Packing Densities of More 2-block Patterns

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The purpose of this talk is to present some new methods for the computation of packing densities of a specific class of layered permutation patterns. We explore the relationship between 2-block patterns whose block lengths have the same ratios, reinforcing the idea that similar-looking patterns will be optimally packed in similar-looking permutations. The final result is the calculation of the packing densities of an infinite class of patterns.

1 Preliminaries

Let $m \leq n$ be positive integers, and let $\pi \in S_m$ and $\sigma \in S_n$ be permutations. We say an m-subset S of $[n]$ is an **occurrence** of π in σ if the restriction of σ to S is isomorphic to π as a linear order, that is, the elements of $\sigma\lceil_S$ are in the same relative order as those of π. We say σ **avoids** π if it has no occurrences of π.

The problem of characterizing the n-permutations that avoid a certain subpattern is well known and has long been the subject of a large body of research (in fact an entire two chapters of [2] are devoted to issues related to avoidance). In 1992, at a SIAM meeting on Discrete Mathematics, an opposite question of sorts was posed by Herb Wilf: what if, instead of trying to characterize permutations that avoid a certain pattern, we look at permutations that have a maximal number of occurrences of a pattern? The subject of permutation packing was born.

The largest body of work on the subject of permutation packing is the Ph.D. Thesis [5] of Alkes Price, whose notation we will adapt where we can. For $\pi \in S_m$ and $\sigma \in S_n$, let $g(\pi, \sigma)$ be the number of occurrences of π in σ. For each $n \in \mathbb{Z}^+$, let

$$g(\pi, n) := \max_{\sigma \in S_n} g(\pi, \sigma).$$

If $\sigma \in S_n$ and $g(\pi, \sigma) = g(\pi, n)$, we say σ is **π-optimal** over S_n. That $g(\pi, n) \leq \binom{n}{m}$ for every $\pi \in S_m$ is clear, since every occurrence of π in σ is by definition an m-subset of $[n]$. It was conjectured by Wilf that $g(\pi, n)$ is asymptotically proportional to $\binom{n}{m}$, and the following stronger result was later proven by Galvin:

1.1 LEMMA (GALVIN): *The sequence* $\left(\frac{g(\pi,n)}{\binom{n}{m}}\right)_{n \geq m}$ *is nonincreasing in* n.

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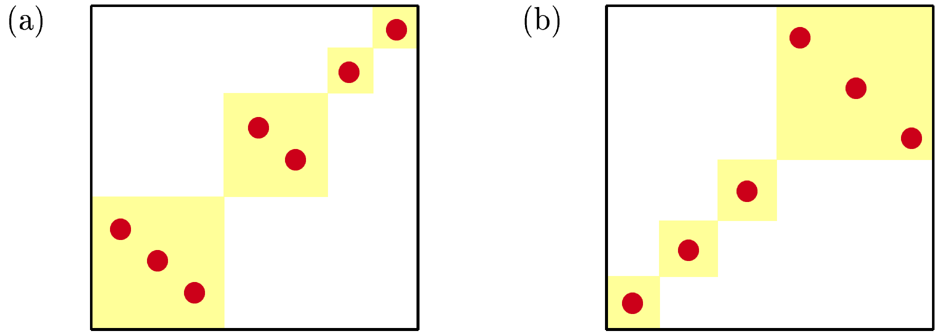

Figure 1: (a) 3215467 is the unique layered permutation of type $(3, 2, 1, 1)$. (b) The structure of $\tau_{\alpha,\beta}$ is shown for $\alpha = \beta = 3$. Layers in both are shaded in.

In particular, a limit exists. We define the **packing density of π** to be

$$\rho(\pi) := \lim_n \frac{g(\pi, n)}{\binom{n}{m}}.$$

In many cases, the packing density of a permutation may be determined by asymptotic means without ever proving any finite structure is optimal. For example, a technique pioneered by Price and used by Peter Hästö in [3] is to characterize the behavior of large permutations with a fixed, small number of layers using partitions of unity. In this technique, the packing density of some patterns may be computed by maximizing the value of a polynomial in several variables instead of actually counting occurrences in specific permutations. One particularly direct way $\rho(\pi)$ is computed is to find an infinite family (σ_k) of permutations $\sigma_k \in S_{n_k}$ that are π-optimal and then compute $\lim_n \frac{g(\pi,\sigma_k)}{\binom{n_k}{m}}$ by ordinary means. In this case the object of the packing density problem extends to the problem of characterizing the structure of their optimal permutations.

The structure most often imposed on patterns in order to determine their packing density is that of layering. A pattern π is called **layered** if it can be decomposed into an array of descending sequences of consecutive integers, which are then ordered increasingly by first elements. Alternately, layered permutations are those which have no descents of size greater than 1. Notice that a layered permutation is uniquely determined by its ordered list of layer lengths (its *type*); the layered pattern of type $(3, 2, 1, 1)$ is shown in Figure 1(a). An early result due to Stromquist, proven first in [7] for posets and then reproduced in [5] for permutations, gives us the distinct advantage that when π is layered, we are able to restrict our search for a π-optimal permutation to the (much smaller) class of layered permutations on n:

1.2 THEOREM (STROMQUIST): *Let $\pi \in S_m$ be layered. Then,*

$$\begin{aligned} g(\pi, n) &= \max\{g(\pi, \sigma) :: \sigma \in S_n\} \\ &= \max\{g(\pi, \sigma) :: \sigma \in S_n \text{ is layered }\}. \end{aligned}$$

In general the lack of this restriction is a large part of what makes the problem of finding packing densities of unlayered patterns so difficult. The problem we next encounter when computing packing densities of layered patterns is that even in the case that π is layered, the number of layers in a π-optimal permutation need not be bounded; the issue of when this problem occurs was dealt with extensively in [3]. Such is the case with the particular class of patterns with which this paper deals. We will use the same notation as [8] to simplify the text of the proofs:

NOTATION: Let $\tau_{\alpha,\beta}$ denote the layered pattern of type $(1^\alpha, \beta)$. $\tau_{3,3}$ is shown in Figure 1(b).

In [8], an **antilayer**[1] was defined to be a sequence of consecutive layers of size 1, and this concept was applied to deal with the apparent phenomenon that patterns with long sequences of layers of size 1 tend to have optimal permutations with long sequences of layers of size 1, nicely complementing the fact that patterns with large layers tend to have optimal permutations with large layers. In our context, a 2-block pattern is just one of the form $\tau_{\alpha,\beta}$ for $\alpha, \beta \geq 2$. α and β are called the block sizes.

2 A more careful look at $\tau_{\alpha,\beta}$

In [1], the structure of a $\tau_{2,2}$-optimal permutation of size n was explicitly characterized, and in [8], it was proven by an inductive technique that the $\tau_{\alpha,\alpha}$-optimal permutation of size $2n$ always had exactly the same structure, independent of α, namely a single antilayer followed by a single layer of the same length. In general, if it can be proven that there is a $\tau_{\alpha,\beta}$-optimal permutation of size n having a 2-block structure (a single antilayer followed by a single layer), then the maximum number of occurrences of $\tau_{\alpha,\beta}$ in a permutation of length n is

1: $$\max_{0\leq k\leq n} \binom{k}{\alpha}\binom{n-k}{\beta}.$$

In [8], the existence of a 2-block $\tau_{\alpha,\beta}$-optimal permutation on $[n]$ was proven in the case that $\alpha = 2$ and $\beta \geq 3$, for n divisible by $\beta + 2$. Although we were then able to compute the packing density of $\tau_{2,\beta}$ via asymptotic methods, we would like a stronger result characterizing the optimal layout of permutations of all lengths, that is, we would like to know the value of k achieving (1). We make use of the following result.

2.1 PROPOSITION: *Let $\alpha, \beta \in \mathbb{Z}^+$. The function*

2: $$B(k) := \binom{k}{\alpha}\binom{n-k}{\beta}$$

($k \in \mathbb{R}$) is unimodal in k and has its maximum on the real line at $k = \frac{\alpha n-\beta}{\alpha+\beta}$.

2.2 COROLLARY: *Let $n \in \mathbb{N}$ and suppose that we know there is a $\tau_{\alpha,\beta}$-optimal permutation of size n consisting of a single antilayer followed by a single layer. Then,*

$$g(\tau_{\alpha,\beta}, n) \leq B\left(\frac{\alpha n-\beta}{\alpha+\beta}\right) = \binom{\frac{\alpha n-\beta}{\alpha+\beta}}{\alpha}\binom{\frac{\beta n+\beta}{\alpha+\beta}}{\beta}.$$

If $(\alpha+\beta) \,\big|\, n$, then

$$g(\tau_{\alpha,\beta}, n) = B\left(\frac{\alpha n}{\alpha+\beta}\right) = \binom{\frac{\alpha n}{\alpha+\beta}}{\alpha}\binom{\frac{\beta n}{\alpha+\beta}}{\beta}.$$

[1]To justify the terminology, layers and antilayers correspond to antichains and chains, respectively, when translated into the language of posets.

3 Application to Larger Patterns

After proving a short technical lemma, we move on to our first main result relating two 2-block patterns whose block lengths have the same ratio.

3.2 THEOREM: *Let $\alpha, \beta, n \in \mathbb{N}$, and suppose $(\alpha+\beta) \,\big|\, n$. Suppose that there is known to be a $\tau_{\alpha,\beta}$-optimal permutation of size n consisting of a single antilayer followed by a single layer. Then, for every $k \in \mathbb{Z}^+$, we have*

$$g(\tau_{k\alpha,k\beta}, n) = \binom{\frac{\alpha n}{\alpha+\beta}}{k\alpha}\binom{\frac{\beta n}{\alpha+\beta}}{k\beta}.$$

Sketch of Proof. Since $(\alpha+\beta) \,\big|\, n$, that $g(\tau_{k\alpha,k\beta}, n) \geq \binom{\frac{\alpha n}{\alpha+\beta}}{k\alpha}\binom{\frac{\beta n}{\alpha+\beta}}{k\beta}$ follows from constructing a permutation on $[n]$ consisting of a single antilayer of size $\frac{\alpha n}{\alpha+\beta}$ and a single layer of size $\frac{\beta n}{\alpha+\beta}$. The reverse inequality is proven by induction on k. By Corollary 2.2, our base case $k = 1$ is covered, that is, we know

$$g(\tau_{\alpha,\beta}, n) = \binom{\frac{\alpha n}{\alpha+\beta}}{\alpha}\binom{\frac{\beta n}{\alpha+\beta}}{\beta}.$$

From here, we assume that the theorem holds for $1, \ldots, k-1$ and suppose that

$$g(\tau_{k\alpha,k\beta}, n) \geq \binom{\frac{\alpha n}{\alpha+\beta}}{k\alpha}\binom{\frac{\beta n}{\alpha+\beta}}{k\beta} + \delta$$

for some $\delta > 0$. Letting $\sigma \in S_n$ be a $\tau_{k\alpha,k\beta}$-optimal permutation in S_n, we then count the number of occurrences of $\tau_{\alpha,\beta}$ in σ and show that there are too many, thus providing a contradiction to Corollary 2.2. ♦

Notice that Theorm 3.2 also characterizes the structure of a $\tau_{k\alpha,k\beta}$-optimal permutation in the cases where it applies.

3.3 COROLLARY: *Suppose $(\alpha+\beta) \,\big|\, n$. If there is a $\tau_{\alpha,\beta}$-permutation $\sigma \in S_n$ consisting of a single antilayer followed by a single layer, then σ is in fact $\tau_{k\alpha,k\beta}$-optimal for every $k \in \mathbb{N}$.*

If we drop the divisibility condition on n, we can still get an upper bound on $g(\tau_{\alpha,\beta}, n)$ by similar means.

3.4 THEOREM: *Let $\alpha, \beta, n \in \mathbb{N}$, and let x be a real number s.t. $x\alpha, x\beta \in \mathbb{N}$. Suppose there is known to be a $\tau_{\alpha,\beta}$-optimal permutation of size n consisting of a single antilayer followed by a single layer and we know*

$$g(\tau_{x\alpha,x\beta}, n) \leq \binom{\frac{\alpha n-\beta}{\alpha+\beta}}{x\alpha}\binom{\frac{\beta n+\beta}{\alpha+\beta}}{x\beta}.$$

Then for every $r \in \mathbb{N}$ we have

$$g(\tau_{(x+r)\alpha,(x+r)\beta}, n) \leq \binom{\frac{\alpha n-\beta}{\alpha+\beta}}{(x+r)\alpha}\binom{\frac{\beta n+\beta}{\alpha+\beta}}{(x+r)\beta}.$$

Proof. Similar in technique to the proof of upper bound in Theorem 3.2. ♦

4 Concrete results

In the case that $\alpha = 2$, the necessary base cases were proven in [8], so we may now use the results of the previous sections to compute the packing densities of a specific class of patterns. In particular, Corollary 2.2 gives us exactly the characterization we wanted of an infinite family of $\tau_{2,\beta}$-optimal permutations.

4.1 Corollary: *Suppose $\beta \geq 2$ and $(\beta+2) \,\big|\, n$. Then, there is a $\tau_{2,\beta}$-optimal permutation $\sigma_{2,\beta}(n)$ on $[n]$ consisting of a single antilayer of length $\frac{2n}{2+\beta}$, followed by a layer of length $\frac{\beta n}{2+\beta}$. It follows that*

$$g_n(\tau_{2,\beta}) = g(\tau_{2,\beta}, \sigma_{2,\beta}) = \binom{\frac{2n}{2+\beta}}{2}\binom{\frac{\beta n}{2+\beta}}{\beta}.$$

4.2 Theorem: *Suppose $\beta \geq \alpha \geq 2$ and $\alpha \,\big|\, 2\beta$. Then, for every n divisible by $(2 + \frac{2\beta}{\alpha})$, we have*

$$g(\tau_{\alpha,\beta}, n) \leq \binom{\frac{\alpha n - \beta}{\alpha+\beta}}{\alpha}\binom{\frac{\beta n+\beta}{\alpha+\beta}}{\beta}.$$

Sketch of Proof. By Theorem 3.4 (applied with $x = 1$ and $x = \frac{3}{2}$), we need only prove that

$$g(\tau_{2,\beta}, n) \leq \binom{\frac{2n-\beta}{2+\beta}}{2}\binom{\frac{\beta n+\beta}{2+\beta}}{\beta} \quad \text{and} \quad g(\tau_{3,\beta}, n) \leq \binom{\frac{3n-\beta}{3+\beta}}{3}\binom{\frac{\beta n+\beta}{3+\beta}}{\beta}.$$

We know that $g(\tau_{2,\beta}, n) = \binom{\frac{2n}{2+\beta}}{2}\binom{\frac{\beta n}{2+\beta}}{\beta}$ from Corollary 4.1; to prove the other inequality, we prove that $g(\tau_{3,\beta}, n) \leq \binom{\frac{3n}{3+\beta}}{3}\binom{\frac{\beta n}{3+\beta}}{\beta}$. ♦

The main result of this paper, the computation of the packing density, now follows as a corollary.

4.3 Corollary: *Suppose $\beta \geq \alpha \geq 2$ and $\alpha \,\big|\, 2\beta$. Then, the packing density of the pattern $\tau_{\alpha,\beta}$ is*

$$\rho(\tau_{\alpha,\beta}) = \binom{\alpha+\beta}{\alpha}\left(\frac{\alpha}{\alpha+\beta}\right)^{\alpha}\left(\frac{\beta}{\alpha+\beta}\right)^{\beta}. \qquad (3)$$

Sketch of Proof. By letting $\sigma_{\alpha,\beta}(n)$ be the permutation on $[n]$ consisting of an antilayer of length $\left\lfloor \frac{\alpha n-\beta}{\alpha+\beta} \right\rfloor$ followed by a layer of length $\left\lceil \frac{\beta n+\beta}{\alpha+\beta} \right\rceil$, we ensure that $g(\tau_{\alpha,\beta}, n) \geq \binom{\lfloor \frac{\alpha n-\beta}{\alpha+\beta} \rfloor}{\alpha}\binom{\lceil \frac{\beta n+\beta}{\alpha+\beta} \rceil}{\beta}$. The result (3) then follows from Theorem 4.2 and the squeeze theorem. ♦

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