\,v^2 + 4\,v\,x + \sqrt{-3\,x^2 - 2\,x + 1} - 3\,v - 2\,x + 1)\,x^3$$
$$(x^3 + x^2 + 2\,x + 1))\}$$

```
> #This solution defines A=AAA and B=BBB
> AAA:=(-1+2*x^2+(-3*x^2-2*x+1)^(1/2)+x)/(-(-3*x^2-2*x+1)^(1/2)*x^3+
  2*v*x^3+x^4+(-3*x^2-2*x+1)^(1/2)*x^2-2*x^2*v-2*x^3+(-3*x^2-2*x+1)^
  (1/2)*x-2*v*x-(-3*x^2-2*x+1)^(1/2)+2*v+2*x-1)/x^2:
  BBB:=-1/2*(2*x^4+2*x^3+2*x^2+(-3*x^2-2*x+1)^(1/2)+x-1)*(-(-3*x^2-2
  *x+1)^(1/2)*x^4+2*v*x^4+x^5-x^4-(-3*x^2-2*x+1)^(1/2)*x^2+2*x^2*v+x
  ^3-2*v*x-3*x^2-2*v+2*x+2)/(-(-3*x^2-2*x+1)^(1/2)*v*x^3+2*v^2*x^3+v
  *x^4+(-3*x^2-2*x+1)^(1/2)*v*x^2+(-3*x^2-2*x+1)^(1/2)*x^3-2*v^2*x^2
  -4*v*x^3-x^4+(-3*x^2-2*x+1)^(1/2)*v*x-(-3*x^2-2*x+1)^(1/2)*x^2-2*v
  ^2*x+2*x^2*v+2*x^3-(-3*x^2-2*x+1)^(1/2)*v-(-3*x^2-2*x+1)^(1/2)*x+2
  *v^2+4*v*x+(-3*x^2-2*x+1)^(1/2)-3*v-2*x+1)/x^3/(x^3+x^2+2*x+1):
> #Solve eqC
> eqC1:=subs(A=AAA,B=BBB,subs(D1=solve(eqD=0,D1),eqC)):
  coeff(eqC1,C): kc:=solve(%=0,v)[2]:
  CC1:=simplify(solve(subs(v=kc,subs(C=0,eqC1))=0,C1));
```

$$CC1 := -(3\,I1\,x^6 + I1\sqrt{-3\,x^2 - 2\,x + 1}\;x^5 - 4\,I1\,x^5 - I1\sqrt{-3\,x^2 - 2\,x + 1}\;x^4 + 3\,x^6$$
$$+\sqrt{-3\,x^2 - 2\,x + 1}\;x^5 - 2\,I1\,x^4 - I1\sqrt{-3\,x^2 - 2\,x + 1}\;x^3 + 2\,x^5 + \sqrt{-3\,x^2 - 2\,x + 1}\;x^4 + 4\,I1\,x^3$$
$$+I1\sqrt{-3\,x^2 - 2\,x + 1}\;x^2 - 7\,x^4 - 2\sqrt{-3\,x^2 - 2\,x + 1}\;x^3 - x^2\,I1 + 2\,x^3 + 2\sqrt{-3\,x^2 - 2\,x + 1}\;x - 4\,x$$
$$-2\sqrt{-3\,x^2 - 2\,x + 1} + 2) \Big/ (x\,(3\,x^5 + \sqrt{-3\,x^2 - 2\,x + 1}\;x^4 - x^4 - 2\sqrt{-3\,x^2 - 2\,x + 1}\;x^3 - 6\,x^3$$
$$+2\,x^2 + 2\sqrt{-3\,x^2 - 2\,x + 1}\;x + 3\,x - \sqrt{-3\,x^2 - 2\,x + 1} - 1))$$

```
> #Finds C and D_1
> map(simplify,solve({subs(A=AAA,eqC)=0,
  subs(B=BBB,C1=CC1,eqD)=0},{C,D1}));
```

$$\{C = -(-1 + 2\,v - 7\,v\,x + 5\,x - x^2\,v - 4\,x^2 + 2\,I1\,v\,x - I1\sqrt{-3\,x^2 - 2\,x + 1}\;x - I1\,x$$
$$-4\,I1\sqrt{-3\,x^2 - 2\,x + 1}\;x^4 + 4\,I1\sqrt{-3\,x^2 - 2\,x + 1}\;x^2 - 3\,I1\,x^6 + 7\,I1\,x^5 + 3\sqrt{-3\,x^2 - 2\,x + 1}\;x^5$$
$$-7\sqrt{-3\,x^2 - 2\,x + 1}\;x^3 + 2\sqrt{-3\,x^2 - 2\,x + 1}\;x^4 + 3\,I1\sqrt{-3\,x^2 - 2\,x + 1}\;x^5$$
$$-2\,I1\sqrt{-3\,x^2 - 2\,x + 1}\;x^3 + 3\sqrt{-3\,x^2 - 2\,x + 1}\;v\,x - 4\,v\,x^4 + 14\,v\,x^3 - 6\,v\,x^5$$
$$+2\sqrt{-3\,x^2 - 2\,x + 1}\;x^2 - 8\,I1\,v\,x^2 - 3\,x^6 + x^5 + 9\,x^4 - 2\,I1\,x^4 + 5\,x^2\,I1 - 6\,I1\,x^3 - 9\,x^3 + 4\,x^3\,I1\,v$$
$$-6\,I1\,v\,x^5 + 8\,I1\,v\,x^4 - \sqrt{-3\,x^2 - 2\,x + 1})\,(-x + \sqrt{-3\,x^2 - 2\,x + 1} - 1) \Big/ ((\sqrt{-3\,x^2 - 2\,x + 1}\;x^3$$
$$-2\,v\,x^3 - x^4 - \sqrt{-3\,x^2 - 2\,x + 1}\;x^2 + 2\,x^2\,v + 2\,x^3 - \sqrt{-3\,x^2 - 2\,x + 1}\;x + 2\,v\,x + \sqrt{-3\,x^2 - 2\,x + 1}$$
$$-2\,v - 2\,x + 1)\,(3\sqrt{-3\,x^2 - 2\,x + 1}\;x^2 - 6\,x^2\,v - 3\,x^3 + 2\sqrt{-3\,x^2 - 2\,x + 1}\;x - 4\,v\,x + x^2$$
$$-\sqrt{-3\,x^2 - 2\,x + 1} + 2\,v + 3\,x - 1)), \mathrm{D1} = (-x + \sqrt{-3\,x^2 - 2\,x + 1} - 1)\,(v - v^2 - 4\,v\,x + 5\,x^2\,v$$

$$- 2 x^2 + \sqrt{-3 x^2 - 2 x + 1}\, v x^4 - 3 \sqrt{-3 x^2 - 2 x + 1}\, v^2 x + 3 \sqrt{-3 x^2 - 2 x + 1}\, v^2 x^5$$
$$+ 3 \sqrt{-3 x^2 - 2 x + 1}\, v x^6 - 4 \sqrt{-3 x^2 - 2 x + 1}\, v^2 x^4 - 2 \sqrt{-3 x^2 - 2 x + 1}\, v^2 x^3$$
$$+ 4 \sqrt{-3 x^2 - 2 x + 1}\, v^2 x^2 - 4 \sqrt{-3 x^2 - 2 x + 1}\, v x^5 + 3\, I1 \sqrt{-3 x^2 - 2 x + 1}\, v x^6$$
$$- 4\, I1 \sqrt{-3 x^2 - 2 x + 1}\, v x^5 - 2\, I1 \sqrt{-3 x^2 - 2 x + 1}\, v x^4 + 4\, I1 \sqrt{-3 x^2 - 2 x + 1}\, v x^3$$
$$- I1 \sqrt{-3 x^2 - 2 x + 1}\, v x^2 + 6\, I1 \sqrt{-3 x^2 - 2 x + 1}\, x^4 + I1 \sqrt{-3 x^2 - 2 x + 1}\, x^2 - 3\, I1\, x^8 + 10\, I1\, x^7$$
$$- 9\, I1\, x^6 + \sqrt{-3 x^2 - 2 x + 1}\, v^2 - 4\, I1\, x^5 - 7 \sqrt{-3 x^2 - 2 x + 1}\, x^3 + 3 \sqrt{-3 x^2 - 2 x + 1}\, x^4$$
$$- \sqrt{-3 x^2 - 2 x + 1}\, v + 3\, I1 \sqrt{-3 x^2 - 2 x + 1}\, x^7 - 7\, I1 \sqrt{-3 x^2 - 2 x + 1}\, x^6$$
$$+ 2\, I1 \sqrt{-3 x^2 - 2 x + 1}\, x^5 - 5\, I1 \sqrt{-3 x^2 - 2 x + 1}\, x^3 + 9 v \sqrt{-3 x^2 - 2 x + 1}\, x^3$$
$$- 8 \sqrt{-3 x^2 - 2 x + 1}\, v x^2 + 3 \sqrt{-3 x^2 - 2 x + 1}\, v x - 3 v^2 x^2 + 4 v^2 x - 10 v x^4 - 3 v x^3 - 5 v^2 x^5$$
$$+ 8 v^2 x^4 - 2 v^2 x^3 + 5 v x^5 - 7 v x^6 + 4 \sqrt{-3 x^2 - 2 x + 1}\, x^2 + 2\, I1\, v^2 x^2 - 3\, I1\, v x^2 - 6 x^7 + 8 x^6$$
$$+ x^5 + 2 x^4 + 11\, I1\, x^4 - 3 v^2 x^6 + 3 v x^7 + x^2\, I1 - 6\, I1\, x^3 + 5 x^3 + 15 x^3\, I1\, v - 6\, I1\, v^2 x^6 - 9\, I1\, v x^7$$
$$+ 8\, I1\, v^2 x^5 + 21\, I1\, v x^6 + 4\, I1\, v^2 x^4 - 6\, I1\, v x^5 - 8\, I1\, v^2 x^3 - 18\, I1\, v x^4) \Big/ (2 x^2 (-1 + 3 v - 4 v^2$$
$$- 13 v x + 5 x + 5 x^2 v - 3 x^2 - 3 \sqrt{-3 x^2 - 2 x + 1}\, v x^4 + 6 \sqrt{-3 x^2 - 2 x + 1}\, v^2 x$$
$$+ 6 \sqrt{-3 x^2 - 2 x + 1}\, v^2 x^5 + 3 \sqrt{-3 x^2 - 2 x + 1}\, v x^6 - 2 \sqrt{-3 x^2 - 2 x + 1}\, v^2 x^4$$
$$- 12 \sqrt{-3 x^2 - 2 x + 1}\, v^2 x^3 + 4 \sqrt{-3 x^2 - 2 x + 1}\, v^2 x^2 - 10 \sqrt{-3 x^2 - 2 x + 1}\, v x^5$$
$$- 2 \sqrt{-3 x^2 - 2 x + 1}\, v^2 - 3 \sqrt{-3 x^2 - 2 x + 1}\, x^6 + 4 \sqrt{-3 x^2 - 2 x + 1}\, x^5 - 8 \sqrt{-3 x^2 - 2 x + 1}\, x^3$$
$$+ 5 \sqrt{-3 x^2 - 2 x + 1}\, x^4 + 3 \sqrt{-3 x^2 - 2 x + 1}\, v + 20 v \sqrt{-3 x^2 - 2 x + 1}\, x^3$$
$$- 3 \sqrt{-3 x^2 - 2 x + 1}\, v x^2 - 10 \sqrt{-3 x^2 - 2 x + 1}\, v x + 2 v^3 - 4 v^3 x^2 - 6 v^3 x + 12 v^3 x^3 + 2 v^2 x^2$$
$$+ 14 v^2 x - 19 v x^4 + 29 v x^3 + 14 v^2 x^5 + 8 v^2 x^4 - 28 v^2 x^3 - 19 v x^5 + 11 v x^6$$
$$- \sqrt{-3 x^2 - 2 x + 1}\, x^2 + 4 \sqrt{-3 x^2 - 2 x + 1}\, x - 3 x^7 - 5 x^6 + 11 x^5 + 9 x^4 - 6 v^3 x^5 - 6 v^2 x^6$$
$$+ 3 v x^7 + 2 v^3 x^4 - 13 x^3 - \sqrt{-3 x^2 - 2 x + 1}\,))\}$$

```
> CCC:=-(1+8*I1*v*x^2+6*I1*v*x^5-8*I1*v*x^4-4*I1*(-3*x^2-2*x+1)^(1/2
)*x^2-3*I1*(-3*x^2-2*x+1)^(1/2)*x^5+I1*(-3*x^2-2*x+1)^(1/2)*x+I1*x
-2*v-5*x+4*x^2-2*I1*v*x+4*I1*(-3*x^2-2*x+1)^(1/2)*x^4+2*I1*(-3*x^2
-2*x+1)^(1/2)*x^3+3*I1*x^6-7*I1*x^5+2*I1*x^4-3*(-3*x^2-2*x+1)^(1/2
)*x^5-2*(-3*x^2-2*x+1)^(1/2)*x^4+7*(-3*x^2-2*x+1)^(1/2)*x^3-2*(-3*
x^2-2*x+1)^(1/2)*x^2-3*(-3*x^2-2*x+1)^(1/2)*v*x-5*x^2*I1+6*I1*x^3+
3*x^6+6*v*x^5+4*v*x^4-14*v*x^3+(-3*x^2-2*x+1)^(1/2)+7*v*x-4*x^3*I1
*v+x^2*v-x^5-9*x^4+9*x^3)*(x-(-3*x^2-2*x+1)^(1/2)+1)/(-(-3*x^2-2*x
+1)^(1/2)*x^3+2*v*x^3+x^4+(-3*x^2-2*x+1)^(1/2)*x^2-2*x^2*v-2*x^3+(
-3*x^2-2*x+1)^(1/2)*x-2*v*x-(-3*x^2-2*x+1)^(1/2)+2*v+2*x-1)/(-3*(-
3*x^2-2*x+1)^(1/2)*x^2+6*x^2*v+3*x^3-2*(-3*x^2-2*x+1)^(1/2)*x+4*v*
x-x^2+(-3*x^2-2*x+1)^(1/2)-2*v-3*x+1):
DDD:=-1/2*(3*I1*v*x^2-(-3*x^2-2*x+1)^(1/2)*v^2-2*I1*v^2*x^2+6*I1*v
```

```
^2*x^6+9*I1*v*x^7-8*I1*v^2*x^5-21*I1*v*x^6-4*I1*v^2*x^4+6*I1*v*x^5
+8*I1*v^2*x^3+18*I1*v*x^4-I1*(-3*x^2-2*x+1)^(1/2)*x^2-2*I1*(-3*x^2
-2*x+1)^(1/2)*x^5+4*(-3*x^2-2*x+1)^(1/2)*v^2*x^4+2*(-3*x^2-2*x+1)^
(1/2)*v^2*x^3-4*(-3*x^2-2*x+1)^(1/2)*v^2*x^2+3*(-3*x^2-2*x+1)^(1/2
)*v^2*x-3*(-3*x^2-2*x+1)^(1/2)*v^2*x^5-3*(-3*x^2-2*x+1)^(1/2)*v*x^
6+I1*(-3*x^2-2*x+1)^(1/2)*v*x^2-3*I1*(-3*x^2-2*x+1)^(1/2)*v*x^6+4*
I1*(-3*x^2-2*x+1)^(1/2)*v*x^5+2*I1*(-3*x^2-2*x+1)^(1/2)*v*x^4-4*I1
*(-3*x^2-2*x+1)^(1/2)*v*x^3+3*I1*x^8-10*I1*x^7-v+2*x^2+(-3*x^2-2*x
+1)^(1/2)*v-3*I1*(-3*x^2-2*x+1)^(1/2)*x^7+7*I1*(-3*x^2-2*x+1)^(1/2
)*x^6-6*I1*(-3*x^2-2*x+1)^(1/2)*x^4+5*I1*(-3*x^2-2*x+1)^(1/2)*x^3+
9*I1*x^6+4*I1*x^5-11*I1*x^4-3*(-3*x^2-2*x+1)^(1/2)*x^4+7*(-3*x^2-2
*x+1)^(1/2)*x^3-4*(-3*x^2-2*x+1)^(1/2)*x^2+4*(-3*x^2-2*x+1)^(1/2)*
v*x^5-(-3*x^2-2*x+1)^(1/2)*v*x^4-9*(-3*x^2-2*x+1)^(1/2)*v*x^3+8*(-
3*x^2-2*x+1)^(1/2)*v*x^2-3*(-3*x^2-2*x+1)^(1/2)*v*x-x^2*I1+6*I1*x^
3+6*x^7-8*x^6+3*v^2*x^6-3*v*x^7+5*v^2*x^5+7*v*x^6-8*v^2*x^4-5*v*x^
5+2*v^2*x^3+10*v*x^4+3*v^2*x^2+3*v*x^3-4*v^2*x+v^2+4*v*x-15*x^3*I1
*v-5*x^2*v-x^5-2*x^4-5*x^3)*(x-(-3*x^2-2*x+1)^(1/2)+1)/x^2/(1+2*(-
3*x^2-2*x+1)^(1/2)*v^2+2*(-3*x^2-2*x+1)^(1/2)*v^2*x^4+12*(-3*x^2-2
*x+1)^(1/2)*v^2*x^3-4*(-3*x^2-2*x+1)^(1/2)*v^2*x^2-6*(-3*x^2-2*x+1
)^(1/2)*v^2*x-6*(-3*x^2-2*x+1)^(1/2)*v^2*x^5-3*(-3*x^2-2*x+1)^(1/2
)*v*x^6+6*v^3*x^5-2*v^3*x^4-3*v-5*x+3*x^2-3*(-3*x^2-2*x+1)^(1/2)*v
-4*(-3*x^2-2*x+1)^(1/2)*x+3*(-3*x^2-2*x+1)^(1/2)*x^6-4*(-3*x^2-2*x
+1)^(1/2)*x^5-5*(-3*x^2-2*x+1)^(1/2)*x^4+8*(-3*x^2-2*x+1)^(1/2)*x^
3+(-3*x^2-2*x+1)^(1/2)*x^2+10*(-3*x^2-2*x+1)^(1/2)*v*x^5+3*(-3*x^2
-2*x+1)^(1/2)*v*x^4-20*(-3*x^2-2*x+1)^(1/2)*v*x^3+3*(-3*x^2-2*x+1)
^(1/2)*v*x^2+10*(-3*x^2-2*x+1)^(1/2)*v*x-2*v^3-12*v^3*x^3+4*v^3*x^
2+6*v^3*x+3*x^7+5*x^6+6*v^2*x^6-3*v*x^7-14*v^2*x^5-11*v*x^6-8*v^2*
x^4+19*v*x^5+28*v^2*x^3+19*v*x^4-2*v^2*x^2-29*v*x^3-14*v^2*x+(-3*x
^2-2*x+1)^(1/2)+4*v^2+13*v*x-5*x^2*v-11*x^5-9*x^4+13*x^3):
> #solving for A0 when we use all other rules
> solve({eqB11=0,A0=x+x*B0+x*C1,B0=x+x*E+x*A1+x*F,E=x+x*E,F=x+x*E+x*
G+x*F,G=x+x*E+x*H+x*A1+x*F,H=x+x*E+x*H,I1=x+x*H+x*I1,C1=-(3*I1*x^6
+I1*(-3*x^2-2*x+1)^(1/2)*x^5-4*I1*x^5-I1*(-3*x^2-2*x+1)^(1/2)*x^4+
3*x^6+(-3*x^2-2*x+1)^(1/2)*x^5-2*I1*x^4-I1*(-3*x^2-2*x+1)^(1/2)*x^
3+2*x^5+(-3*x^2-2*x+1)^(1/2)*x^4+4*I1*x^3+I1*(-3*x^2-2*x+1)^(1/2)*
x^2-7*x^4-2*(-3*x^2-2*x+1)^(1/2)*x^3-x^2*I1+2*x^3+2*(-3*x^2-2*x+1)
^(1/2)*x-4*x-2*(-3*x^2-2*x+1)^(1/2)+2)/x/((-3*x^2-2*x+1)^(1/2)*x^4
+3*x^5-2*(-3*x^2-2*x+1)^(1/2)*x^3-x^4-6*x^3+2*(-3*x^2-2*x+1)^(1/2)
*x+2*x^2-(-3*x^2-2*x+1)^(1/2)+3*x-1)},{A0,A1,B0,C1,E,F,G,H,I1});
```

$$\{A0 = -(6x^8 - 14x^7 - 9\sqrt{-3x^2-2x+1}\,x^5 - 11x^6 + 6\sqrt{-3x^2-2x+1}\,x^4 + 17x^5 + \sqrt{-3x^2-2x+1}\,x^3 - 7x^4 - 5\sqrt{-3x^2-2x+1}\,x^2 - 2x^3 + 4\sqrt{-3x^2-2x+1}\,x + 7x^2$$

$$-\sqrt{-3x^2-2x+1}-5x+1)\Big/(2x^2(x+1)^2(3x-1)(x-1)(x^2-2x+1)),$$

$$A1=\frac{\sqrt{-3x^2-2x+1}\,x^2+x^3+\sqrt{-3x^2-2x+1}\,x-2x^2-\sqrt{-3x^2-2x+1}-2x+1}{2x^4(x+1)(x-1)^2},$$

$$B0=-\frac{2x^6-\sqrt{-3x^2-2x+1}\,x+x^2+\sqrt{-3x^2-2x+1}+2x-1}{2(x-1)^2(x+1)x^3},\ C1=(18x^7$$

$$+9\sqrt{-3x^2-2x+1}\,x^5-3x^6-9\sqrt{-3x^2-2x+1}\,x^4-16x^5+3\sqrt{-3x^2-2x+1}\,x^3+20x^4$$

$$+7\sqrt{-3x^2-2x+1}\,x^2-8x^3-8\sqrt{-3x^2-2x+1}\,x-11x^2+2\sqrt{-3x^2-2x+1}+10x-2)\Big/$$

$$(2x^3(x+1)^2(3x-1)(x-1)(x^2-2x+1)),\ E=-\frac{x}{x-1},$$

$$F=-\frac{-1+2x^2+\sqrt{-3x^2-2x+1}+x}{2x^2(x-1)^2(x+1)},\ G=\frac{2x^4+2x^3+2x^2+\sqrt{-3x^2-2x+1}+x-1}{2x^3(x+1)(x-1)},$$

$$H=\frac{x}{(x-1)^2},\ I1=-\frac{x(x^2-x+1)}{(x^2-2x+1)(x-1)}\}$$

```
> FinalA0:=-1/2*(6*x^8-14*x^7-9*(-3*x^2-2*x+1)^(1/2)*x^5-11*x^6+6*(-
  3*x^2-2*x+1)^(1/2)*x^4+17*x^5+(-3*x^2-2*x+1)^(1/2)*x^3-7*x^4-5*(-3
  *x^2-2*x+1)^(1/2)*x^2-2*x^3+4*(-3*x^2-2*x+1)^(1/2)*x+7*x^2-(-3*x^2
  -2*x+1)^(1/2)-5*x+1)/x^2/(x+1)^2/(3*x-1)/(x-1)/(x^2-2*x+1);
```

$$FinalA0:=-(6x^8-14x^7-9\sqrt{-3x^2-2x+1}\,x^5-11x^6+6\sqrt{-3x^2-2x+1}\,x^4+17x^5$$

$$+\sqrt{-3x^2-2x+1}\,x^3-7x^4-5\sqrt{-3x^2-2x+1}\,x^2-2x^3+4\sqrt{-3x^2-2x+1}\,x+7x^2$$

$$-\sqrt{-3x^2-2x+1}-5x+1)\Big/(2x^2(x+1)^2(3x-1)(x-1)(x^2-2x+1))$$

```
> taylor(FinalA0,x,20);
```

$$x+2x^2+6x^3+19x^4+60x^5+179x^6+526x^7+1527x^8+4426x^9+12828x^{10}+37241x^{11}+$$

$$108301x^{12}+315509x^{13}+920632x^{14}+2690246x^{15}+7871635x^{16}+23059538x^{17}+O(x^{18})$$

```
> factor(coeff(FinalA0,(-3*x^2-2*x+1)^(1/2)))*(-3*x^2-2*x+1)^(1/2)+f
  actor(coeff(FinalA0,(-3*x^2-2*x+1)^(1/2),0));
```

$$\frac{(9x^5-6x^4-x^3+5x^2-4x+1)\sqrt{-3x^2-2x+1}}{2x^2(x+1)^2(3x-1)(x-1)^3}-\frac{2x^6-6x^5+x^4+3x^3-4x^2+3x-1}{2(x-1)^3(x+1)x^2}$$

```
> M:=(1-x-sqrt(1-2*x-3*x^2))/2/x^2:
  MM:=n->coeff(taylor(M,x,n+10),x,n): #g.f. of Motzkin numbers;
> ff:=factor(subs((-3*x^2-2*x+1)^(1/2)=1-x-2*x^2*fM,FinalA0)):
  factor(coeff(ff,fM,1))*fM+factor(coeff(ff,fM,0));
```

$$-\frac{(9x^5-6x^4-x^3+5x^2-4x+1)fM}{(x+1)^2(3x-1)(x-1)^3}-\frac{3x^6-7x^5-x^4+x^3-x^2+2x-1}{(x+1)^2(3x-1)(x-1)^3}$$

```
> with(genfunc):
> simplify(rgf_expand(-(9*x^5-6*x^4-x^3+5*x^2-4*x+1)/(x+1)^2/(3*x-1)
```

```
/(x-1)^3,x,n));
simplify(rgf_expand(-(3*x^6-7*x^5-x^4+x^3-x^2+2*x-1)/(x+1)^2/(3*x-
1)/(x-1)^3,x,n));
```

$$\frac{(-1)^n n}{8}+\frac{n^2}{4}-\frac{9\,3^n}{32}+\frac{39\,(-1)^{(n+1)}}{32}-n+\frac{1}{2}$$

$$\frac{(-1)^{(n+1)} n}{8}-\frac{n^2}{4}+\frac{27\,3^n}{32}+\frac{29\,(-1)^n}{32}+\frac{n}{4}+\frac{1}{4}$$

```
> CoeffxnFinalA0:=n->add(((4*j-39)/32*(-1)^j+1/4*(j^2-4*j+2)-1/32*3^
(j+2))*MM(n-j),j=0..n)+(29-4*n)/32*(-1)^n-1/4*(n^2-n-1)+1/32*3^(n+
3);
```

$$CoeffxnFinalA0 := n \to \mathrm{add}\left(\left(\frac{(4\,j-39)\,(-1)^j}{32}+\frac{j^2}{4}-j+\frac{1}{2}-\frac{3^{(j+2)}}{32}\right)\mathrm{MM}(n-j), j=0\,..\,n\right)$$
$$+\frac{1}{32}\,(29-4\,n)\,(-1)^n-\frac{1}{4}\,n^2+\frac{1}{4}\,n+\frac{1}{4}+\frac{1}{32}\,3^{(n+3)}$$

```
> #checking the formula up to n=20:
> seq(CoeffxnFinalA0(i+1),i=0..20); taylor(FinalA0,x,23);
```

1, 2, 6, 19, 60, 179, 526, 1527, 4426, 12828, 37241, 108301, 315509, 920632, 2690246, 7871635, 23059538, 67624102, 198507563, 583232749

$$x+2\,x^2+6\,x^3+19\,x^4+60\,x^5+179\,x^6+526\,x^7+1527\,x^8+4426\,x^9+12828\,x^{10}+37241\,x^{11}+108301\,x^{12}+315509\,x^{13}+920632\,x^{14}+2690246\,x^{15}+7871635\,x^{16}+23059538\,x^{17}+67624102\,x^{18}+198507563\,x^{19}+583232749\,x^{20}+\mathrm{O}(x^{21})$$

```
> #-----------------------------------------------------------------
-----------------------------------------
> #Case 0010
> restart: A1:=x+x*A2+x*B1; C:=(x+x*A2)/(1-x);
eqA:=-A+x/(1-v)+x/v*(A-A2)+x/(1-v)*A+x/(1-v)*C;
eqB:=-B+x/(1-v)+x/v*(A-A2)+x/v*(B-B1)+x/(1-v)*B;
```

$$A1 := A2\,x+B1\,x+x$$

$$C := \frac{A2\,x+x}{1-x}$$

$$eqA := -A+\frac{x}{1-v}+\frac{x\,(A-A2)}{v}+\frac{x\,A}{1-v}+\frac{x\,(A2\,x+x)}{(1-v)\,(1-x)}$$

$$eqB := -B+\frac{x}{1-v}+\frac{x\,(A-A2)}{v}+\frac{x\,(B-B1)}{v}+\frac{x\,B}{1-v}$$

```
> coeff(eqA,A); ka:=solve(%=0,v)[2];
AAA2:=solve(subs(A=0,v=ka,eqA)=0,A2);
AAA:=solve(subs(A2=AAA2,eqA)=0,A);
```

$$-1+\frac{x}{v}+\frac{x}{1-v}$$

$$ka := \frac{1}{2}-\frac{\sqrt{1-4\,x}}{2}$$

$$AAA2 := -\frac{-1+\sqrt{1-4\,x}}{1+\sqrt{1-4\,x}-2\,x}$$

$$AAA := -\frac{(\sqrt{1-4\,x}+2\,v-1)\,x}{(1+\sqrt{1-4\,x}-2\,x)\,(v^2-v+x)}$$

```
> limit(subs(B=0,subs(A=AAA,A2=AAA2,eqB)),v=ka);
BBB1:=solve(%=0,B1);
```

$$\frac{-3\,B1\,\sqrt{1-4\,x}\,x+4\,B1\,x^2+B1\,\sqrt{1-4\,x}-5\,x\,B1+4\,x^2+B1-2\,x}{2\,\sqrt{1-4\,x}\,x-\sqrt{1-4\,x}+4\,x-1}$$

$$BBB1 := \frac{2\,x\,(2\,x-1)}{-1+3\,\sqrt{1-4\,x}\,x-4\,x^2-\sqrt{1-4\,x}+5\,x}$$

```
> FinalA:=1/factor(simplify(rationalize(1/subs(A2=AAA2,B1=BBB1,A1))+
1)-1); taylor(FinalA,x,20);
```

$$FinalA := \frac{x}{\sqrt{1-4\,x}}$$

$$x+2\,x^2+6\,x^3+20\,x^4+70\,x^5+252\,x^6+924\,x^7+3432\,x^8+12870\,x^9+48620\,x^{10}+184756\,x^{11}+705432\,x^{12}+2704156\,x^{13}+10400600\,x^{14}+40116600\,x^{15}+155117520\,x^{16}+601080390\,x^{17}+2333606220\,x^{18}+9075135300\,x^{19}+\mathrm{O}(x^{20})$$

```
> seq(binomial(2*n,n),n=0..19);
```

1, 2, 6, 20, 70, 252, 924, 3432, 12870, 48620, 184756, 705432, 2704156, 10400600, 40116600, 155117520, 601080390, 2333606220, 9075135300

```
> #-------------------------------------------------------------------
-------------------------------------------------
```

```
> ##0011
```

```
> restart: A1:=x+x*A2+x*B1; C:=(x+x*A2)/(1-x);
eqA:=-A+x/(1-v)+x/v*(A-A2)+x/(1-v)*A+x/(1-v)*C;
eqB:=-B+x/(1-v)+x/v*(A-A2)+x/v*(B-B1)+x*v/(1-v)*A+x*v/(1-v)*C+x*B1
;
```

$$A1 := A2\,x+B1\,x+x$$

$$C := \frac{A2\,x+x}{1-x}$$

$$eqA := -A+\frac{x}{1-v}+\frac{x\,(A-A2)}{v}+\frac{x\,A}{1-v}+\frac{x\,(A2\,x+x)}{(1-v)\,(1-x)}$$

$$eqB := -B + \frac{x}{1-v} + \frac{x\,(A-A2)}{v} + \frac{x\,(B-B1)}{v} + \frac{x\,v\,A}{1-v} + \frac{x\,v\,(A2\,x+x)}{(1-v)\,(1-x)} + x\,B1$$

```
> coeff(eqA,A); ka:=solve(%=0,v)[2];
  AAA2:=solve(subs(A=0,v=ka,eqA)=0,A2);
  AAA:=solve(subs(A2=AAA2,eqA)=0,A);
```

$$-1 + \frac{x}{v} + \frac{x}{1-v}$$

$$ka := \frac{1}{2} - \frac{\sqrt{1-4\,x}}{2}$$

$$AAA2 := -\frac{-1+\sqrt{1-4\,x}}{1+\sqrt{1-4\,x}-2\,x}$$

$$AAA := -\frac{(\sqrt{1-4\,x}+2\,v-1)\,x}{(1+\sqrt{1-4\,x}-2\,x)\,(v^2-v+x)}$$

```
> eqB1:=subs(A=AAA,A2=AAA2,eqB); BBB1:=solve(subs(v=x,eqB1)=0,B1);
```

$$eqB1 := -B + \frac{x}{1-v} + \frac{x\left(-\frac{(\sqrt{1-4\,x}+2\,v-1)\,x}{(1+\sqrt{1-4\,x}-2\,x)\,(v^2-v+x)} + \frac{-1+\sqrt{1-4\,x}}{1+\sqrt{1-4\,x}-2\,x}\right)}{v} + \frac{x\,(B-B1)}{v} - \frac{x^2\,v\,(\sqrt{1-4\,x}+2\,v-1)}{(1-v)\,(1+\sqrt{1-4\,x}-2\,x)\,(v^2-v+x)} + \frac{x\,v\left(-\frac{x\,(-1+\sqrt{1-4\,x})}{1+\sqrt{1-4\,x}-2\,x}+x\right)}{(1-v)\,(1-x)} + x\,B1$$

$$BBB1 := -\frac{-2\,x^3+\sqrt{1-4\,x}\,x+2\,x^2-\sqrt{1-4\,x}-3\,x+1}{x\,(1+\sqrt{1-4\,x}-2\,x)\,(-1+x)}$$

```
> FinalA:=simplify(subs(A2=AAA2,B1=BBB1,A1)); taylor(FinalA,x,20);
```

$$FinalA := -\frac{\sqrt{1-4\,x}\,x-2\,x^2-\sqrt{1-4\,x}-x+1}{(-1+x)\,(1+\sqrt{1-4\,x}-2\,x)}$$

$$x + 2\,x^2 + 6\,x^3 + 20\,x^4 + 68\,x^5 + 233\,x^6 + 805\,x^7 + 2807\,x^8 + 9879\,x^9 + 35073\,x^{10} + 125513\,x^{11} + 452389\,x^{12} + 1641029\,x^{13} + 5986994\,x^{14} + 21954974\,x^{15} + 80884424\,x^{16} + 299233544\,x^{17} + 1111219334\,x^{18} + 4140813374\,x^{19} + O(x^{20})$$

```
> FinalA:=simplify(simplify(rationalize(FinalA)+1)-1);
  coeff(FinalA,sqrt(1-4*x))*sqrt(1-4*x)+factor(coeff(FinalA,sqrt(1-4
  *x),0));
```

$$FinalA := -\frac{2\,x^3+2\,\sqrt{1-4\,x}\,x+2\,x^2-\sqrt{1-4\,x}-4\,x+1}{2\,x^2\,(-1+x)}$$

$$-\frac{(2x-1)\sqrt{1-4x}}{2x^2(-1+x)}-\frac{2x^3+2x^2-4x+1}{2x^2(-1+x)}$$

```
> CC:=(1-sqrt(1-4*x))/2/x: #generating function Catalan numbers
> cc:=n->1/(n+1)*binomial(2*n,n);
```

$$cc := n \to \frac{\mathrm{binomial}(2n, n)}{n+1}$$

```
> ff:=subs(sqrt(1-4*x)=1-2*x*fC,FinalA):
  factor(coeff(ff,fC,1))*fC+factor(coeff(ff,fC,0));
```

$$\frac{(2x-1)fC}{x(-1+x)}-\frac{x^2+x-1}{x(-1+x)}$$

```
> seq(cc(n+2)-add(cc(j),j=0..n+1)+1,n=0..20);
```

1, 2, 6, 20, 68, 233, 805, 2807, 9879, 35073, 125513, 452389, 1641029, 5986994, 21954974, 80884424, 299233544, 1111219334, 4140813374, 15478839554, 58028869154

```
> #-----------------------------------------------------------------
  ---------------------------
> ##0012
> restart:
  eqA:=-A+x/(1-v)+x/v*(A-A2)+sum(m*x^2*v^(m-2)/(1-2*x),m=2..infinity
  ); AAA2:=solve(subs(v=x,eqA)=0,A2);
  AAA:=factor(solve(subs(A2=AAA2,eqA)=0,A));
```

$$eqA := -A+\frac{x}{1-v}+\frac{x(A-A2)}{v}+\frac{x^2(v-2)}{(-1+2x)(v-1)^2}$$

$$AAA2 := -\frac{x(x^2-x+1)}{(-1+2x)(x-1)^2}$$

$$AAA := \frac{(2vx^2-2vx-x^2+v+x-1)x}{(x-1)^2(-1+2x)(v-1)^2}$$

```
> eqB:=-B+x/(1-v)+x/v^2*(AAA-AAA2-v*A3)+x/v*(B-B2)+sum(m*x^2*v^(m-2)
  /(1-2*x),m=2..infinity);
```

$$eqB := -B+\frac{x}{1-v}+\frac{x\left(-vA3+\frac{(2vx^2-2vx-x^2+v+x-1)x}{(x-1)^2(-1+2x)(v-1)^2}+\frac{x(x^2-x+1)}{(-1+2x)(x-1)^2}\right)}{v^2}+\frac{x(B-B2)}{v}+\frac{x^2(v-2)}{(-1+2x)(v-1)^2}$$

```
> BBB2:=solve(subs(v=x,eqB)=0,B2);
```

$$BBB2 := -\frac{2A3x^4-7A3x^3+x^4+9A3x^2-3x^3-5A3x+2x^2+A3-2x}{(x-1)^3(-1+2x)}$$

```
> AAA3:=subs(v=0,diff(AAA,v)); BBB2:=factor(subs(A3=AAA3,BBB2));
```

$$AAA3 := \frac{(2x^2-2x+1)x}{(x-1)^2(-1+2x)} + \frac{2(-x^2+x-1)x}{(x-1)^2(-1+2x)}$$

$$BBB2 := -\frac{x(x^3-3x^2+x-1)}{(x-1)^3(-1+2x)}$$

```
> solve({A1=x+x*AAA2+x*B1,B1=x+x*AAA3+x*BBB2+x*B1},{B1,A1});
```

$$\{A1 = -\frac{x(2x^4-5x^3+8x^2-4x+1)}{2x^5-9x^4+16x^3-14x^2+6x-1}, B1 = -\frac{x(x^4-4x^3+7x^2-3x+1)}{2x^5-9x^4+16x^3-14x^2+6x-1}\}$$

```
> FinalA:=factor(-x*(2*x^4-5*x^3+8*x^2-4*x+1)/(2*x^5-9*x^4+16*x^3-14
*x^2+6*x-1));
```

$$FinalA := -\frac{x(2x^4-5x^3+8x^2-4x+1)}{(x-1)^4(-1+2x)}$$

```
> taylor(FinalA,x,14);
```

$$x + 2x^2 + 6x^3 + 19x^4 + 55x^5 + 144x^6 + 348x^7 + 793x^8 + 1733x^9 + 3678x^{10} + 7650x^{11} + 15695x^{12} + 31907x^{13} + O(x^{14})$$

```
> with(genfunc): simplify(rgf_expand(FinalA,x,n));
```

$$-\frac{n^3}{3} - \frac{n^2}{2} + 4\,2^n - \frac{19n}{6} - 3$$

```
> seq(2^(n+3)-1/6*(2*n^2+7*n+24)*(n+1)-3,n=0..20);
```

1, 2, 6, 19, 55, 144, 348, 793, 1733, 3678, 7650, 15695, 31907, 64476, 129784, 260597, 522449, 1046410, 2094622, 4191371, 8385231

```
> #-------------------------------------------------------------------
-------------------------------------------------
> ###0100=0110
> FinalA:=(1-3*x)^2/2/x^2/sqrt(1-4*x)-(1-3*x)*(1-x)/2/x^2;
```

$$FinalA := \frac{(1-3x)^2}{2x^2\sqrt{-4x+1}} - \frac{(1-3x)(1-x)}{2x^2}$$

```
> taylor(FinalA,x,20);
```

$$x + 2x^2 + 6x^3 + 21x^4 + 78x^5 + 297x^6 + 1144x^7 + 4433x^8 + 17238x^9 + 67184x^{10} + 262276x^{11} + 1025202x^{12} + 4011660x^{13} + 15712335x^{14} + 61590780x^{15} + 241610745x^{16} + 948453990x^{17} + O(x^{18})$$

```
> seq(1/2*binomial(2*n+4,n+2)-3*binomial(2*n+2,n+1)+9/2*binomial(2*n
,n),n=1..10);
```

1, 2, 6, 21, 78, 297, 1144, 4433, 17238, 67184

```
> factor(simplify(1/2*binomial(2*n+4,n+2)-3*binomial(2*n+2,n+1)+9/2*
binomial(2*n,n),GAMMA));
```

$$\frac{1}{8}\frac{(n^2-n+6)\Gamma(2n+5)}{(2n+3)(2n+1)\Gamma(n+3)^2}$$

```
> seq(1/8*(n^2+n+6)/(2*n+5)/(2*n+3)*binomial(2*n+6,n+3),n=0..20);
```

1, 2, 6, 21, 78, 297, 1144, 4433, 17238, 67184, 262276, 1025202, 4011660, 15712335, 61590780, 241610745, 948453990, 3725581860, 14643037860, 57585238230, 226578907620

```
> #-------------------------------------------------------------------
  ----------------------------------------------
> ##0101=0111
> restart: eqA:=-A+x/(1-v)+x/v*(A-A1)+x/(1-v)*B;
  eqB:=-B+x/(1-v)+x/v*(A-A1)+x/v*(B-B1)+x/(1-v)*B;
```

$$eqA := -A + \frac{x}{1-v} + \frac{x\,(A - A1)}{v} + \frac{x\,B}{1-v}$$

$$eqB := -B + \frac{x}{1-v} + \frac{x\,(A - A1)}{v} + \frac{x\,(B - B1)}{v} + \frac{x\,B}{1-v}$$

```
> eqB1:=subs(A=solve(eqA=0,A),eqB); factor(coeff(eqB1,B));
  alias(vv=RootOf(v^3-v^2*x+v*x^2-v^2+2*v*x-x^2=0,v));
  vvv:=allvalues(vv):
```

$$eqB1 := -B + \frac{x}{1-v} + \frac{x\left(-\frac{x\,(A1\,v + B\,v - A1 + v)}{(-1+v)\,(v-x)} - A1\right)}{v} + \frac{x\,(B - B1)}{v} + \frac{x\,B}{1-v}$$

$$-\frac{v^3 - v^2\,x + v\,x^2 - v^2 + 2\,v\,x - x^2}{(-1+v)\,(v-x)\,v}$$

$$vv$$

```
> factor(taylor(v^3-v^2*x+v*x^2-v^2+2*v*x-x^2,v,10)); a1:=-1-x:
  b1:=x*(x+2): c1:=-x^2: Q1:=factor((3*b1-a1^2)/9);
  R1:=factor((9*a1*b1-27*c1-2*a1^3)/54);
  tt:=simplify(R1/sqrt(-Q1^3));
  xx:=2*sqrt(-Q1)*cos(arccos(tt)/3+j*2*Pi/3)-a1/3; ##xx :=
  2/3*(-2*x^2-4*x+1)^(1/2)*cos(1/3*Pi+1/3*arccos(1/2*(-7*x^3+6*x^2-1
  2*x+2)/((-2*x^2-4*x+1)^3)^(1/2))+2/3*j*Pi)+1/3*x+1/3;
```

$$-x^2 + x\,(x+2)\,v + (-x-1)\,v^2 + v^3$$

$$Q1 := \frac{2}{9}x^2 + \frac{4}{9}x - \frac{1}{9}$$

$$R1 := -\frac{7}{54}x^3 + \frac{1}{9}x^2 - \frac{2}{9}x + \frac{1}{27}$$

$$tt := -\frac{7\,x^3 - 6\,x^2 + 12\,x - 2}{2\sqrt{-(2\,x^2 + 4\,x - 1)^3}}$$

$$xx := \frac{2}{3}\sqrt{-2\,x^2 - 4\,x + 1}\,\cos\left(\frac{\pi}{3} - \frac{1}{3}\arccos\left(\frac{7\,x^3 - 6\,x^2 + 12\,x - 2}{2\sqrt{-(2\,x^2 + 4\,x - 1)^3}}\right) + \frac{2\,j\,\pi}{3}\right) + \frac{x}{3} + \frac{1}{3}$$

```
> subs(signum(9*x^6+12*x^5+60*x^4-12*x^3)=1,simplify(series(vvv[2],x
,4))) assuming x>0 and x<0.01; simplify(series(subs(j=2,xx),x,4))
assuming x>0 and x<0.01;
```

$$x + x^{(3/2)} + x^2 + \frac{3\,x^{(5/2)}}{2} + \frac{5\,x^3}{2} + \frac{35\,x^{(7/2)}}{8} + \mathrm{O}(x^4)$$

$$x + x^{(3/2)} + x^2 + \frac{3\,x^{(5/2)}}{2} + \frac{5\,x^3}{2} + \frac{35\,x^{(7/2)}}{8} + \mathrm{O}(x^4)$$

```
>  simplify(series(vvv[1]*vvv[2]*vvv[3],x,5)) assuming x>0 and
x<0.01; simplify(series(vvv[1]+vvv[2]+vvv[3],x,5)) assuming x>0
and x<0.01;
```

$$x^2 + \mathrm{O}(x^5)$$

$$1 + x$$

```
> solve({subs(v=u1,B=0,eqB1),subs(v=u2,B=0,eqB1)},{A1,B1});
```

$$\{A1 = -\frac{u1\,u2\,x + u1\,u2 - u1\,x - u2\,x}{(-1+u1)\,(-1+u2)\,x},\ B1 = \frac{u1\,u2}{x\,(u1\,u2 - u1 - u2 + 1)}\}$$

```
> simplify(series(subs(u1=vvv[2],u2=vvv[3],-(u1*u2*x+u1*u2-u1*x-u2*x
)/(-1+u1)/(-1+u2)/x),x,11));
```

$$x + 2\,x^2 + 1347\,x^7 + 21\,x^4 + 5798\,x^8 + \mathrm{O}(x^{10}) + 6\,x^3 + 322\,x^6 + 80\,x^5 + 25512\,x^9$$

```
> factor(simplify(-(x^2/u3*x+x^2/u3-x*(1+x-u3))/(1-(1+x-u3)+x^2/u3)/
x));
```

$$-\frac{u3^2 - u3\,x + x^2 - u3 + x}{u3^2 - u3\,x + x^2}$$

```
> simplify(series(subs(u3=vvv[1],-(u3^2-u3*x+x^2-u3+x)/(u3^2-u3*x+x^
2)),x,10));
```

$$x + 2\,x^2 + 1347\,x^7 + 21\,x^4 + 5798\,x^8 + \mathrm{O}(x^{10}) + 6\,x^3 + 322\,x^6 + 80\,x^5 + 25512\,x^9$$

```
> alias(ww=RootOf(f=x*(f+1)/(1-x*(f+1)^2),f)); taylor(ww,x,10);
```

$$vv,\ ww$$

$$x + 2\,x^2 + 6\,x^3 + 21\,x^4 + 80\,x^5 + 322\,x^6 + 1347\,x^7 + 5798\,x^8 + 25512\,x^9 + \mathrm{O}(x^{10})$$

```
> ss:=factor(simplify(subs(f=-(u3^2-u3*x+x^2-u3+x)/(u3^2-u3*x+x^2),-
f+x*(f+1)/(1-x*(f+1)^2))));
simplify(series(subs(u3=vvv[1],ss),x,10));
```

$$ss := \frac{(u3^3 - u3^2\,x + u3\,x^2 - u3\,x + x^2)\,(u3^3 - u3^2\,x + u3\,x^2 - u3^2 + 2\,u3\,x - x^2)}{(u3^2 - u3\,x + x^2)\,(u3^4 - 2\,u3^3\,x + 3\,u3^2\,x^2 - 2\,u3\,x^3 + x^4 - u3^2\,x + 2\,u3\,x^2 - x^3)}$$

$$\mathrm{O}(x^{10})$$

```
> seq(add(1/i*binomial(n,i-1)*binomial(2*n+2-i,i-1),i=1..n+1),n=0..1
0);
```

$$1,\ 2,\ 6,\ 21,\ 80,\ 322,\ 1347,\ 5798,\ 25512,\ 114236,\ 518848$$

```
> #-----------------------------------------------------------------
```

```
-----------------------------
> ##0102
> restart: D1:=x/(1-2*x); E:=x/(1-x):
  eqA:=-A+x/(1-v)+x/v*(A-A1)+x/(1-v)*B;
  eqB:=-B+x/(1-v)*(1+D1+C)+x/v*(B-B1);
  eqC:=-C+x/(1-v)*(1+E+C)+x/v*(C-C1);
```

$$D1 := \frac{x}{1-2x}$$

$$eqA := -A + \frac{x}{1-v} + \frac{x\,(A-A1)}{v} + \frac{x\,B}{1-v}$$

$$eqB := -B + \frac{x\left(1+\frac{x}{1-2x}+C\right)}{1-v} + \frac{x\,(B-B1)}{v}$$

$$eqC := -C + \frac{x\left(1+\frac{x}{1-x}+C\right)}{1-v} + \frac{x\,(C-C1)}{v}$$

```
> coeff(eqC,C); kc:=solve(%=0,v)[2];
  CC1:=solve(subs(v=kc,subs(C=0,eqC))=0,C1);
  CCC:=solve(subs(C1=CC1,eqC)=0,C);
```

$$-1 + \frac{x}{1-v} + \frac{x}{v}$$

$$kc := \frac{1}{2} - \frac{\sqrt{1-4x}}{2}$$

$$CC1 := \frac{-1+\sqrt{1-4x}}{\sqrt{1-4x}\,x - \sqrt{1-4x} + x - 1}$$

$$CCC := \frac{(\sqrt{1-4x}+2v-1)\,x}{(\sqrt{1-4x}\,x - \sqrt{1-4x} + x - 1)\,(v^2 - v + x)}$$

```
> BB1:=solve(subs(v=x,subs(C=CCC,eqB))=0,B1);
  BBB:=simplify(solve(subs(B1=BB1,C=CCC,eqB)=0,B));
```

$$BB1 := -\frac{\sqrt{1-4x}\,x^3 - 2\sqrt{1-4x}\,x^2 + x^3 + 3\sqrt{1-4x}\,x + 2x^2 - \sqrt{1-4x} - 3x + 1}{(-1+x)\,(-1+2x)\,(\sqrt{1-4x}\,x - \sqrt{1-4x} + x - 1)}$$

$$BBB := (\sqrt{1-4x}\,v^2 x^2 + \sqrt{1-4x}\,v x^2 + \sqrt{1-4x}\,x^3 + 5v^2 x^2 - 3\sqrt{1-4x}\,v x - 2\sqrt{1-4x}\,x^2 - 6v^2 x - 7v x^2 + x^3 + \sqrt{1-4x}\,v + 3\sqrt{1-4x}\,x + 2v^2 + 9xv + 2x^2 - \sqrt{1-4x} - 3v - 3x + 1)\,x \Big/ ((-1+x)\,(-1+v)\,(-1+2x)\,(\sqrt{1-4x}\,x - \sqrt{1-4x} + x - 1)\,(v^2 - v + x))$$

```
> FinalA:=solve(subs(v=x,subs(B=BBB,eqA))=0,A1);
  taylor(FinalA,x,20);
```

$$FinalA := -(2\sqrt{1-4x}\,x^5 - 6\sqrt{1-4x}\,x^4 + 2x^5 + 11\sqrt{1-4x}\,x^3 - 2x^4 - 10\sqrt{1-4x}\,x^2 - 3x^3$$

$$+5\sqrt{1-4x}\,x+8x^2-\sqrt{1-4x}-5x+1)\Big/((-1+x)^3(-1+2x)(\sqrt{1-4x}\,x-\sqrt{1-4x}+x-1))$$

$$x+2x^2+6x^3+20x^4+66x^5+213x^6+683x^7+2211x^8+7291x^9+24552x^{10}+84305x^{11}+294297x^{12}+1041213x^{13}+3723752x^{14}+13434874x^{15}+48825190x^{16}+178535378x^{17}+656304995x^{18}+2423830156x^{19}+O(x^{20})$$

```
> FinalA:=factor(simplify(simplify(rationalize(FinalA)+1)-1));
  factor(coeff(FinalA,sqrt(1-4*x)))*sqrt(1-4*x)+factor(coeff(FinalA,
  sqrt(1-4*x),0)); taylor(FinalA,x,20);
```

$$FinalA := (-4x^6+2\sqrt{1-4x}\,x^4+12x^5-7\sqrt{1-4x}\,x^3-24x^4+9\sqrt{1-4x}\,x^2+27x^3-5\sqrt{1-4x}\,x-19x^2+\sqrt{1-4x}+7x-1)\Big/(2(-1+2x)(-1+x)^4x)$$

$$\frac{\sqrt{1-4x}}{2x(-1+x)}-\frac{4x^6-12x^5+24x^4-27x^3+19x^2-7x+1}{2(-1+2x)(-1+x)^4x}$$

$$x+2x^2+6x^3+20x^4+66x^5+213x^6+683x^7+2211x^8+7291x^9+24552x^{10}+84305x^{11}+294297x^{12}+1041213x^{13}+3723752x^{14}+13434874x^{15}+48825190x^{16}+178535378x^{17}+656304995x^{18}+O(x^{19})$$

```
> CC:=(1-sqrt(1-4*x))/2/x: #generating function Catalan numbers
> cc:=n->1/(n+1)*binomial(2*n,n);
```

$$cc := n \to \frac{\mathrm{binomial}(2n,n)}{n+1}$$

```
> ff:=subs(sqrt(1-4*x)=1-2*x*fC,FinalA):
  factor(coeff(ff,fC,1))*fC+factor(coeff(ff,fC,0));
```

$$-\frac{fC}{-1+x}-\frac{(2x^2-2x+1)(x^3-2x^2+3x-1)}{(-1+2x)(-1+x)^4}$$

```
> with(genfunc):
  simplify(rgf_expand(-(2*x^2-2*x+1)*(x^3-2*x^2+3*x-1)/(-1+2*x)/(-1+
  x)^4,x,n));
```

$$-\frac{n^3}{6}+2^n-\frac{11n}{6}-1$$

```
> seq(add(cc(j),j=0..n+1)+2^(n+1)-(n+1)*(n^2+2*n+12)/6-1,n=0..18);
```

1, 2, 6, 20, 66, 213, 683, 2211, 7291, 24552, 84305, 294297, 1041213, 3723752, 13434874, 48825190, 178535378, 656304995, 2423830156

```
>
> #-----------------------------------------------------------------
  ----------------------------------
> #0112
> restart: C:=x/(1-2*x): eqB:=-B+x/(1-v)+x/v*(B-B1)+x/(1-v)*(B+C);
  kb:=solve(coeff(eqB,B)=0,v)[2];
```

```
BB1:=solve(subs(B=0,v=kb,eqB)=0,B1);
BBB:=solve(subs(B1=BB1,eqB)=0,B);
```

$$eqB := -B + \frac{x}{1-v} + \frac{x\,(B - B1)}{v} + \frac{x\left(B + \frac{x}{1-2\,x}\right)}{1-v}$$

$$kb := \frac{1}{2} - \frac{\sqrt{1-4\,x}}{2}$$

$$BB1 := -\frac{\sqrt{1-4\,x}\;x - \sqrt{1-4\,x} - x + 1}{2\sqrt{1-4\,x}\;x - \sqrt{1-4\,x} + 2\,x - 1}$$

$$BBB := -\frac{(\sqrt{1-4\,x}\;x + 2\,x\,v - \sqrt{1-4\,x} - 2\,v - x + 1)\,x}{(2\sqrt{1-4\,x}\;x - \sqrt{1-4\,x} + 2\,x - 1)\,(v^2 - v + x)}$$

```
> FinalA:=factor(simplify(simplify(rationalize(x/(1-x)+subs(v=x,BBB)
*x/(1-x))+1)-1)); taylor(FinalA,x,13);
```

$$FinalA := -\frac{\sqrt{1-4\,x}\;x^2 + 4\,x^3 - 2\sqrt{1-4\,x}\;x - 5\,x^2 + \sqrt{1-4\,x} + 4\,x - 1}{2\,(-1+2\,x)\,(x-1)\,x}$$

$$x + 2\,x^2 + 6\,x^3 + 20\,x^4 + 67\,x^5 + 223\,x^6 + 742\,x^7 + 2484\,x^8 + 8399\,x^9 + 28731\,x^{10} + 99451\,x^{11} + O(x^{12})$$

```
> CC:=(1-sqrt(1-4*x))/2/x: cc:=n->1/(n+1)*binomial(2*n,n);
ff:=subs(sqrt(1-4*x)=1-2*x*fC,FinalA):
factor(coeff(ff,fC,1))*fC+factor(coeff(ff,fC,0));
```

$$cc := n \rightarrow \frac{\mathrm{binomial}(2\,n, n)}{n+1}$$

$$\frac{(x-1)\,fC}{-1+2\,x} - \frac{2\,x^2 - 2\,x + 1}{(-1+2\,x)\,(x-1)}$$

```
> with(genfunc):
simplify(rgf_expand((2*x^2-2*x+1)/(-1+2*x)/(x-1),x,n));
```

$$-1 + 2^n$$

```
> seq(cc(n+1)-2^(n+1)+1+add(2^(n-j)*cc(j),j=0..n),n=0..10);
```

$$1, 2, 6, 20, 67, 223, 742, 2484, 8399, 28731, 99451$$

```
> #-----------------------------------------------------------------
---------------------------
> #0120,0122
> v0:=(1-sqrt(1-4*x))/2: c1:=v0/(1-v0):
cc:=solve(c=x/(1-v)+x/v*(c-c1)+x/(1-v)*c,c);
b1:=x/(1-2*x)+x/(1-2*x)*subs(v=2*x,cc);
bb:=solve(b=x/(1-v)+2*x/v*(b-b1)+x/(1-v)*cc,b);
a1:=x/(1-x)*(1+subs(v=x,bb));
```

$$cc := -\frac{(\sqrt{1-4x}+2v-1)x}{(1+\sqrt{1-4x})(v^2-v+x)}$$

$$b1 := \frac{x}{1-2x} - \frac{x^2(\sqrt{1-4x}+4x-1)}{(1-2x)(1+\sqrt{1-4x})(4x^2-x)}$$

$$bb := \frac{(v\sqrt{1-4x}-2\sqrt{1-4x}\,x-4xv+v)x}{(1+\sqrt{1-4x})(-1+4x)(v^2-v+x)}$$

$$a1 := \frac{x\left(1+\frac{-\sqrt{1-4x}\,x-4x^2+x}{x(1+\sqrt{1-4x})(-1+4x)}\right)}{-x+1}$$

```
> FinalA:=factor(simplify(simplify(rationalize(a1)+1)-1));
  FinalA:=(1-2*x)/2/(1-x)*(1/(1-4*x)^(1/2)-1); taylor(FinalA,x,20);
```

$$FinalA := -\frac{(-1+2x)(\sqrt{1-4x}+4x-1)}{2(-1+4x)(x-1)}$$

$$FinalA := \frac{(1-2x)\left(\frac{1}{\sqrt{1-4x}}-1\right)}{2(-x+1)}$$

$$x+2x^2+6x^3+21x^4+77x^5+287x^6+1079x^7+4082x^8+15522x^9+59280x^{10}+227240x^{11}+873886x^{12}+3370030x^{13}+13027730x^{14}+50469890x^{15}+195892565x^{16}+761615285x^{17}+2965576715x^{18}+11563073315x^{19}+O(x^{20})$$

```
> seq(1/2*binomial(2*n+2,n+1)-1/2*add(binomial(2*j,j),j=1..n),n=0..2
  0);
```

1, 2, 6, 21, 77, 287, 1079, 4082, 15522, 59280, 227240, 873886, 3370030, 13027730, 50469890, 195892565, 761615285, 2965576715, 11563073315, 45141073925, 176423482325

```
> #-------------------------------------------------------------------
  ------------------------------
```

```
> ##0123
```

```
> restart:
  eqB:=-B+x/(1-v)+2*x/v*(B-B1)+sum(m*x^2/(1-2*x)*v^(m-1),m=1..infini
  ty); BB1:=solve(subs(v=2*x,eqB)=0,B1);
  BBB:=solve(subs(B1=BB1,eqB)=0,B);
  FinalA:=factor(subs(v=x,x/(1-v)*(1+BBB))); taylor(FinalA,x,20);
```

$$eqB := -B + \frac{x}{1-v} + \frac{2x(B-B1)}{v} + \frac{x^2}{(1-2x)(v-1)^2}$$

$$BB1 := -\frac{x(4x^2-3x+1)}{(-1+2x)^3}$$

$$BBB := \frac{(6\,v\,x^2 - 4\,v\,x - 4\,x^2 + v + 3\,x - 1)\,x}{(v-1)^2\,(-1+2\,x)^3}$$

$$FinalA := -\frac{x\,(8\,x^5 - 22\,x^4 + 30\,x^3 - 21\,x^2 + 7\,x - 1)}{(x-1)^3\,(-1+2\,x)^3}$$

$$x + 2\,x^2 + 6\,x^3 + 21\,x^4 + 73\,x^5 + 238\,x^6 + 724\,x^7 + 2075\,x^8 + 5667\,x^9 + 14892\,x^{10} + 37942\,x^{11} + 94273\,x^{12} + 229453\,x^{13} + 548954\,x^{14} + 1294440\,x^{15} + 3014775\,x^{16} + 6946951\,x^{17} + 15859864\,x^{18} + 35913898\,x^{19} + O(x^{20})$$

```
>
> seq(2^(n-1)*(n^2-3*n+4)+n*(n+1)/2-1,n=0..10);
```

$$1, 2, 6, 21, 73, 238, 724, 2075, 5667, 14892, 37942$$

```
> #-------------------------------------------------------------------
  ----------------------------------------
> #1000,1100
> FinalA:=(1-x)*(1-2*x)^2/2/x^3+(2*x^5-56*x^4+78*x^3-44*x^2+11*x-1)/
  2/x^3/sqrt(1-4*x)^3; taylor(FinalA,x,20);
```

$$FinalA := \frac{(1-x)\,(1-2\,x)^2}{2\,x^3} + \frac{2\,x^5 - 56\,x^4 + 78\,x^3 - 44\,x^2 + 11\,x - 1}{2\,x^3\,(-4\,x+1)^{(3/2)}}$$

$$x + 2\,x^2 + 6\,x^3 + 22\,x^4 + 89\,x^5 + 377\,x^6 + 1628\,x^7 + 7072\,x^8 + 30706\,x^9 + 132872\,x^{10} + 572356\,x^{11} + 2453508\,x^{12} + 10467461\,x^{13} + 44458195\,x^{14} + 188049840\,x^{15} + 792436020\,x^{16} + O(x^{17})$$

```
> f:=n->(2*n+1)!/n!^2:
  factor(subs(n=n+1,simplify(f(n-2)-28*f(n-1)+39*f(n)-22*f(n+1)+11/2
  *f(n+2)-1/2*f(n+3),GAMMA)));
  seq(1/2*(n^5+2*n^4+23*n^3+46*n^2+120*n+48)/(n+1)/(n+2)/(n+3)/(n+4)
  *binomial(2*n,n),n=0..10);
>
```

$$\frac{1}{2}\,\frac{(n^5 + 2\,n^4 + 23\,n^3 + 46\,n^2 + 120\,n + 48)\,\Gamma(2\,n+1)}{(n+1)\,(n+2)\,(n+3)\,(n+4)\,\Gamma(n+1)^2}$$

$$1, 2, 6, 22, 89, 377, 1628, 7072, 30706, 132872, 572356$$

```
> #-------------------------------------------------------------------
  ----------------------------------------
> #case 1001,1011,1101
> seq(1/(n+1)*sum(binomial(n+1,j)*binomial(2*n+2,n-2*j),j=0..n/2),n=
  0..20);
```

$$1, 2, 6, 22, 89, 382, 1708, 7870, 37108, 178184, 868318, 4283402, 21347902, 107330004, 543707480, 2772469998, 14219396908, 73303128344, 379621891640, 1974078923416, 10303600000553$$

```
>
> #-------------------------------------------------------------------
```

```
----------------------------------------
> #1002
> restart: HH:=x/(1-x): GG:=(x+x*HH+x*E1)/(1-x): FF:=x/(1-2*x):
  eqE:=-E+x/(1-v)*(1+HH+GG+E)+x/v*(E-E1);
  ke:=solve(coeff(eqE,E)=0,v)[2]; EE1:=solve(subs(v=ke,eqE)=0,E1);
  EEE:=solve(subs(E1=EE1,eqE)=0,E);
```

$$eqE := -E + \frac{x\left(1+\frac{x}{1-x}+\frac{x\,E1+\frac{x^2}{1-x}+x}{1-x}+E\right)}{1-v}+\frac{x\,(E-E1)}{v}$$

$$ke := \frac{1}{2}-\frac{\sqrt{1-4\,x}}{2}$$

$$EE1 := \frac{-1+\sqrt{1-4\,x}}{-1+\sqrt{1-4\,x}\,x-2\,x^2-\sqrt{1-4\,x}+3\,x}$$

$$EEE := \frac{(\sqrt{1-4\,x}+2\,v-1)\,x}{(-1+x)\,(-2\,x+1+\sqrt{1-4\,x})\,(v^2-v+x)}$$

```
> eqD:=-D1+x/(1-v)*(1+D1)+x/v*(D1-D11)+x*EEE;
  DD1:=factor(simplify(simplify(rationalize(solve(limit(eqD,v=ke)=0,
  D11))+1)-1));
  DDD:=factor(simplify(simplify(rationalize(solve(subs(D11=DD1,eqD)=
  0,D1))+1)-1));
```

$$eqD := -\mathrm{D1}+\frac{x\,(1+\mathrm{D1})}{1-v}+\frac{x\,(\mathrm{D1}-\mathrm{D11})}{v}+\frac{x^2\,(\sqrt{1-4\,x}+2\,v-1)}{(-1+x)\,(-2\,x+1+\sqrt{1-4\,x})\,(v^2-v+x)}$$

$$DD1 := -\frac{2\sqrt{1-4\,x}\,x+4\,x^2-\sqrt{1-4\,x}-5\,x+1}{(-1+4\,x)\,(-1+x)}$$

$$\begin{aligned}DDD := {} & (-v+2\,v^2+9\,x\,v+6\sqrt{1-4\,x}\,x^2\,v+\sqrt{1-4\,x}\,v-8\,v^3\,x^2-8\,v^2\,x^3+6\,v^3\,x+30\,v^2\,x^2\\ & +8\,v\,x^3-15\,v^2\,x+\sqrt{1-4\,x}\,v^3-2\sqrt{1-4\,x}\,v^2-v^3+4\sqrt{1-4\,x}\,v^3\,x^2-6\sqrt{1-4\,x}\,v^3\,x\\ & -12\sqrt{1-4\,x}\,v^2\,x^2+4\sqrt{1-4\,x}\,v\,x^3+13\sqrt{1-4\,x}\,v^2\,x-7\sqrt{1-4\,x}\,v\,x-8\,x^4-4\sqrt{1-4\,x}\,x^3\\ & +2\sqrt{1-4\,x}\,x^2-22\,v\,x^2+10\,x^3-2\,x^2)\Big/(2\,(-1+4\,x)\,(-1+x)\,(v^2-v+x)^2)\end{aligned}$$

```
> factor(coeff(DDD,sqrt(1-4*x),0));
```

$$-\frac{(v-1+x)\,(2\,v^2\,x-v^2-4\,v\,x+2\,x^2+v)}{2\,(v^2-v+x)^2\,(-1+x)}$$

```
> eqC:=-C+x/(1-v)*(1+FF+subs(E1=EE1,GG)+EEE)+x/v*(C-C1);
  CC1:=solve(limit(eqC,v=x)=0,C1);
  CCC:=simplify(solve(subs(C1=CC1,eqC)=0,C));
```

$$eqC := -C + x\left(1 + \frac{x}{1-2x} + \frac{\dfrac{x\left(-1+\sqrt{1-4x}\right)}{-1+\sqrt{1-4x}\,x - 2x^2 - \sqrt{1-4x} + 3x} + \dfrac{x^2}{1-x} + x}{1-x} + \frac{\left(\sqrt{1-4x} + 2v - 1\right)x}{(-1+x)\left(-2x+1+\sqrt{1-4x}\right)\left(v^2 - v + x\right)}\right) / (1-v) + \frac{x(C - C1)}{v}$$

$$CC1 := -\frac{\sqrt{1-4x}\,x^3 - 2x^4 - 2\sqrt{1-4x}\,x^2 + x^3 + 3\sqrt{1-4x}\,x + 2x^2 - \sqrt{1-4x} - 3x + 1}{-4x^4 + 12x^3 - 13x^2 + 6x - 1 + 2\sqrt{1-4x}\,x^3 - 5\sqrt{1-4x}\,x^2 + 4\sqrt{1-4x}\,x - \sqrt{1-4x}}$$

$$CCC := \left(\sqrt{1-4x}\,v^2x^2 - 2v^2x^3 + \sqrt{1-4x}\,x^2v + \sqrt{1-4x}\,x^3 + 5v^2x^2 + 2vx^3 - 2x^4 - 3\sqrt{1-4x}\,vx - 2\sqrt{1-4x}\,x^2 - 6v^2x - 7vx^2 + x^3 + \sqrt{1-4x}\,v + 3\sqrt{1-4x}\,x + 2v^2 + 9xv + 2x^2 - \sqrt{1-4x} - 3v - 3x + 1\right)x \Big/ \left((-1+v)\left(v^2 - v + x\right)\left(-4x^4 + 12x^3 - 13x^2 + 6x - 1 + 2\sqrt{1-4x}\,x^3 - 5\sqrt{1-4x}\,x^2 + 4\sqrt{1-4x}\,x - \sqrt{1-4x}\right)\right)$$

```
> eqB:=-B+x/(1-v)*(1+DDD)+x/v*(B-B1)+x*CCC;
  BB1:=simplify(solve(limit(eqB,v=x)=0,B1));
  BBB:=simplify(solve(subs(B1=BB1,eqB)=0,B));
```

$$eqB := -B + x\left(1 + \left(-v + 2v^2 + 9xv + 6\sqrt{1-4x}\,x^2v + \sqrt{1-4x}\,v - 8v^3x^2 - 8v^2x^3 + 6v^3x + 30v^2x^2 + 8vx^3 - 15v^2x + \sqrt{1-4x}\,v^3 - 2\sqrt{1-4x}\,v^2 - v^3 + 4\sqrt{1-4x}\,v^3x^2 - 6\sqrt{1-4x}\,v^3x - 12\sqrt{1-4x}\,v^2x^2 + 4\sqrt{1-4x}\,vx^3 + 13\sqrt{1-4x}\,v^2x - 7\sqrt{1-4x}\,vx - 8x^4 - 4\sqrt{1-4x}\,x^3 + 2\sqrt{1-4x}\,x^2 - 22vx^2 + 10x^3 - 2x^2\right) \Big/ \left(2(-1+4x)(-1+x)\left(v^2 - v + x\right)^2\right)\right) / (1-v) + \frac{x(B - B1)}{v} + x^2\left(\sqrt{1-4x}\,v^2x^2 - 2v^2x^3 + \sqrt{1-4x}\,x^2v + \sqrt{1-4x}\,x^3 + 5v^2x^2 + 2vx^3 - 2x^4 - 3\sqrt{1-4x}\,vx - 2\sqrt{1-4x}\,x^2 - 6v^2x - 7vx^2 + x^3 + \sqrt{1-4x}\,v + 3\sqrt{1-4x}\,x + 2v^2 + 9xv + 2x^2 - \sqrt{1-4x} - 3v - 3x + 1\right) \Big/ \left((-1+v)\left(v^2 - v + x\right)\left(-4x^4 + 12x^3 - 13x^2 + 6x - 1 + 2\sqrt{1-4x}\,x^3 - 5\sqrt{1-4x}\,x^2 + 4\sqrt{1-4x}\,x - \sqrt{1-4x}\right)\right)$$

$$BB1 := x\left(4\sqrt{1-4x}\,x^4 + 8x^5 + \sqrt{1-4x}\,x^3 - 46x^4 - 5\sqrt{1-4x}\,x^2 + 63x^3 + 4\sqrt{1-4x}\,x - 37x^2 - \sqrt{1-4x} + 10x - 1\right) \Big/ \left(-116x^4 + 68x^5 - 16x^6 + 101x^3 - 47x^2 + 11x - 1 + 43\sqrt{1-4x}\,x^3 - 30\sqrt{1-4x}\,x^4 + 8\sqrt{1-4x}\,x^5 - 29\sqrt{1-4x}\,x^2 + 9\sqrt{1-4x}\,x - \sqrt{1-4x}\right)$$

$$BBB := -x\left(-v^2 + 3xv + 8\sqrt{1-4x}\,v^4x^5 - 28\sqrt{1-4x}\,v^2x^4 + 10\sqrt{1-4x}\,vx^5 - 24\sqrt{1-4x}\,v^4x^4 - 24\sqrt{1-4x}\,v^3x^5 + 8\sqrt{1-4x}\,v^2x^6 + 37\sqrt{1-4x}\,v^4x^3 + 68\sqrt{1-4x}\,v^3x^4 + 4\sqrt{1-4x}\,v^2x^5 - 8\sqrt{1-4x}\,vx^6 + 4\sqrt{1-4x}\,x^6 + \sqrt{1-4x}\,x^5 - 90\sqrt{1-4x}\,v^3x^3 - 11\sqrt{1-4x}\,vx^4 - 25\sqrt{1-4x}\,x^2v^4 + 44\sqrt{1-4x}\,x^3v^2 + 8\sqrt{1-4x}\,xv^4 - 3\sqrt{1-4x}\,x^2v + 8x^7 - 46x^6 - v^4 - 8v^4x^5 - 16v^3x^6 - 2v^4x^4 + 76v^3x^5 + 32v^2x^6 - 16vx^7 - 82v^3x^4$$

$$- 168\, v^2 x^5 + 36\, v\, x^6 + 36\, v\, x^5 + 21\, x^3 v^4 + 16\, x^3 v^3 + 232\, x^4 v^2 - 111\, x^4 v - 21\, x^2 v^4 + 8\, x\, v^4$$
$$- \sqrt{1-4x}\, v^4 + 24\, v^3 x^2 - 132\, v^2 x^3 - 14\, v^3 x + 25\, v^2 x^2 + 85\, v\, x^3 + 3\, v^2 x + 2\sqrt{1-4x}\, v^3$$
$$- \sqrt{1-4x}\, v^2 + 2\, v^3 + 54\sqrt{1-4x}\, v^3 x^2 - 16\sqrt{1-4x}\, v^3 x - 25\sqrt{1-4x}\, v^2 x^2 + 5\sqrt{1-4x}\, v\, x^3$$
$$+ 7\sqrt{1-4x}\, v^2 x + \sqrt{1-4x}\, v\, x + 63\, x^5 - 37\, x^4 - 5\sqrt{1-4x}\, x^4 + 4\sqrt{1-4x}\, x^3 - \sqrt{1-4x}\, x^2$$
$$- 27\, v\, x^2 + 10\, x^3 - x^2) \Big/ ((-1+v)(v^2 - v + x)^2 (-116\, x^4 + 68\, x^5 - 16\, x^6 + 101\, x^3 - 47\, x^2 + 11\, x$$
$$- 1 + 43\sqrt{1-4x}\, x^3 - 30\sqrt{1-4x}\, x^4 + 8\sqrt{1-4x}\, x^5 - 29\sqrt{1-4x}\, x^2 + 9\sqrt{1-4x}\, x$$
$$- \sqrt{1-4x}))$$

```
> eqA:=-A+x/(1-v)*(1+BBB)+x/v*(A-A1):
  FinalA:=simplify(solve(limit(eqA,v=x)=0,A1));
```

$$FinalA := -(-16\, x^8 + 2\sqrt{1-4x}\, x^6 + 108\, x^7 - 28\sqrt{1-4x}\, x^5 - 274\, x^6 + 57\sqrt{1-4x}\, x^4 + 402\, x^5$$
$$- 58\sqrt{1-4x}\, x^3 - 365\, x^4 + 32\sqrt{1-4x}\, x^2 + 206\, x^3 - 9\sqrt{1-4x}\, x - 70\, x^2 + \sqrt{1-4x} + 13\, x - 1$$
$$) \Big/ (-1 + 13\, x + 8\sqrt{1-4x}\, x^7 - 46\sqrt{1-4x}\, x^6 + 111\sqrt{1-4x}\, x^5 - 16\, x^8 + 100\, x^7 - 268\, x^6$$
$$+ 401\, x^5 - 365\, x^4 - 145\sqrt{1-4x}\, x^4 + 110\sqrt{1-4x}\, x^3 - 48\sqrt{1-4x}\, x^2 + 206\, x^3 - 70\, x^2$$
$$+ 11\sqrt{1-4x}\, x - \sqrt{1-4x})$$

```
> FinalA:=factor(simplify(simplify(rationalize(FinalA)+1)-1)):
  FinalA:=factor(coeff(FinalA,sqrt(1-4*x)))*sqrt(1-4*x)+factor(coeff
  (FinalA,sqrt(1-4*x),0)); taylor(FinalA,x,13);
```

$$FinalA := -\frac{(2x-1)(2x^2-4x+1)\sqrt{1-4x}}{2x^2(-1+x)(-1+4x)}$$
$$- \frac{(2x^2-2x+1)(2x^6-12x^5+25x^4-34x^3+24x^2-8x+1)}{2x^2(-1+x)^5(2x-1)}$$
$$x + 2x^2 + 6x^3 + 22x^4 + 87x^5 + 348x^6 + 1382x^7 + 5450x^8 + 21421x^9 + 84125x^{10} + O(x^{11})$$

```
> with(genfunc):
  factor(simplify(rgf_expand(-1/2*(2*x^2-2*x+1)*(2*x^6-12*x^5+25*x^4
  -34*x^3+24*x^2-8*x+1)/x^2/(-1+x)^5/(2*x-1),x,n)));
```

$$-\frac{n^4}{24} + \frac{n^3}{12} - \frac{11\, n^2}{24} + 2^n - \frac{7\, n}{12} - \frac{1}{2}$$

```
> seq(1/2*binomial(2*n+6,n+3)-5/2*binomial(2*n+4,n+2)+5/2*binomial(2
  *n+2,n+1)+1/2*sum(binomial(2*j,j),j=0..n)+2^(n+1)-1/2*(n^4+2*n^3+
  11*n^2+34*n+36),n=0..20);
```

1, 2, 6, 22, 87, 348, 1382, 5450, 21421, 84125, 330453, 1298685, 5106130, 20083379, 79015633, 310963725, 1224124884, 4820192192, 18985852508, 74804350676, 294820185101

```
> #-------------------------------------------------------------------
  ---------------------------------------------
> #1010=1110
> restart: alias(vv=RootOf(v^3-(v*x+v-x)*(v-x)=0,v));
```

```
VVV:=allvalues(vv):
```

$$vv$$

```
> simplify(series(VVV[1],x,5)) assuming x>0 and x<0.1;
subs(signum(9*x^6+12*x^5+60*x^4-12*x^3)=1,simplify(series(VVV[2],x
,5))) assuming x>0 and x<0.1;
subs(signum(9*x^6+12*x^5+60*x^4-12*x^3)=1,simplify(series(VVV[3],x
,5))) assuming x>0 and x<0.1;
```

$$1 - 16\,x^4 - 5\,x^3 - 2\,x^2 + \mathrm{O}(x^5) - x$$

$$x + x^{(3/2)} + x^2 + \frac{3\,x^{(5/2)}}{2} + \frac{5\,x^3}{2} + \frac{35\,x^{(7/2)}}{8} + 8\,x^4 + \frac{241\,x^{(9/2)}}{16} + \mathrm{O}(x^5)$$

$$x - x^{(3/2)} + x^2 - \frac{3\,x^{(5/2)}}{2} + \frac{5\,x^3}{2} - \frac{35\,x^{(7/2)}}{8} + 8\,x^4 - \frac{241\,x^{(9/2)}}{16} + \mathrm{O}(x^5)$$

```
> simplify(series(VVV[3]*VVV[2]*VVV[1],x,5)) assuming x>0 and
x<0.1;simplify(series(VVV[3]+VVV[2]+VVV[1],x,5)) assuming x>0 and
x<0.1;
```

$$x^2 + \mathrm{O}(x^5)$$

$$1 + x$$

```
> eqD:=-D1+x/(1-v)+x/v*(D1-D11)+x/(1-v)*(D1+EE+FF);
kd:=solve(coeff(eqD,D1)=0,v)[2];
solve({subs(v=kd,subs(D1=0,eqD))=0,EE=x+x*D11+x*EE+x*FF,FF=x+x*EE+
x*FF},{D11,EE,FF});
```

$$eqD := -\mathrm{D1} + \frac{x}{1-v} + \frac{x\,(\mathrm{D1} - \mathrm{D11})}{v} + \frac{x\,(\mathrm{D1} + EE + FF)}{1-v}$$

$$kd := \frac{1}{2} - \frac{\sqrt{1-4\,x}}{2}$$

$$\{\mathrm{D11} = \frac{2\sqrt{1-4\,x}\,x + 2\,x^2 - \sqrt{1-4\,x} - 4\,x + 1}{2\,x^3},\ EE = \frac{\sqrt{1-4\,x}\,x - \sqrt{1-4\,x} - 3\,x + 1}{2\,x^2},$$

$$FF = -\frac{\sqrt{1-4\,x} + 2\,x - 1}{2\,x}\}$$

```
> FF:=(1-2*x-sqrt(1-4*x))/2/x; EE:=(1-3*x-(1-x)*sqrt(1-4*x))/2/x^2;
DD1:=((2*x-1)*(1-4*x)^(1/2)+2*x^2-4*x+1)/2/x^3;
DDD:=solve(subs(D11=DD1,eqD)=0,D1);
```

$$FF := \frac{1 - 2\,x - \sqrt{1-4\,x}}{2\,x}$$

$$EE := \frac{1 - 3\,x - (-x+1)\sqrt{1-4\,x}}{2\,x^2}$$

$$DD1 := \frac{(2x-1)\sqrt{1-4x}+2x^2-4x+1}{2x^3}$$

$$DDD := -\frac{\sqrt{1-4x}\,v\,x-\sqrt{1-4x}\,v-2\sqrt{1-4x}\,x-3xv-2x^2+\sqrt{1-4x}+v+4x-1}{2x^2(v^2-v+x)}$$

```
> eqC:=-C+x/(1-v)+x/v*(C-C1)+x*DDD+x/(1-v)*B+x/v*(B-B1):
  CC:=solve(eqC=0,C):
```

```
> eqB:=-B+x/(1-v)+x*CC+x/(1-v)*B+x/v*(B-B1):
  eq1:=subs(v=u1,subs(B=0,eqB)): eq2:=subs(u1=u2,eq1):
```

```
> ss:=solve({eq1=0,eq2=0},{B1,C1}): BB1:=factor(op(op(ss)[1])[2]);
  simplify(series(subs(u1=VVV[2],u2=VVV[3],BB1),x,8)) assuming x>0
  and x<0.1;
```

$$BB1 := -u1\,u2\,(2x+\sqrt{1-4x}-3)\,(2\,u1\,u2\,x^2+\sqrt{1-4x}\,u1\,x+\sqrt{1-4x}\,u2\,x-4\,u1\,u2\,x-2x^2\,u1-2x^2\,u2-\sqrt{1-4x}\,u1-\sqrt{1-4x}\,u2-2\sqrt{1-4x}\,x+4\,u1\,u2+5\,x\,u1+5\,x\,u2+4x^2+\sqrt{1-4x}-3\,u1-3\,u2-8x+3)\Big/(x\,(x^2-2x+2)\,(-1+u1)\,(-2\,u1+\sqrt{1-4x}+1)\,(-1+u2)\,(-2\,u2+\sqrt{1-4x}+1))$$

$$11x^3+3x^2+791x^6+184x^5+44x^4+x+O(x^7)$$

```
> BB1:=simplify(-x^2/u3*(2*x+(1-4*x)^(1/2)-3)*(2*x^2/u3*x^2+(1-4*x)^
  (1/2)*(1+x-u3)*x-4*x^2/u3*x-2*(1+x-u3)*x^2-(1-4*x)^(1/2)*(1+x-u3)-
  2*(1-4*x)^(1/2)*x+4*x^2/u3+5*(1+x-u3)*x+4*x^2+(1-4*x)^(1/2)-3*(1+x
  -u3)-8*x+3)/x/(x^2-2*x+2)/(1-(1+x-u3)+x^2/u3)/(4*x^2/u3-2*(1+x-u3)
  *((1-4*x)^(1/2)+1)+(sqrt(1-4*x)+1)^2));
  simplify(series(subs(u1=VVV[2],u2=VVV[3],u3=VVV[1],BB1),x,8));
```

$$BB1 := (\sqrt{1-4x}\,x\,u3^2-\sqrt{1-4x}\,x^2\,u3-2x^2\,u3^2+2x^3\,u3-2x^4-\sqrt{1-4x}\,u3^2+2\sqrt{1-4x}\,x\,u3+5\,x\,u3^2-7x^2\,u3+4x^3-3\,u3^2+6\,x\,u3-4x^2)\,(2x+\sqrt{1-4x}-3)\,x\Big/(2\,(x^2-2x+2)\,(u3^2-u3\,x+x^2)\,(\sqrt{1-4x}\,u3^2-\sqrt{1-4x}\,x\,u3+u3^2-3\,x\,u3+2x^2))$$

$$3468x^7+791x^6+184x^5+44x^4+11x^3+3x^2+O(x^8)+x$$

```
> factor(expand(subs(u3^3=(u3*x+u3-x)*(u3-x),expand(subs(u3^4=u3*(u3
  *x+u3-x)*(u3-x),expand(denom(BB1))))))));
```

$$-2\,(x^2-2x+2)\,(2\sqrt{1-4x}\,x\,u3^2-2\sqrt{1-4x}\,x^2\,u3+x^3\sqrt{1-4x}-2x^2\,u3^2+2x^3\,u3-2x^4-\sqrt{1-4x}\,u3^2+2\sqrt{1-4x}\,x\,u3-x^2\sqrt{1-4x}+4\,x\,u3^2-6x^2\,u3+3x^3-u3^2+2\,x\,u3-x^2)$$

```
> BB1:=factor(-((1-4*x)^(1/2)*x*u3^2-(1-4*x)^(1/2)*x^2*u3-2*x^2*u3^2
  +2*x^3*u3-2*x^4-(1-4*x)^(1/2)*u3^2+2*(1-4*x)^(1/2)*x*u3+5*x*u3^2-7
  *x^2*u3+4*x^3-3*u3^2+6*x*u3-4*x^2)*(2*x+(1-4*x)^(1/2)-3)*x/2/(x^2-
  2*x+2)/(2*(1-4*x)^(1/2)*x*u3^2-2*(1-4*x)^(1/2)*x^2*u3+(1-4*x)^(1/2
  )*x^3-2*x^2*u3^2+2*x^3*u3-2*x^4-(1-4*x)^(1/2)*u3^2+2*(1-4*x)^(1/2)
  *x*u3-(1-4*x)^(1/2)*x^2+4*x*u3^2-6*x^2*u3+3*x^3-u3^2+2*x*u3-x^2));
  simplify(series(subs(u3=VVV[1],BB1),x,8));
```

$$BB1 := -(\sqrt{1-4x}\,x\,u3^2-\sqrt{1-4x}\,x^2\,u3-2x^2\,u3^2+2x^3\,u3-2x^4-\sqrt{1-4x}\,u3^2$$

$$+2\sqrt{1-4x}\,x\,u3+5\,x\,u3^2-7\,x^2\,u3+4\,x^3-3\,u3^2+6\,x\,u3-4\,x^2)\,(2\,x+\sqrt{1-4x}-3)\,x\,/\,(2$$
$$(x^2-2\,x+2)\,(2\sqrt{1-4x}\,x\,u3^2-2\sqrt{1-4x}\,x^2\,u3+x^3\sqrt{1-4x}-2\,x^2\,u3^2+2\,x^3\,u3-2\,x^4$$
$$-\sqrt{1-4x}\,u3^2+2\sqrt{1-4x}\,x\,u3-x^2\sqrt{1-4x}+4\,x\,u3^2-6\,x^2\,u3+3\,x^3-u3^2+2\,x\,u3-x^2))$$

$$3468\,x^7+791\,x^6+184\,x^5+44\,x^4+11\,x^3+3\,x^2+\mathrm{O}(x^8)+x$$

```
> CC1:=simplify(op(op(ss)[2])[2]);
  simplify(series(subs(u1=VVV[2],u2=VVV[3],u3=VVV[1],CC1),x,8));
```

$$CC1 := -2\,(-3\,x^2-5\,u1\,u2-3\,x^2\sqrt{1-4x}+2\,x^3\sqrt{1-4x}-6\,u1\,u2\,x+7\,u1^2\,u2\,x-4\,u1^2\,u2\,x^3$$
$$+5\,u1^2\,u2+5\,u1\,u2^2-4\,u1\,u2^2\,x^3+16\,u1\,u2\,x^3-5\,u1\,u2^2\,x^2+7\,u1\,u2^2\,x+15\,u1\,u2\,x^2$$
$$-8\,u1^2\,u2^2\,x-5\,u1^2\,u2\,x^2+2\sqrt{1-4x}\,u1\,x^2-\sqrt{1-4x}\,u1\,u2-\sqrt{1-4x}\,u1^2\,x^2$$
$$-2\sqrt{1-4x}\,u1\,x^3+\sqrt{1-4x}\,u1^2\,u2-2\sqrt{1-4x}\,u2\,x^3+\sqrt{1-4x}\,u1\,u2^2-u2^2\,x^2\sqrt{1-4x}$$
$$+2\,u2\,x^2\sqrt{1-4x}-6\,u1^2\,u2^2-4\sqrt{1-4x}\,u1\,u2\,x+3\sqrt{1-4x}\,u1^2\,u2\,x+2\sqrt{1-4x}\,u1\,u2\,x^3$$
$$+\sqrt{1-4x}\,u1\,u2^2\,x^2+3\sqrt{1-4x}\,u1\,u2^2\,x-\sqrt{1-4x}\,u1\,u2\,x^2+\sqrt{1-4x}\,u1^2\,u2\,x^2+4\,u2^2\,x^3$$
$$+4\,u1^2\,x^3+4\,x\,u1+u1^2\,x^2-4\,u1^2\,x-12\,x^3\,u1+4\,x\,u2+u2^2\,x^2-4\,u2^2\,x-12\,x^3\,u2-2\,x^2\,u2$$
$$-2\,x^2\,u1-2\sqrt{1-4x}\,u2^2\,x+2\sqrt{1-4x}\,u2\,x-2\sqrt{1-4x}\,u1^2\,x+2\sqrt{1-4x}\,u1\,x+8\,x^3)$$
$$(\sqrt{1-4x}-2)\,/\,((\sqrt{1-4x}\,u2-2\,u2^2-\sqrt{1-4x}+3\,u2-1)$$
$$(\sqrt{1-4x}\,u1-2\,u1^2-\sqrt{1-4x}+3\,u1-1)\,x^2\,(3+4\,x))$$

$$x+4\,x^2+16\,x^3+65\,x^4+270\,x^5+\mathrm{O}(x^6)$$

```
> CC1:=1/factor(simplify(simplify(1/simplify(subs(u1=(1+x-u3-sqrt((1
  +x-u3)^2-4*x^2/u3))/2,u2=(1+x-u3+sqrt((1+x-u3)^2-4*x^2/u3))/2,-2*(
  -3*(1-4*x)^(1/2)*x^2-3*x^2+2*(1-4*x)^(1/2)*u1*u2*x^3-(1-4*x)^(1/2)
  *u1*u2*x^2+(1-4*x)^(1/2)*u1*u2^2*x^2+3*(1-4*x)^(1/2)*u1*u2^2*x+(1-
  4*x)^(1/2)*u1^2*u2*x^2+3*(1-4*x)^(1/2)*u1^2*u2*x-4*(1-4*x)^(1/2)*u
  1*u2*x-4*u1^2*u2*x^3-4*u1*u2^2*x^3-8*u1^2*u2^2*x+2*(1-4*x)^(1/2)*u
  2*x+4*u2^2*x^3+2*(1-4*x)^(1/2)*u1*x+5*u1*u2^2+5*u1^2*u2-5*u1*u2-(1
  -4*x)^(1/2)*u2^2*x^2+(1-4*x)^(1/2)*u1*u2^2+(1-4*x)^(1/2)*u1^2*u2-(
  1-4*x)^(1/2)*u1^2*x^2-2*(1-4*x)^(1/2)*u2*x^3-2*(1-4*x)^(1/2)*u2^2*
  x+2*(1-4*x)^(1/2)*u2*x^2-2*(1-4*x)^(1/2)*u1*x^3+2*(1-4*x)^(1/2)*u1
  *x^2-(1-4*x)^(1/2)*u1*u2+2*(1-4*x)^(1/2)*x^3-2*u2*x^2-12*u2*x^3-4*
  u2^2*x+4*u2*x+u2^2*x^2-5*u1*u2^2*x^2+7*u1*u2^2*x+7*u1^2*u2*x-6*u1*
  u2*x-5*u1^2*u2*x^2+16*u1*u2*x^3+15*u1*u2*x^2-6*u1^2*u2^2+4*u1^2*x^
  3+4*u1*x+u1^2*x^2-12*u1*x^3-4*u1^2*x-2*u1*x^2-2*(1-4*x)^(1/2)*u1^2
  *x+8*x^3)*((1-4*x)^(1/2)-2)/((1-4*x)^(1/2)*u2-2*u2^2-(1-4*x)^(1/2)
  +3*u2-1)/((1-4*x)^(1/2)*u1-2*u1^2-(1-4*x)^(1/2)+3*u1-1)/x^2/(3+4*x
  )))+1)-1));simplify(series(subs(u3=VVV[1],CC1),x,8));
```

$$CC1 := (5\sqrt{1-4x}\,x\,u3^2-4\sqrt{1-4x}\,x^2\,u3+2\sqrt{1-4x}\,u3^4+\sqrt{1-4x}\,x\,u3^4$$
$$-4\,x^2\sqrt{1-4x}\,u3^3+4\,x^3\sqrt{1-4x}\,u3^2-3\,x^4\sqrt{1-4x}\,u3-4\sqrt{1-4x}\,x\,u3^3$$
$$+5\,x^2\sqrt{1-4x}\,u3^2-5\,x^3\sqrt{1-4x}\,u3-2\,x^2\,u3^3-8\,x\,u3^3+x^4\,u3-2\,x^3\,u3^2+4\,x^5\,u3-8\,x^4\,u3^2$$

$$+8x^3 u3^3 - 14x^2 u3 - 15x^3 u3 + 11x^2 u3^2 + 13x u3^2 - 4u3^3 - x u3^4 - 2\sqrt{1-4x}\, u3^3 + 6x^3 + 8x^4 + 4u3^4 - 4x^2 u3^4)(\sqrt{1-4x} - 2) \Big/ (x(3+4x)(-x+u3)(u3^2 - u3\,x + x^2)(\sqrt{1-4x}\, u3 + u3 - 2x))$$

$$x + 4x^2 + 16x^3 + 65x^4 + 270x^5 + 1146x^6 + O(x^7)$$

```
> CC1:=factor(simplify(subs(u3^3=(u3*x+u3-x)*(u3-x),simplify(subs(u3
^4=u3*(u3*x+u3-x)*(u3-x),expand(numer(CC1))))))/simplify(subs(u3^3
=(u3*x+u3-x)*(u3-x),simplify(subs(u3^4=u3*(u3*x+u3-x)*(u3-x),expan
d(denom(CC1))))))); simplify(series(subs(u3=VVV[1],CC1),x,8));
```

$$CC1 := -(\sqrt{1-4x}\, u3^2 - \sqrt{1-4x}\, x\, u3 + x^2\sqrt{1-4x} + \sqrt{1-4x}\, x + u3^2 - x\, u3 + x^2 - x)\, x \Big/ (2\sqrt{1-4x}\, x\, u3^2 - 2\sqrt{1-4x}\, x^2 u3 + x^3\sqrt{1-4x} - 2x^2 u3^2 + 2x^3 u3 - 2x^4 - \sqrt{1-4x}\, u3^2 + 2\sqrt{1-4x}\, x\, u3 - x^2\sqrt{1-4x} + 4x u3^2 - 6x^2 u3 + 3x^3 - u3^2 + 2x u3 - x^2)$$

$$x + 4x^2 + 16x^3 + 65x^4 + 270x^5 + 1146x^6 + 4959x^7 + O(x^8)$$

```
> BBxx:=factor(subs(v=x,solve(subs(B1=BB1,C1=CC1,eqB)=0,B)));
```

$$BBxx := -(\sqrt{1-4x}\, x\, u3^2 - \sqrt{1-4x}\, x^2 u3 + x^3\sqrt{1-4x} + 2\sqrt{1-4x}\, x\, u3 - 2x^2\sqrt{1-4x} + x u3^2 + x^2 u3 - 3x^3 - \sqrt{1-4x}\, u3 + \sqrt{1-4x}\, x - 4x u3 + 4x^2 + u3 - x) \Big/ (2\sqrt{1-4x}\, x\, u3^2 - 2\sqrt{1-4x}\, x^2 u3 + x^3\sqrt{1-4x} - 2x^2 u3^2 + 2x^3 u3 - 2x^4 - \sqrt{1-4x}\, u3^2 + 2\sqrt{1-4x}\, x\, u3 - x^2\sqrt{1-4x} + 4x u3^2 - 6x^2 u3 + 3x^3 - u3^2 + 2x u3 - x^2)$$

```
> FinalA:=factor(x/(1-x)+x/(1-x)*BBxx);ss:=simplify(series(subs(u3=V
VV[1],FinalA),x,17));
```

$$FinalA := -(\sqrt{1-4x}\, u3^2 - \sqrt{1-4x}\, x\, u3 - 2x u3^2 + 2x^2 u3 - 2x^3 - \sqrt{1-4x}\, u3 + \sqrt{1-4x}\, x + u3^2 - 5x u3 + 4x^2 + u3 - x)\, x \Big/ (2\sqrt{1-4x}\, x\, u3^2 - 2\sqrt{1-4x}\, x^2 u3 + x^3\sqrt{1-4x} - 2x^2 u3^2 + 2x^3 u3 - 2x^4 - \sqrt{1-4x}\, u3^2 + 2\sqrt{1-4x}\, x\, u3 - x^2\sqrt{1-4x} + 4x u3^2 - 6x^2 u3 + 3x^3 - u3^2 + 2x u3 - x^2)$$

$$ss := 2x^2 + x + 1662x^7 + O(x^{17}) + 7426x^8 + 72630819x^{14} + 1634321883x^{16} + 343674750x^{15} + 153821x^{10} + 709709x^{11} + 379x^6 + 89x^5 + 6x^3 + 22x^4 + 33624x^9 + 15432829x^{13} + 3298692x^{12}$$

```
>
> ff:=factor(expand(simplify(simplify(subs(v=solve(f=x/(1-x)+x/(1-x)
*BBxx,u3)[1],v^3-(v*x+v-x)*(v-x))+1)-1))):
> ff:=(factor((-2*f*x^2+4*f*x-(1-4*x)^(1/2)*f+(1-4*x)^(1/2)*x+2*(1-4
*x)^(1/2)*f*x-2*x^2+x-f)^6*((factor(coeff(ff,(16*(1-4*x)^(1/2)*f^2
*x^3-12*f^2*x^4-24*(1-4*x)^(1/2)*f^2*x^2+20*(1-4*x)^(1/2)*f*x^3+48
*f^2*x^3-24*f*x^4+8*(1-4*x)^(1/2)*f^2*x-20*(1-4*x)^(1/2)*f*x^2+4*(
1-4*x)^(1/2)*x^3-40*f^2*x^2+52*f*x^3-12*x^4-2*x^2*(1-4*x)^(1/2)+8*
f^2*x-12*f*x^2+16*x^3+8*(1-4*x)^(1/2)*x+14*x^2-2*(1-4*x)^(1/2)-12*
x+2)^(1/2),1))*(16*(1-4*x)^(1/2)*f^2*x^3-12*f^2*x^4-24*(1-4*x)^(1/
```

```
2)*f^2*x^2+20*(1-4*x)^(1/2)*f*x^3+48*f^2*x^3-24*f*x^4+8*(1-4*x)^(1
/2)*f^2*x-20*(1-4*x)^(1/2)*f*x^2+4*(1-4*x)^(1/2)*x^3-40*f^2*x^2+52
*f*x^3-12*x^4-2*x^2*(1-4*x)^(1/2)+8*f^2*x-12*f*x^2+16*x^3+8*(1-4*x
)^(1/2)*x+14*x^2-2*(1-4*x)^(1/2)-12*x+2)^(1/2))^2-(-coeff(ff,(16*(
1-4*x)^(1/2)*f^2*x^3-12*f^2*x^4-24*(1-4*x)^(1/2)*f^2*x^2+20*(1-4*x
)^(1/2)*f*x^3+48*f^2*x^3-24*f*x^4+8*(1-4*x)^(1/2)*f^2*x-20*(1-4*x)
^(1/2)*f*x^2+4*(1-4*x)^(1/2)*x^3-40*f^2*x^2+52*f*x^3-12*x^4-2*x^2*
(1-4*x)^(1/2)+8*f^2*x-12*f*x^2+16*x^3+8*(1-4*x)^(1/2)*x+14*x^2-2*(
1-4*x)^(1/2)-12*x+2)^(1/2),0))^2))):
> ff:=factor(simplify(ff/32/x^9));
```

$$
\begin{aligned}
ff := & -4\sqrt{1-4x}\,x^5 + f^4\sqrt{1-4x} - f^6 x\sqrt{1-4x} + 25 f^5 x^2\sqrt{1-4x} - 2 f^5 x\sqrt{1-4x} \\
& + 62 f^4 x^2\sqrt{1-4x} - 12 f^4 x\sqrt{1-4x} + 32 f x^4 - 81 f x^5 - 4 f x^3 - 6 x^7 - 55 f^6 x^3\sqrt{1-4x} \\
& + 12 f^6 x^2\sqrt{1-4x} - 123 f^5 x^3\sqrt{1-4x} + 40 f^3 x^8 - 298 f^3 x^7 + 30 f^2 x^8 + 642 f^3 x^6 - 137 f^2 x^7 \\
& + 12 f x^8 - 751 f^3 x^5 + 286 f^2 x^6 - 37 f x^7 + 540 f^3 x^4 - 347 f^2 x^5 + 75 f x^6 - 225 f^3 x^3 + 48 f^3 x^2 \\
& - 4 f^3 x + 2\sqrt{1-4x}\,x^6 + \sqrt{1-4x}\,x^4 - 77 f^6 x^3 + 14 f^6 x^2 - 169 f^5 x^3 - f^6 x + 29 f^5 x^2 - 2 f^5 x \\
& + 84 f^4 x^2 - 14 f^4 x + 300\sqrt{1-4x}\,f^5 x^4 - 466\sqrt{1-4x}\,f^4 x^5 + 376\sqrt{1-4x}\,f^4 x^4 \\
& - 189\sqrt{1-4x}\,f^4 x^3 + 24\sqrt{1-4x}\,f x^4 - 4\sqrt{1-4x}\,f x^3 - 70\sqrt{1-4x}\,f^3 x^7 + 232\sqrt{1-4x}\,f^3 x^6 \\
& - 35\sqrt{1-4x}\,f^2 x^7 - 341\sqrt{1-4x}\,f^3 x^5 + 102\sqrt{1-4x}\,f^2 x^6 - 7\sqrt{1-4x}\,f x^7 \\
& + 298\sqrt{1-4x}\,f^3 x^4 - 157\sqrt{1-4x}\,f^2 x^5 + 25\sqrt{1-4x}\,f x^6 - 153\sqrt{1-4x}\,f^3 x^3 \\
& + 131\sqrt{1-4x}\,f^2 x^4 - 41\sqrt{1-4x}\,f x^5 + 40\sqrt{1-4x}\,f^3 x^2 - 4\sqrt{1-4x}\,f^3 x + 2 f^6 x^8 - 49 f^6 x^7 \\
& + 12 f^5 x^8 + 196 f^6 x^6 - 209 f^5 x^7 + 30 f^4 x^8 - 294 f^6 x^5 + 647 f^5 x^6 - 352 f^4 x^7 + 210 f^6 x^4 - 805 f^5 x^5 \\
& + 872 f^4 x^6 - 996 f^4 x^5 + 668 f^4 x^4 - 293 f^4 x^3 + 2 x^8 + 10 x^6 - 6 x^5 + x^4 + 215 f^2 x^4 - 60 f^2 x^3 \\
& + 6 f^2 x^2 - 48\sqrt{1-4x}\,f^2 x^3 + f^4 + 504 f^5 x^4 - 7\sqrt{1-4x}\,f^6 x^7 + 56\sqrt{1-4x}\,f^6 x^6 \\
& - 35\sqrt{1-4x}\,f^5 x^7 - 126\sqrt{1-4x}\,f^6 x^5 + 207\sqrt{1-4x}\,f^5 x^6 - 70\sqrt{1-4x}\,f^4 x^7 \\
& + 120\sqrt{1-4x}\,f^6 x^4 - 371\sqrt{1-4x}\,f^5 x^5 + 304\sqrt{1-4x}\,f^4 x^6 + 6\sqrt{1-4x}\,f^2 x^2
\end{aligned}
$$

```
> factor(simplify((coeff(ff,sqrt(1-4*x))*sqrt(1-4*x))^2-(coeff(ff,sq
rt(1-4*x),0))^2));
```

$$
\begin{aligned}
& -4 x^6 (f^6 x^7 + 6 f^5 x^7 - 10 f^5 x^6 + 15 f^4 x^7 + 3 f^5 x^5 - 46 f^4 x^6 + 20 f^3 x^7 + 50 f^4 x^5 - 84 f^3 x^6 + 15 f^2 x^7 \\
& \quad - 21 f^4 x^4 + 139 f^3 x^5 - 76 f^2 x^6 + 6 f x^7 + 3 f^4 x^3 - 117 f^3 x^4 + 159 f^2 x^5 - 34 f x^6 + x^7 + 50 f^3 x^3 \\
& \quad - 170 f^2 x^4 + 86 f x^5 - 6 x^6 - 11 f^3 x^2 + 94 f^2 x^3 - 115 f x^4 + 19 x^5 + f^3 x - 23 f^2 x^2 + 92 f x^3 - 32 x^4 \\
& \quad + 2 f^2 x - 45 f x^2 + 27 x^3 + 11 f x - 9 x^2 - f + x) (f^2 x + 2 f x - f + x)^3
\end{aligned}
$$

```
> simplify(series(subs(f=subs(u3=VVV[1],FinalA),
f^6*x^7+6*f^5*x^7-10*f^5*x^6+15*f^4*x^7+3*f^5*x^5-46*f^4*x^6+20*f^
3*x^7+50*f^4*x^5-84*f^3*x^6+15*f^2*x^7-21*f^4*x^4+139*f^3*x^5-76*f
^2*x^6+6*f*x^7+3*f^4*x^3-117*f^3*x^4+159*f^2*x^5-34*f*x^6+x^7+50*f
^3*x^3-170*f^2*x^4+86*f*x^5-6*x^6-11*f^3*x^2+94*f^2*x^3-115*f*x^4+
```

```
19*x^5+f^3*x-23*f^2*x^2+92*f*x^3-32*x^4+2*f^2*x-45*f*x^2+27*x^3+11
*f*x-9*x^2-f+x ),x,10)) assuming x>0 and x<0.1;
```

$$O(x^{10})$$

```
> fff:=factor(series(f^6*x^7+6*f^5*x^7-10*f^5*x^6+15*f^4*x^7+3*f^5*x
^5-46*f^4*x^6+20*f^3*x^7+50*f^4*x^5-84*f^3*x^6+15*f^2*x^7-21*f^4*x
^4+139*f^3*x^5-76*f^2*x^6+6*f*x^7+3*f^4*x^3-117*f^3*x^4+159*f^2*x^
5-34*f*x^6+x^7+50*f^3*x^3-170*f^2*x^4+86*f*x^5-6*x^6-11*f^3*x^2+94
*f^2*x^3-115*f*x^4+19*x^5+f^3*x-23*f^2*x^2+92*f*x^3-32*x^4+2*f^2*x
-45*f*x^2+27*x^3+11*f*x-9*x^2-f+x,f,10));
```

$$fff := x\,(x^6 - 6\,x^5 + 19\,x^4 - 32\,x^3 + 27\,x^2 - 9\,x + 1) + (x-1)\,(6\,x^6 - 28\,x^5 + 58\,x^4 - 57\,x^3 + 35\,x^2 - 10\,x + 1)\,f + x\,(15\,x^6 - 76\,x^5 + 159\,x^4 - 170\,x^3 + 94\,x^2 - 23\,x + 2)\,f^2 + x\,(20\,x^6 - 84\,x^5 + 139\,x^4 - 117\,x^3 + 50\,x^2 - 11\,x + 1)\,f^3 + x^3\,(15\,x^4 - 46\,x^3 + 50\,x^2 - 21\,x + 3)\,f^4 + x^5\,(6\,x^2 - 10\,x + 3)\,f^5 + x^7\,f^6$$

```
> factor(taylor(numer(BBxx),u3,10));factor(taylor(denom(BBxx),u3,10)
);
```

$$-(x-1)\,x\,(\sqrt{1-4\,x}\,x - \sqrt{1-4\,x} - 3\,x + 1) + (x^2\sqrt{1-4\,x} - x^2 + \sqrt{1-4\,x} + 4\,x - 2\sqrt{1-4\,x}\,x - 1)\,u3 - x\,(\sqrt{1-4\,x} + 1)\,u3^2$$

$$x^2\,(x-1)\,(\sqrt{1-4\,x} - 2\,x + 1) - 2\,x\,(\sqrt{1-4\,x}\,x - x^2 - \sqrt{1-4\,x} + 3\,x - 1)\,u3 + (4\,x - 2\,x^2 - 1 + 2\sqrt{1-4\,x}\,x - \sqrt{1-4\,x})\,u3^2$$

```
> #-------------------------------------------------------------------
-----------------------------------------------
> #1012
> alias(vv=RootOf(v^3-(v*x+v-x)*(v-x)=0,v)); VVV:=allvalues(vv):
> simplify(series(VVV[1],x,5)) assuming x>0 and x<0.1;
subs(signum(9*x^6+12*x^5+60*x^4-12*x^3)=1,simplify(series(VVV[2],x
,5))) assuming x>0 and x<0.1;
subs(signum(9*x^6+12*x^5+60*x^4-12*x^3)=1,simplify(series(VVV[3],x
,5))) assuming x>0 and x<0.1;
```

$$vv$$

$$1 - 16\,x^4 - 5\,x^3 - 2\,x^2 + O(x^5) - x$$

$$x + x^{(3/2)} + x^2 + \frac{3\,x^{(5/2)}}{2} + \frac{5\,x^3}{2} + \frac{35\,x^{(7/2)}}{8} + 8\,x^4 + \frac{241\,x^{(9/2)}}{16} + O(x^5)$$

$$x - x^{(3/2)} + x^2 - \frac{3\,x^{(5/2)}}{2} + \frac{5\,x^3}{2} - \frac{35\,x^{(7/2)}}{8} + 8\,x^4 - \frac{241\,x^{(9/2)}}{16} + O(x^5)$$

```
> FinalA:=-((3*x^2-3*x+1)*v0^2+(-2*x^3+2*x-1)*v0+2*x^4-2*x^2+x)/((1-
x)*(1-2*x)*(v0^2-v0*x+x^2));
simplify(series(subs(v0=VVV[1],FinalA),x,12));
```

$$FinalA := -\frac{(3x^2-3x+1)\,v0^2+(-2x^3+2x-1)\,v0+2x^4-2x^2+x}{(-x+1)(-2x+1)(v0^2-v0\,x+x^2)}$$

$$2x^2+x+1494x^7+O(x^{12})+6413x^8+125185x^{10}+566257x^{11}+356x^6+87x^5+6x^3+22x^4+28088x^9$$

```
> alias(ww=RootOf(x*(2*x^2-2*x+1)*(x^6-2*x^5+13*x^4-24*x^3+19*x^2-7*x+1)+(1-2*x)*(1-x)*(x^7-x^6+24*x^5-51*x^4+50*x^3-27*x^2+8*x-1)*f+x*(x^3+6*x^2-6*x+2)*(1-2*x)^2*(1-x)^2*f^2+x*(1-2*x)^3*(1-x)^3*f^3=0,f)): series(ww,x,13);
```

$$x+2x^2+6x^3+22x^4+87x^5+356x^6+1494x^7+6413x^8+28088x^9+125185x^{10}+566257x^{11}+2593591x^{12}+O(x^{13})$$

```
> #-----------------------------------------------------------------------------------------------
> #1020,1022
> FinalA:=(2-15*x+34*x^2-22*x^3)/2/x^3/(1-x)/sqrt(1-4*x)+(2*x^4+4*x^3-16*x^2+11*x-2)/2/x^3/(1-x);
```

$$FinalA := \frac{-22x^3+34x^2-15x+2}{2x^3(-x+1)\sqrt{1-4x}}+\frac{2x^4+4x^3-16x^2+11x-2}{2x^3(-x+1)}$$

```
> taylor(FinalA,x,20);
```

$$x+2x^2+6x^3+22x^4+88x^5+363x^6+1507x^7+6239x^8+25687x^9+105145x^{10}+428145x^{11}+1735649x^{12}+7010239x^{13}+28228774x^{14}+113391334x^{15}+454561834x^{16}+O(x^{17})$$

```
> series((2-15*x+34*x^2-22*x^3)/2/x^3/(1-x),x,10);
```

$$x^{-3}-\frac{13}{2}x^{-2}+\frac{21}{2}x^{-1}-\frac{1}{2}-\frac{1}{2}x-\frac{1}{2}x^2-\frac{1}{2}x^3-\frac{1}{2}x^4-\frac{1}{2}x^5-\frac{1}{2}x^6-\frac{1}{2}x^7-\frac{1}{2}x^8-\frac{1}{2}x^9+O(x^{10})$$

```
> seq(binomial(2*n+8,n+4)-13/2*binomial(2*n+6,n+3)+21/2*binomial(2*n+4,n+2)-1/2*add(binomial(2*j,j),j=0..n+1)-1/2,n=0..20);
```

1, 2, 6, 22, 88, 363, 1507, 6239, 25687, 105145, 428145, 1735649, 7010239, 28228774, 113391334, 454561834, 1819243834, 7271148424, 29029141984, 115789690144, 461508680644

```
> #-----------------------------------------------------------------------------------------------
> #1023
> FinalA:=-(2*x^2-2*x+1)*sqrt(1-4*x)/2/x/(1-x)^2/(1-2*x)-(16*x^9-76*x^8+212*x^7-342*x^6)/2/x/(1-x)^5/(1-2*x)^3-(374*x^5-286*x^4+151*x^3-53*x^2+11*x-1)/2/x/(1-x)^5/(1-2*x)^3;
```

$$FinalA := -\frac{(2x^2-2x+1)\sqrt{1-4x}}{2x(1-x)^2(1-2x)}-\frac{16x^9-76x^8+212x^7-342x^6}{2x(1-x)^5(1-2x)^3}$$

$$-\frac{374\,x^5-286\,x^4+151\,x^3-53\,x^2+11\,x-1}{2\,x\,(1-x)^5\,(1-2\,x)^3}$$

```
> taylor(FinalA,x,12);
```

$$x+2\,x^2+6\,x^3+22\,x^4+87\,x^5+341\,x^6+1280\,x^7+4595\,x^8+15961\,x^9+54418\,x^{10}+\mathrm{O}(x^{11})$$

```
> CC:=(1-sqrt(1-4*x))/2/x: cc:=n->1/(n+1)*binomial(2*n,n):
ff:=subs(sqrt(1-4*x)=1-2*x*fC,FinalA):
factor(coeff(ff,fC,1))*fC+factor(coeff(ff,fC,0));
```

$$-\frac{(2\,x^2-2\,x+1)\,fC}{(-1+x)^2\,(-1+2\,x)}-\frac{8\,x^8-38\,x^7+102\,x^6-151\,x^5+144\,x^4-91\,x^3+37\,x^2-9\,x+1}{(-1+x)^5\,(-1+2\,x)^3}$$

```
> with(genfunc):
> aa1:=rgf_expand(-(2*x^2-2*x+1)/(x-1)^2/(2*x-1),x,n);
aa2:=factor(subs(n=n+1,simplify(rgf_expand(-(8*x^8-38*x^7+102*x^6-
151*x^5+144*x^4-91*x^3+37*x^2-9*x+1)/(x-1)^5/(2*x-1)^3,x,n))));
```

$$aa1 := 2\,2^n-n-1$$

$$aa2 := \frac{n^4}{8}+\frac{11\,n^3}{12}+\frac{43\,n^2}{8}+\frac{199\,n}{12}+24+\frac{2^{(n+1)}\,n^2}{4}-\frac{3\,2^{(n+1)}\,n}{4}-13\,2^{(n+1)}$$

```
> seq(add((2^(j+1)-j-1)*cc(n+1-j),j=0..n+1)+1/24*(3*n^4+22*n^3+129*n
^2+398*n)+24+2^(n-1)*(n^2-3*n-52),n=0..20);
```

1, 2, 6, 22, 87, 341, 1280, 4595, 15961, 54418, 184415, 627008, 2151350, 7470265, 26267185, 93459565, 336023816, 1218952946, 4455066424, 16384909926, 60580008381

```
> #-----------------------------------------------------------------
-------------------------------------------
> #1102
> restart: GG:=x/(1-x): FF:=(x+x*GG+x*D11)/(1-x): EE:=x/(1-2*x):
eqD:=-D1+x/(1-v)*(1+D1+GG+FF)+x/v*(D1-D11);
kd:=solve(coeff(eqD,D1)=0,v)[2]; DD1:=solve(subs(v=kd,eqD)=0,D11);
DDD:=solve(subs(D11=DD1,eqD)=0,D1);
```

$$eqD := -\mathrm{D1}+\frac{x\left(1+\mathrm{D1}+\frac{x}{1-x}+\frac{x\,\mathrm{D11}+\frac{x^2}{1-x}+x}{1-x}\right)}{1-v}+\frac{x\,(\mathrm{D1}-\mathrm{D11})}{v}$$

$$kd := \frac{1}{2}-\frac{\sqrt{1-4\,x}}{2}$$

$$DD1 := \frac{-1+\sqrt{1-4\,x}}{-1+\sqrt{1-4\,x}\,x-2\,x^2-\sqrt{1-4\,x}+3\,x}$$

$$DDD := \frac{(\sqrt{1-4\,x}+2\,v-1)\,x}{(-1+x)\,(-2\,x+1+\sqrt{1-4\,x})\,(v^2-v+x)}$$

```
> eqC:=-C+x/(1-v)*(1+EE+DDD+subs(D11=DD1,FF))+x/v*(C-C1);
  CC1:=solve(subs(v=x,eqC)=0,C1); CCC:=solve(subs(C1=CC1,eqC)=0,C);
```

$$eqC := -C + x\left(1 + \frac{x}{1-2x} + \frac{(\sqrt{1-4x}+2v-1)x}{(-1+x)(-2x+1+\sqrt{1-4x})(v^2-v+x)} + \frac{\frac{x(-1+\sqrt{1-4x})}{-1+\sqrt{1-4x}\,x-2x^2-\sqrt{1-4x}+3x} + \frac{x^2}{1-x} + x}{1-x}\right) / (1-v) + \frac{x(C-C1)}{v}$$

$$CC1 := -\frac{\sqrt{1-4x}\,x^3 - 2x^4 - 2\sqrt{1-4x}\,x^2 + x^3 + 3\sqrt{1-4x}\,x + 2x^2 - \sqrt{1-4x} - 3x + 1}{(-1+\sqrt{1-4x}\,x-2x^2-\sqrt{1-4x}+3x)(-1+2x)(-1+x)}$$

$$CCC := (\sqrt{1-4x}\,v^2x^2 - 2v^2x^3 + \sqrt{1-4x}\,vx^2 + \sqrt{1-4x}\,x^3 + 5v^2x^2 + 2vx^3 - 2x^4 - 3\sqrt{1-4x}\,vx - 2\sqrt{1-4x}\,x^2 - 6v^2x - 7vx^2 + x^3 + \sqrt{1-4x}\,v + 3\sqrt{1-4x}\,x + 2v^2 + 9xv + 2x^2 - \sqrt{1-4x} - 3v - 3x + 1)x \Big/ ((-1+\sqrt{1-4x}\,x-2x^2-\sqrt{1-4x}+3x)(v^2-v+x)(-1+x)(-1+2x)(-1+v))$$

```
> eqB:=-B+x/(1-v)*(1+B)+x/v*(B-B1)+x*CCC;
  BB1:=solve(limit(eqB,v=kd)=0,B1);
  BBB:=solve(subs(B1=BB1,eqB)=0,B);
```

$$eqB := -B + \frac{x(1+B)}{1-v} + \frac{x(B-B1)}{v} + x^2(\sqrt{1-4x}\,v^2x^2 - 2v^2x^3 + \sqrt{1-4x}\,vx^2 + \sqrt{1-4x}\,x^3 + 5v^2x^2 + 2vx^3 - 2x^4 - 3\sqrt{1-4x}\,vx - 2\sqrt{1-4x}\,x^2 - 6v^2x - 7vx^2 + x^3 + \sqrt{1-4x}\,v + 3\sqrt{1-4x}\,x + 2v^2 + 9xv + 2x^2 - \sqrt{1-4x} - 3v - 3x + 1) \Big/ ((-1+\sqrt{1-4x}\,x-2x^2-\sqrt{1-4x}+3x)(v^2-v+x)(-1+x)(-1+2x)(-1+v))$$

$$BB1 := -x(\sqrt{1-4x}\,x^3 - 4x^4 - 5\sqrt{1-4x}\,x^2 + 17x^3 + 4\sqrt{1-4x}\,x - 15x^2 - \sqrt{1-4x} + 6x - 1) \Big/ (1 + 6x^4\sqrt{1-4x} - 8x^5 - 17\sqrt{1-4x}\,x^3 + 30x^4 + 17\sqrt{1-4x}\,x^2 - 43x^3 - 7\sqrt{1-4x}\,x + 29x^2 + \sqrt{1-4x} - 9x)$$

$$BBB := 2(v^2 - 2xv - \sqrt{1-4x}\,v^3 + 4v^3x^5 - 4v^2x^6 - 32v^3x^4 + 9v^2x^5 + 16vx^6 - 67vx^5 + 2\sqrt{1-4x}\,v^3x^5 - 10\sqrt{1-4x}\,v^3x^4 - 3\sqrt{1-4x}\,v^2x^5 + 2\sqrt{1-4x}\,vx^6 + 21\sqrt{1-4x}\,v^3x^3 + 4\sqrt{1-4x}\,v^2x^4 - 15\sqrt{1-4x}\,vx^5 + 38\sqrt{1-4x}\,vx^4 + 7\sqrt{1-4x}\,v^3x - 33\sqrt{1-4x}\,vx^3 - 4x^7 + 3\sqrt{1-4x}\,x^6 - 14x^5\sqrt{1-4x} + 14x^4\sqrt{1-4x} + \sqrt{1-4x}\,v^2 + 47v^3x^3 + 12v^2x^4 - 30v^3x^2 - 30v^2x^3 + 88vx^4 + 9v^3x + 23v^2x^2 - 55vx^3 - 8v^2x - v^3 + 21x^6 - 6\sqrt{1-4x}\,x^3 + \sqrt{1-4x}\,x^2 - 34x^5 + 24x^4 + 17vx^2 - 18\sqrt{1-4x}\,v^3x^2 - 12\sqrt{1-4x}\,v^2x^3 + 13\sqrt{1-4x}\,v^2x^2 - 6\sqrt{1-4x}\,v^2x + 13\sqrt{1-4x}\,vx^2 - 2\sqrt{1-4x}\,vx - 8x^3 + x^2)x \Big/ ($$

$$(-1+2x)(-1+x)(v^2-v+x)^2(-1+\sqrt{1-4x}\,x-2x^2-\sqrt{1-4x}+3x)$$
$$(3\sqrt{1-4x}\,x-4x^2-\sqrt{1-4x}+5x-1))$$

```
> eqA:=-A+x/(1-v)*(1+BBB)+x/v*(A-A1):
  AA1:=simplify(solve(limit(eqA,v=x)=0,A1));
```

$$AA1 := -x\,(8x^5\sqrt{1-4x}-8x^6-24x^4\sqrt{1-4x}+46x^5+34\sqrt{1-4x}\,x^3-84x^4-24\sqrt{1-4x}\,x^2+80x^3+8\sqrt{1-4x}\,x-40x^2-\sqrt{1-4x}+10x-1)\Big/(10\sqrt{1-4x}\,x^6-8x^7-45x^5\sqrt{1-4x}+54x^6+82x^4\sqrt{1-4x}-141x^5-77\sqrt{1-4x}\,x^3+188x^4+39\sqrt{1-4x}\,x^2-139x^3-10\sqrt{1-4x}\,x+57x^2+\sqrt{1-4x}-12x+1)$$

```
> FinalA:=factor(simplify(simplify(rationalize(AA1)+1)-1));
  factor(coeff(FinalA,sqrt(1-4*x)))*sqrt(1-4*x)+factor(coeff(FinalA,
  sqrt(1-4*x),0)); taylor(FinalA,x,13);
```

$$FinalA := -(4\sqrt{1-4x}\,x^6+16x^7-35x^5\sqrt{1-4x}-64x^6+77x^4\sqrt{1-4x}+155x^5-76\sqrt{1-4x}\,x^3-195x^4+39\sqrt{1-4x}\,x^2+140x^3-10\sqrt{1-4x}\,x-57x^2+\sqrt{1-4x}+12x-1)\Big/(2x^2(-1+4x)(-1+2x)(-1+x)^3)$$
$$-\frac{(4x^5-31x^4+46x^3-30x^2+9x-1)\sqrt{1-4x}}{2(-1+x)^2(-1+2x)(-1+4x)x^2}-\frac{4x^6-15x^5+35x^4-40x^3+25x^2-8x+1}{2(-1+x)^3(-1+2x)x^2}$$
$$x+2x^2+6x^3+22x^4+87x^5+349x^6+1393x^7+5520x^8+21764x^9+85576x^{10}+O(x^{11})$$

```
>
> with(genfunc):
> aa1:=rgf_expand(1/2*(4*x^5-31*x^4+46*x^3-30*x^2+9*x-1)/(-1+x)^2/(-
  1+2*x),x,n); seq(aa1,n=0..3);
  taylor(1/2*(4*x^5-31*x^4+46*x^3-30*x^2+9*x-1)/(-1+x)^2/(-1+2*x),x,
  10);
  aa2:=factor(simplify(rgf_expand(-1/2*(4*x^6-15*x^5+35*x^4-40*x^3+2
  5*x^2-8*x+1)/(-1+x)^3/(-1+2*x)/x^2,x,n)));
```

$$aa1 := \frac{2^n}{8}-\frac{3n}{2}+4$$
$$\frac{33}{8},\frac{11}{4},\frac{3}{2},\frac{1}{2}$$
$$\frac{1}{2}-\frac{5}{2}x+\frac{5}{2}x^2+\frac{1}{2}x^3+\frac{1}{2}x^5+3x^6+\frac{19}{2}x^7+24x^8+\frac{109}{2}x^9+O(x^{10})$$
$$aa2 := \frac{n^2}{2}-2^{(-1+n)}+1$$

```
> seq(1/2*binomial(2*n+6,n+3)-21/4*binomial(2*n+4,n+2)+binomial(2*n+
  2,n+1)+1/2*(n+1)^2-2^n+1+1/4*add((2^(j-1)-6*j+16)*binomial(2*n+6-2
  *j,n+3-j),j=1..n+3),n=0..20);
```

1, 2, 6, 22, 87, 349, 1393, 5520, 21764, 85576, 336083, 1319461, 5180736, 20347913, 79950236,

314271500, 1235886385, 4862263266, 19137302398, 75352962521, 296819163725

```
> #-------------------------------------------------------------------
  ------------------------------------------------
> #1120
> FinalA:=((3*x-1)*sqrt(1-4*x)+5*x^2-5*x+1)/x/(1-4*x);
  taylor(FinalA,x,20);
```

$$FinalA := \frac{(3x-1)\sqrt{1-4x}+5x^2-5x+1}{x(1-4x)}$$

$$x+2x^2+6x^3+22x^4+88x^5+364x^6+1522x^7+6374x^8+26640x^9+110980x^{10}+460716x^{11}+1906172x^{12}+7862416x^{13}+32341144x^{14}+132707626x^{15}+543376774x^{16}+2220650656x^{17}+9060011284x^{18}+O(x^{19})$$

```
> seq(4^(n)-n/2/(2*n+3)*binomial(2*n+4,n+2),n=0..10);
```

1, 2, 6, 22, 88, 364, 1522, 6374, 26640, 110980, 460716

```
> #-------------------------------------------------------------------
  ------------------------------------------------
> ##,1200,1220
> restart: EE:=(x+x*D11)/(1-x):
  eqD:=-D1+x/(1-v)*(1+D1+EE)+x/v*(D1-D11);
  kd:=solve(coeff(eqD,D1)=0,v)[2];
  DD1:=solve(subs(D1=0,v=kd,eqD)=0,D11);
  DDD:=solve(subs(D11=DD1,eqD)=0,D1);
```

$$eqD := -D1+\frac{x\left(1+D1+\frac{D11\,x+x}{1-x}\right)}{1-v}+\frac{x(D1-D11)}{v}$$

$$kd := \frac{1}{2}-\frac{\sqrt{1-4x}}{2}$$

$$DD1 := -\frac{-1+\sqrt{1-4x}}{1+\sqrt{1-4x}-2x}$$

$$DDD := -\frac{(\sqrt{1-4x}+2v-1)x}{(1+\sqrt{1-4x}-2x)(v^2-v+x)}$$

```
> eqC:=-C+x/(1-v)*(1+C)+x*DDD+x/v*(C-C1);
  CC1:=solve(limit(subs(C=0,eqC),v=kd)=0,C1);
  CCC:=factor(simplify(solve(subs(C1=CC1,eqC)=0,C)));
```

$$eqC := -C+\frac{x(1+C)}{1-v}-\frac{x^2(\sqrt{1-4x}+2v-1)}{(1+\sqrt{1-4x}-2x)(v^2-v+x)}+\frac{x(C-C1)}{v}$$

$$CC1 := -\frac{x(1+\sqrt{1-4x}-2x)}{-1+3\sqrt{1-4x}\,x-4x^2-\sqrt{1-4x}+5x}$$

$$CCC := -2x\left(v^2 - 2xv + 2x^4 - v^3 - 4v^3x^2 - \sqrt{1-4x}\,v^3 - 2\sqrt{1-4x}\,x^3 + \sqrt{1-4x}\,v^2 - 2v^3x^3 + 2vx^4 + 5v^3x + v^2x^2 - 7vx^3 - 4v^2x + 9vx^2 - 2\sqrt{1-4x}\,v^2x^3 + 3\sqrt{1-4x}\,v^3x - \sqrt{1-4x}\,v^2x^2 - \sqrt{1-4x}\,vx^3 - 2\sqrt{1-4x}\,v^2x + 5\sqrt{1-4x}\,vx^2 - 2\sqrt{1-4x}\,vx - 4x^3 + x^2 + \sqrt{1-4x}\,x^2\right) \Big/ \left(\left(-1 + 3\sqrt{1-4x}\,x - 4x^2 - \sqrt{1-4x} + 5x\right)\left(1 + \sqrt{1-4x} - 2x\right)\left(v^2 - v + x\right)^2\right)$$

```
> eqB:=-B+x/(1-v)*(1+CCC)+2*x/v*(B-B1);
  BB1:=solve(subs(v=2*x,eqB)=0,B1);
  BBB:=solve(subs(B1=BB1,eqB)=0,B);
```

$$eqB := -B + x\left(1 - 2x\left(v^2 - 2xv + 2x^4 - v^3 - 4v^3x^2 - \sqrt{1-4x}\,v^3 - 2\sqrt{1-4x}\,x^3 + \sqrt{1-4x}\,v^2 - 2v^3x^3 + 2vx^4 + 5v^3x + v^2x^2 - 7vx^3 - 4v^2x + 9vx^2 - 2\sqrt{1-4x}\,v^2x^3 + 3\sqrt{1-4x}\,v^3x - \sqrt{1-4x}\,v^2x^2 - \sqrt{1-4x}\,vx^3 - 2\sqrt{1-4x}\,v^2x + 5\sqrt{1-4x}\,vx^2 - 2\sqrt{1-4x}\,vx - 4x^3 + x^2 + \sqrt{1-4x}\,x^2\right) \Big/ \left(\left(-1 + 3\sqrt{1-4x}\,x - 4x^2 - \sqrt{1-4x} + 5x\right)\left(1 + \sqrt{1-4x} - 2x\right)\left(v^2 - v + x\right)^2\right)\right) / (1-v) + \frac{2x(B - B1)}{v}$$

$$BB1 := \frac{2x\left(9\sqrt{1-4x}\,x^2 - 10x^3 - 6\sqrt{1-4x}\,x + 19x^2 + \sqrt{1-4x} - 8x + 1\right)}{\left(-1 + 3\sqrt{1-4x}\,x - 4x^2 - \sqrt{1-4x} + 5x\right)\left(1 + \sqrt{1-4x} - 2x\right)(-1 + 4x)}$$

$$BBB := 2x\left(v^2 - 2xv + 4v^3x^4 - 2v^2x^4 - 4vx^5 - 10x^5 + 2\sqrt{1-4x}\,v^3x^3 + 4\sqrt{1-4x}\,v^2x^4 - 13\sqrt{1-4x}\,v^3x^2 + 6\sqrt{1-4x}\,vx^4 + 19x^4 - v^3 + 9\sqrt{1-4x}\,x^4 - 25v^3x^2 - \sqrt{1-4x}\,v^3 - 6\sqrt{1-4x}\,x^3 + \sqrt{1-4x}\,v^2 + 18v^3x^3 - 6v^2x^3 + 34vx^4 + 9v^3x + 18v^2x^2 - 45vx^3 - 8v^2x + 17vx^2 + 2\sqrt{1-4x}\,v^2x^3 + 7\sqrt{1-4x}\,v^3x + 8\sqrt{1-4x}\,v^2x^2 - 23\sqrt{1-4x}\,vx^3 - 6\sqrt{1-4x}\,v^2x + 13\sqrt{1-4x}\,vx^2 - 2\sqrt{1-4x}\,vx - 8x^3 + x^2 + \sqrt{1-4x}\,x^2\right) \Big/ \left((-1 + 4x)\left(-1 + 3\sqrt{1-4x}\,x - 4x^2 - \sqrt{1-4x} + 5x\right)\left(1 + \sqrt{1-4x} - 2x\right)\left(v^2 - v + x\right)^2\right)$$

```
> eqA:=-A+x/(1-v)*(1+BBB)+x/v*(A-A1);
  FinalA:=solve(subs(v=x,eqA)=0,A1); taylor(FinalA,x,13);
```

$$eqA := -A + x\left(1 + 2x\left(v^2 - 2xv + 4v^3x^4 - 2v^2x^4 - 4vx^5 - 10x^5 + 2\sqrt{1-4x}\,v^3x^3 + 4\sqrt{1-4x}\,v^2x^4 - 13\sqrt{1-4x}\,v^3x^2 + 6\sqrt{1-4x}\,vx^4 + 19x^4 - v^3 + 9\sqrt{1-4x}\,x^4 - 25v^3x^2 - \sqrt{1-4x}\,v^3 - 6\sqrt{1-4x}\,x^3 + \sqrt{1-4x}\,v^2 + 18v^3x^3 - 6v^2x^3 + 34vx^4 + 9v^3x + 18v^2x^2 - 45vx^3 - 8v^2x + 17vx^2 + 2\sqrt{1-4x}\,v^2x^3 + 7\sqrt{1-4x}\,v^3x + 8\sqrt{1-4x}\,v^2x^2 - 23\sqrt{1-4x}\,vx^3 - 6\sqrt{1-4x}\,v^2x + 13\sqrt{1-4x}\,vx^2 - 2\sqrt{1-4x}\,vx - 8x^3 + x^2 + \sqrt{1-4x}\,x^2\right) \Big/ \left((-1 + 4x)\left(-1 + 3\sqrt{1-4x}\,x - 4x^2 - \sqrt{1-4x} + 5x\right)\left(1 + \sqrt{1-4x} - 2x\right)\left(v^2 - v + x\right)^2\right)\right) / (1-v) + \frac{x(A - A1)}{v}$$

$$FinalA := 2x$$

$$\left(14\sqrt{1-4x}\,x^3-20x^4-20\sqrt{1-4x}\,x^2+44x^3+8\sqrt{1-4x}\,x-34x^2-\sqrt{1-4x}+10x-1\right)\Big/$$
$$\left((-1+x)(-1+4x)\left(-1+3\sqrt{1-4x}\,x-4x^2-\sqrt{1-4x}+5x\right)\left(1+\sqrt{1-4x}-2x\right)\right)$$
$$x+2x^2+6x^3+22x^4+89x^5+377x^6+1628x^7+7072x^8+30707x^9+132891x^{10}+572567x^{11}$$
$$+2455303x^{12}+O(x^{13})$$

```
> FinalA:=factor(simplify(simplify(rationalize(FinalA)+1)-1));
  factor(coeff(FinalA,sqrt(1-4*x)))*sqrt(1-4*x)+factor(coeff(FinalA,
  sqrt(1-4*x),0));
```

$$FinalA := -\frac{22\sqrt{1-4x}\,x^3+40x^4-29\sqrt{1-4x}\,x^2-78x^3+10\sqrt{1-4x}\,x+49x^2-\sqrt{1-4x}-12x+1}{2x(-1+x)(-1+4x)^2}$$
$$-\frac{(22x^3-29x^2+10x-1)\sqrt{1-4x}}{2x(-1+x)(-1+4x)^2}-\frac{(5x-1)(2x-1)}{2x(-1+4x)}$$

```
> f:=n->(2*n+1)!/n!^2:
  seq((n+4)/2/(n+2)*binomial(2*n+2,n+1)+add(f(j),j=0..n-1)-4^n,n=0..
  10);
```

$$1, 2, 6, 22, 89, 377, 1628, 7072, 30707, 132891, 572567$$

```
>
> #-----------------------------------------------------------------
  ----------------------------------------
> #1202
> restart: eqD:=-D1+x/(1-v)*(1+C)+x/v*(C+D1-C1-D11);
  eqC:=-C+x/(1-v)*(1+C)+x*D1+x/v*(C-C1);
  eqC1:=subs(D1=solve(eqD=0,D1),eqC); factor(coeff(eqC1,C));
  alias(vv=RootOf(v^3-v^2*x+v*x^2-v^2+2*v*x-x^2=0,v)):
  VVV:=allvalues(vv):
```

$$eqD := -D1+\frac{x(1+C)}{1-v}+\frac{x(C+D1-C1-D11)}{v}$$
$$eqC := -C+\frac{x(1+C)}{1-v}+x\,D1+\frac{x(C-C1)}{v}$$
$$eqC1 := -C+\frac{x(1+C)}{1-v}-\frac{x^2(C1\,v+D11\,v+C-C1-D11+v)}{(-1+v)(v-x)}+\frac{x(C-C1)}{v}$$
$$-\frac{v^3-v^2x+v\,x^2-v^2+2\,v\,x-x^2}{(-1+v)(v-x)\,v}$$

```
> solve({subs(C=0,v=u1,eqC1),subs(C=0,v=u2,eqC1)},{C1,D11});
```

$$\left\{C1=\frac{u1\,u2}{x(u1\,u2-u1-u2+1)},\right.$$

$$D11 = -\frac{u1\,u2\,x^2 + 2\,u1\,u2\,x - u1\,x^2 - u2\,x^2 + u1\,u2 - u1\,x - u2\,x + x^2}{(-1+u1)\,(-1+u2)\,x^2}\}$$

```
> CC1:=factor(u1*u2/x/(u1*u2-u1-u2+1));
DD1:=factor(-(u1*u2*x^2+2*u1*u2*x-u1*x^2-u2*x^2+u1*u2-u1*x-u2*x+x^
2)/(-1+u1)/(-1+u2)/x^2);
```

$$CC1 := \frac{u1\,u2}{x\,(-1+u2)\,(-1+u1)}$$

$$DD1 := -\frac{(u2\,x + u2 - x)\,(u1\,x + u1 - x)}{(-1+u1)\,(-1+u2)\,x^2}$$

```
> simplify(series(subs(u1=VVV[2],u2=VVV[3],CC1),x,10)) assuming x>0
and x<0.1;
```

$$10\,x^3 + 3\,x^2 + 2563\,x^7 + 11181\,x^8 + 602\,x^6 + 146\,x^5 + 37\,x^4 + O(x^9) + x$$

```
> solve({subs(C1=CC1,D11=DD1,eqC),subs(C1=CC1,D11=DD1,eqD)},{C,D1});
```

$$\{C = -x\,(u1\,u2 - u1\,v - u2\,v + v^2) \big/ (u1\,u2\,v^3 - u1\,u2\,v^2\,x + u1\,u2\,v\,x^2 - u1\,u2\,v^2 + 2\,u1\,u2\,v\,x - u1\,u2\,x^2 - u1\,v^3 + u1\,v^2\,x - u1\,v\,x^2 - u2\,v^3 + u2\,v^2\,x - u2\,v\,x^2 + u1\,v^2 - 2\,u1\,v\,x + u1\,x^2 + u2\,v^2 - 2\,u2\,v\,x + u2\,x^2 + v^3 - v^2\,x + v\,x^2 - v^2 + 2\,v\,x - x^2),\ D1 = (u1\,u2\,v^2\,x - u1\,u2\,v\,x^2 + u1\,u2\,x^3 + u1\,u2\,v^2 - u1\,u2\,v\,x + 2\,u1\,u2\,x^2 - u1\,v^2\,x + u1\,v\,x^2 - u1\,x^3 - u2\,v^2\,x + u2\,v\,x^2 - u2\,x^3 - u1\,u2\,v + u1\,u2\,x + u1\,v\,x - u1\,x^2 + u2\,v\,x - u2\,x^2 - v\,x^2 + x^3) \big/ ((u1\,u2 - u1 - u2 + 1)\,x\,(v^3 - v^2\,x + v\,x^2 - v^2 + 2\,v\,x - x^2))\}$$

```
> CCC:=factor(-x*(u1*u2-u1*v-u2*v+v^2)/(u1*u2*v^3-u1*u2*v^2*x+u1*u2*
v*x^2-u1*u2*v^2+2*u1*u2*v*x-u1*u2*x^2-u1*v^3+u1*v^2*x-u1*v*x^2-u2*
v^3+u2*v^2*x-u2*v*x^2+u1*v^2-2*u1*v*x+u1*x^2+u2*v^2-2*u2*v*x+u2*x^
2+v^3-v^2*x+v*x^2-v^2+2*v*x-x^2));
DDD:=factor((u1*u2*v^2*x-u1*u2*v*x^2+u1*u2*x^3+u1*u2*v^2-u1*u2*v*x
+2*u1*u2*x^2-u1*v^2*x+u1*v*x^2-u1*x^3-u2*v^2*x+u2*v*x^2-u2*x^3-u1*
u2*v+u1*u2*x+u1*v*x-u1*x^2+u2*v*x-u2*x^2-v*x^2+x^3)/(u1*u2-u1-u2+1
)/x/(v^3-v^2*x+v*x^2-v^2+2*v*x-x^2));
```

$$CCC := -\frac{x\,(u2 - v)\,(u1 - v)}{(v^3 - v^2\,x + v\,x^2 - v^2 + 2\,v\,x - x^2)\,(-1+u2)\,(-1+u1)}$$

$$DDD := (u1\,u2\,v^2\,x - u1\,u2\,v\,x^2 + u1\,u2\,x^3 + u1\,u2\,v^2 - u1\,u2\,v\,x + 2\,u1\,u2\,x^2 - u1\,v^2\,x + u1\,v\,x^2 - u1\,x^3 - u2\,v^2\,x + u2\,v\,x^2 - u2\,x^3 - u1\,u2\,v + u1\,u2\,x + u1\,v\,x - u1\,x^2 + u2\,v\,x - u2\,x^2 - v\,x^2 + x^3) \big/ ((-1+u2)\,(-1+u1)\,x\,(v^3 - v^2\,x + v\,x^2 - v^2 + 2\,v\,x - x^2))$$

```
> eqB:=-B+x/(1-v)*(1+CCC)+2*x/v*(B-B1):
BB1:=factor(solve(limit(eqB,v=2*x)=0,B1));
BBB:=factor(solve(subs(B1=BB1,eqB)=0,B));
```

$$BB1 := -\frac{3\,u1\,u2\,x + u1\,u2 - 3\,u1\,x - 3\,u2\,x + x}{(6\,x - 1)\,(-1+u2)\,(-1+u1)}$$

$$BBB := x\,(3\,u1\,u2\,v^2 - 3\,u1\,u2\,v\,x + 3\,u1\,u2\,x^2 - 2\,u1\,u2\,v + u1\,u2\,x - u1\,v^2 + 5\,u1\,v\,x - 3\,u1\,x^2 - u2\,v^2 + 5\,u2\,v\,x - 3\,u2\,x^2 - 4\,v^2\,x + 2\,v\,x^2 + v^2 - 2\,v\,x + x^2) \Big/ ((-1+u1)\,(-1+u2)\,(v^3 - v^2\,x + v\,x^2 - v^2 + 2\,v\,x - x^2)\,(6\,x-1))$$

```
> eqA:=-A+x/(1-v)*(1+BBB)+x/v*(A-A1):
  AA1:=factor(solve(limit(eqA,v=x)=0,A1));
```

$$AA1 := -\frac{6\,u1\,u2\,x^2 + 2\,u1\,u2\,x - 6\,u1\,x^2 - 6\,u2\,x^2 - u1\,u2 + 2\,u1\,x + 2\,u2\,x + 4\,x^2 - x}{(-1+x)\,(-1+u1)\,(-1+u2)\,(6\,x-1)}$$

```
> simplify(series(subs(u1=VVV[2],u2=VVV[3],AA1),x,11)) assuming x>0
  and x<0.1;
```

$$1665\,x^7 + 157564\,x^{10} + 379\,x^6 + 89\,x^5 + 22\,x^4 + \mathrm{O}(x^{11}) + 34098\,x^9 + 7474\,x^8 + 6\,x^3 + 2\,x^2 + x$$

```
> simplify(series(subs(u1=VVV[2],u2=VVV[3],u1*u2*VVV[1]),x,11))
  assuming x>0 and x<0.1;
  simplify(series(subs(u1=VVV[2],u2=VVV[3],u1+u2+VVV[1]),x,11))
  assuming x>0 and x<0.1;
```

$$\mathrm{O}(x^{11}) + x^2$$

$$1 + x$$

```
> AA1 :=
  factor(-(6*x^2/u3*x^2+2*x^2/u3*x-6*(1+x-u3)*x^2-x^2/u3+2*(1+x-u3)*
  x+4*x^2-x)/(-1+x)/(1-(1+x-u3)+x^2/u3)/(6*x-1));
```

$$AA1 := -\frac{x\,(6\,u3^2\,x - 6\,u3\,x^2 + 6\,x^3 - 2\,u3^2 + 2\,x^2 + u3 - x)}{(-1+x)\,(u3^2 - u3\,x + x^2)\,(6\,x-1)}$$

```
> simplify(series(subs(u3=VVV[1],AA1),x,12)) assuming x>0 and x<0.1;
```

$$x + 379\,x^6 + 89\,x^5 + 22\,x^4 + 6\,x^3 + 2\,x^2 + 7474\,x^8 + 1665\,x^7 + 34098\,x^9 + 735708\,x^{11} + 157564\,x^{10} + \mathrm{O}(x^{12})$$

```
> ff:=simplify(subs(v=solve(f=-x*(6*u3^2*x-6*u3*x^2+6*x^3-2*u3^2+2*x
  ^2+u3-x)/(-1+x)/(u3^2-u3*x+x^2)/(6*x-1),u3)[1],v^3-v^2*x+v*x^2-v^2
  +2*v*x-x^2)):
  factor((coeff(ff,(-108*f^2*x^4+252*f^2*x^3-216*f*x^4-183*f^2*x^2+2
  52*f*x^3-108*x^4+42*f^2*x-24*f*x^2-3*f^2-14*f*x+28*x^2+2*f-8*x+1)^
  (1/2),1)*(-108*f^2*x^4+252*f^2*x^3-216*f*x^4-183*f^2*x^2+252*f*x^3
  -108*x^4+42*f^2*x-24*f*x^2-3*f^2-14*f*x+28*x^2+2*f-8*x+1)^(1/2))^2
  -(coeff(ff,(-108*f^2*x^4+252*f^2*x^3-216*f*x^4-183*f^2*x^2+252*f*x
  ^3-108*x^4+42*f^2*x-24*f*x^2-3*f^2-14*f*x+28*x^2+2*f-8*x+1)^(1/2),
  0))^2);
```

$$-x^4\,(6\,x-1)^2\,(6\,f^3\,x^4 - 19\,f^3\,x^3 + 12\,f^2\,x^4 + 21\,f^3\,x^2 - 25\,f^2\,x^3 + 8\,f\,x^4 - 9\,f^3\,x + 14\,f^2\,x^2 - 11\,f\,x^3 + 2\,x^4 + f^3 - f^2\,x + 4\,f\,x^2 - 2\,x^3 - f\,x + x^2) \Big/ (6\,f\,x^2 - 7\,f\,x + 6\,x^2 + f - 2\,x)^3$$

```
> factor(taylor(subs(f=f,6*f^3*x^4-19*f^3*x^3+12*f^2*x^4+21*f^3*x^2-
```

```
25*f^2*x^3+8*f*x^4-9*f^3*x+14*f^2*x^2-11*f*x^3+2*x^4+f^3-f^2*x+4*f
*x^2-2*x^3-f*x+x^2),f,10));
```

$$x^2\,(2\,x^2-2\,x+1)+x\,(-1+x)\,(8\,x^2-3\,x+1)\,f+x\,(12\,x-1)\,(-1+x)^2\,f^2+(6\,x-1)\,(-1+x)^3\,f^3$$

```
> taylor(subs(f=x+379*x^6+89*x^5+22*x^4+6*x^3+2*x^2+7474*x^8+1665*x^
7+34098*x^9+735708*x^11+157564*x^10,
x^2*(2*x^2-2*x+1)+x*(-1+x)*(8*x^2-3*x+1)*f+x*(12*x-1)*(-1+x)^2*f^2
+(6*x-1)*(-1+x)^3*f^3), x,10);
```

$$\mathrm{O}(x^{10})$$

```
> #-----------------------------------------------------------------
----------------
> #1203
> FinalA:=(1-2*x)/2/(1-x)^2/sqrt(1-4*x)-(16*x^8-76*x^7+188*x^6-270*x
^5)/2/(1-x)^5/(1-2*x)^3-(246*x^4-145*x^3+53*x^2-11*x+1)/2/(1-x)^5/
(1-2*x)^3;
```

$$FinalA := \frac{1-2\,x}{2\,(1-x)^2\sqrt{1-4\,x}}-\frac{16\,x^8-76\,x^7+188\,x^6-270\,x^5}{2\,(1-x)^5\,(1-2\,x)^3}-\frac{246\,x^4-145\,x^3+53\,x^2-11\,x+1}{2\,(1-x)^5\,(1-2\,x)^3}$$

```
> taylor(FinalA,x,10);
```

$$x+2\,x^2+6\,x^3+22\,x^4+88\,x^5+358\,x^6+1435\,x^7+5632\,x^8+21745\,x^9+\mathrm{O}(x^{10})$$

```
> with(genfunc):
simplify(rgf_expand(-(16*x^8-76*x^7+188*x^6-270*x^5)/2/(1-x)^5/(1-
2*x)^3-(246*x^4-145*x^3+53*x^2-11*x+1)/2/(1-x)^5/(1-2*x)^3,x,n));
```

$$\frac{n^4}{24}+\frac{2^n\,n^2}{4}-\frac{n^3}{12}-\frac{13\,2^n\,n}{4}-\frac{n^2}{24}+10\,2^n-\frac{53\,n}{12}-\frac{19}{2}$$

```
>
> seq((n+1)/24*(n^3+n^2-2*n-108)+2^(n-1)*(n^2-11*n+28)-19/2+1/2*bino
mial(2*n+2,n+1)-1/2*add((j-1)*binomial(2*n+2-2*j,n+1-j),j=2..n+1),
n=0..20);
```

1, 2, 6, 22, 88, 358, 1435, 5632, 21745, 83188, 317244, 1210801, 4634360, 17801713, 68624309, 265385781, 1029098893, 3999657155, 15574396842, 60742472734, 237227822368

```
> #-----------------------------------------------------------------
-------------------------------------------------------
> #1220
> FinalA:=(22*x^3-29*x^2+10*x-1)/2/x/(1-x)/sqrt(1-4*x)^3+(1-5*x)*(1-
2*x)/2/x/(1-4*x);
```

$$FinalA := \frac{22\,x^3-29\,x^2+10\,x-1}{2\,x\,(1-x)\,(1-4\,x)^{(3/2)}}+\frac{(1-5\,x)\,(1-2\,x)}{2\,x\,(1-4\,x)}$$

```
> taylor(FinalA,x,20);
```

$$x + 2x^2 + 6x^3 + 22x^4 + 89x^5 + 377x^6 + 1628x^7 + 7072x^8 + 30707x^9 + 132891x^{10} + 572567x^{11} + 2455303x^{12} + 10480437x^{13} + 44542229x^{14} + 188552702x^{15} + 795271994x^{16} + 3343239677x^{17} + 14012749109x^{18} + O(x^{19})$$

```
>
> f:=n->(2*n+1)!/n!^2:
  seq((n+4)/(n+2)*binomial(2*n+1,n)+add(f(j),j=0..n-1)-4^(n),n=0..10
  );
```

$$1, 2, 6, 22, 89, 377, 1628, 7072, 30707, 132891, 572567$$

```
> #-----------------------------------------------------------------
  ----------------------------------------
> #1230
> FinalA:=-1/2*x*(2*x^2-2*x+1)/(-1+4*x)/(-1+2*x)/(-1+x)^2*(1-4*x)^(1
  /2)-1/2*x*(16*x^3-26*x^2+16*x-3)/(-1+4*x)/(-1+2*x)/(-1+x)^2;
```

$$FinalA := -\frac{x(2x^2-2x+1)\sqrt{1-4x}}{2(-1+4x)(2x-1)(-1+x)^2} - \frac{x(16x^3-26x^2+16x-3)}{2(-1+4x)(2x-1)(-1+x)^2}$$

```
> taylor(FinalA,x,20);
```

$$x + 2x^2 + 6x^3 + 22x^4 + 89x^5 + 374x^6 + 1583x^7 + 6668x^8 + 27866x^9 + 115570x^{10} + 476266x^{11} + 1952798x^{12} + 7975328x^{13} + 32471190x^{14} + 131883508x^{15} + 534612570x^{16} + 2163742213x^{17} + 8746058250x^{18} + 35314607143x^{19} + O(x^{20})$$

```
> with(genfunc):
  aa1:=rgf_expand((1/2*x*(2*x^2-2*x+1)/(-1+2*x)/(-1+x)^2,x,n));;
  aa2:=rgf_expand(-1/2*x*(16*x^3-26*x^2+16*x-3)/(-1+4*x)/(-1+2*x)/(-
  1+x)^2,x,n);;
```

$$aa1 := -\frac{2^n}{2} + \frac{n}{2}$$

$$aa2 := \frac{2^n}{2} + \frac{4^n}{6} - \frac{n}{2} + \frac{1}{3}$$

```
> seq(1/3*(2*4^n+1)+add((j/2-2^(j-1))*binomial(2*n+2-2*j,n+1-j),j=1.
  .n),n=0..20);
```

$$1, 2, 6, 22, 89, 374, 1583, 6668, 27866, 115570, 476266, 1952798, 7975328, 32471190, 131883508, 534612570, 2163742213, 8746058250, 35314607143, 142464126480, 574280881003$$

```
> #-----------------------------------------------------------------
  ----------------------------------------
>
```