The Möbius function of permutations ordered by pattern containment

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We consider the poset Σ of all permutations under the partial order $p \leq q$ if p is a pattern in q. We are able to give a formula for the Möbius function of the subposet of layered permutations. This result is related to a theorem of Björner [2, 3] about subword order.

To give more details, let $\mathbb{N}$ and $\mathbb{P}$ denote the nonnegative and positive integers, respectively. If $n \in \mathbb{N}$ then we will use the notation $[n] = \{1, 2, \ldots, n\}$. Given any $S \subseteq \mathbb{P}$ of cardinality $\#S = n$ then there is a corresponding *standardization map* which is the unique order-preserving bijection $\mathrm{st} : S \to [n]$. We also use $|S|$ for cardinality and apply st element-wise to any object with label set S.

Let Σ_n denote the symmetric group consisting of all permutations $p = a_1 a_2 \ldots a_n$ of $[n]$. If $p = a_1 \ldots a_k \in \Sigma_k$ and $q = b_1 \ldots b_n \in \Sigma_n$ then we say that q *contains* p *as a pattern* and write $q \geq p$ if there is a subword $q' = b_{i_1} \ldots b_{i_k}$ of q with $\mathrm{st}(q') = p$. Otherwise we say q *avoids* p.

Let Σ be the poset (partially ordered set) obtained by ordering $\cup_{n\geq 0}\Sigma_n$ by pattern containment. Clearly this is a ranked poset with the nth rank consisting of Σ_n. Previous research about Σ has concentrated on determining when certain subposets defined by pattern avoidance are partially well-ordered, see for example [1]. Here, we are going to study the *Möbius function* μ of Σ. This is the function defined recursively on intervals $[p, q]$ of Σ by

$$\sum_{r\in[p,q]} \mu(r, q) = \delta_{p,q} \tag{1}$$

where $\delta_{p,q}$ is the Kronecker delta. The Möbius function is a fundamental invariant of any locally finite (where every interval finite) poset P because it encodes enumerative and topological information about P [5].

As is often the case when dealing with permutation patterns, it is easier to first considered layered permutations. A permutation p is *layered* if it has a factorization (as a word) such that elements in each factor are in decreasing order and elements taken from different factors are in increasing order in p. The factors are called the *layers* of p. For example, $p = 3\ 2\ 1\ 5\ 4\ 9\ 8\ 7\ 6$ is layered with layers $3\ 2\ 1$, $5\ 4$, and $9\ 8\ 7\ 6$. If $p \in \Sigma_n$ is layered then it has a corresponding composition (ordered partition) of n, namely $v = (v(1), v(2), \ldots, v(s))$ where the $v(i)$ are the layer lengths. In the example, the composition is $v = (3, 2, 4)$.

Consider the subposet Λ of Σ consisting of all layered permutations. One can reinterpret the partial order in terms of compositions as follows. If $v = (v(1), \ldots, v(s))$ and $w = (w(1), \ldots, w(t))$ then $v \leq w$ in Λ if and only if there is a sequence of indices $\{i_1 < \ldots < i_s\}$ with $v(j) \leq w(i_j)$ for $1 \leq j \leq s$.

From this viewpoint, the partial order on Λ is reminiscent of subword order, and indeed there is a connection. The Möbius function of subword order has been determined by Björner [2, 3] as follows. Let A^* denote the set of all finite words w (repetitions allowed) over an alphabet A. An *embedding* of a word $v = a_1 \ldots a_k$ into a word $w = b_1 \ldots b_n$ is a sequence of indices $I = \{i_1 < \ldots < i_k\}$ such that $a_j = b_{i_j}$ for $1 \leq j \leq k$. For example, if $v = a\ b\ b\ a$ and $w = a\ a\ b\ a\ b\ b\ a$ then $\{1, 3, 5, 7\}$ and $\{2, 5, 6, 7\}$ are embeddings of v into w. *Subword order* is the partial order on A^* defined by $v \leq w$ if and only if there is an embedding of v into w. This makes A^* into a ranked poset where the rank of a word w is its length $\ell(w)$.

To describe the Möbius function of A^*, we need to consider a special type of embedding. A word $w = a_1 \ldots a_n \in A^*$ has *repetition set*

$$R(w) = \{i \mid a_i = a_{i+1}\}. \tag{2}$$

Call an embedding I of v into w *normal* if $I \supseteq R(w)$. Of the two embeddings in the previous paragraph, only the first is normal since it contains $R(w) = \{1, 5\}$. Let $\binom{w}{v}_n$ denote the number of normal embeddings of v in w.

Theorem 1 (Björner) *The Möbius function of A^* is given by*

$$\mu(v, w) = (-1)^{\ell(v,w)} \binom{w}{v}_n \tag{3}$$

where $\ell(v, w) = \ell(w) - \ell(v)$. ∎

The special case of this theorem when $v = \emptyset$ follows from the work of Farmer [4] and was also discovered by Viennot [6].

We have been able to determine the Möbius function of Λ. To write down the formula, a few more definitions are needed. A sequence $\epsilon = (\epsilon(1), \epsilon(2), \ldots, \epsilon(n)) \in \mathbb{N}^*$ has *support*

$$S(\epsilon) = \{i \mid \epsilon(i) > 0\}.$$

Consider a composition $v = (v(1), \ldots, v(k)) \in \mathbb{P}^*$ An *expansion of* v is a sequence $\epsilon_v = (\epsilon_v(1), \ldots, \epsilon_v(n))$ such that if we restrict ϵ_v to its support, we recover v. For example, some expansions of $v = (1, 2, 1, 1, 2)$ are $\epsilon_v = (1, 0, 2, 1, 1, 0, 0, 2)$, $\epsilon_v = (1, 0, 2, 1, 0, 0, 1, 2)$, and $\epsilon_v = (0, 0, 1, 2, 1, 1, 2, 0)$. An *embedding* of v into $w = (w(1), \ldots, w(n))$ is an expansion $\epsilon_{vw} = (\epsilon_{vw}(1), \ldots, \epsilon_{vw}(n))$ of v such that

$$\epsilon_{vw}(i) \leq w(i) \text{ for all } 1 \leq i \leq n. \tag{4}$$

If $w = (1, 1, 2, 1, 1, 1, 3, 3)$ then the first two expansions of v above are embeddings, while the third is not since the inequality in the definition is violated for $i = 4$. Note that $v \leq w$ in Λ precisely when there is an embedding of the composition v into the composition w.

To define normal embeddings in this context, we will need the concept of a run. If $w = (w(1), \ldots, w(n)) \in \mathbb{P}^*$ then a *run of k's* is a maximal interval of indices $[r, t]$ such that

$$w(r) = w(r+1) = \ldots = w(t) = k.$$

Now define an embedding η_{vw} of v into w to be *normal* if it satisfies the following conditions.

1. For $1 \leq i \leq \ell(w)$ we have

$$\eta_{vw}(i) = w(i), w(i) - 1, \text{ or } 0.$$

2. For all $k \geq 1$ and every run $[r, t]$ of k's in w we have $\eta_{vw}(i) \neq 0$ for
   (a) $r \leq i < t$ if $k = 1$, or
   (b) $i = t$ if $k \geq 2$.

Of the two embeddings in the example of the previous paragraph, the first one is normal but the second one is not, both because $\epsilon_v(5)$ violates condition 2(a) and because $\epsilon_v(7)$ violates condition 1. Also note that conditions 1 and 2(a) together imply that the ones in v and w satisfy Björner's repetition set criterion.

Define the *defect* of a normal embedding η_{vw} to be

$$d(\eta_{vw}) = \#\{i \mid \eta_{vw}(i) = w(i) - 1\},$$

and the *sign* of the embedding to be

$$(-1)^{\eta_{vw}} = (-1)^{d(\eta_{vw})}.$$

Theorem 2 (Sagan-Vatter) *The Möbius function of* Λ *is given by*

$$\mu(v, w) = \sum_{\eta_{vw}} (-1)^{\eta_{vw}}$$

where the sum is over all normal embeddings η_{vw} *of* v *into* w. ■

This theorem is proved by means of a sign-reversing involution. There is also a common generalization of Theorem 1 and Theorem 2 which will be discussed in the talk.

References

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