Enumerative and structural applications of profile classes

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The three flavors of profile classes

The best-studied example of a profile class of permutations is the set of skew-merged permutations. A permutation is said to be skew-merged if it is the union of an increasing subsequence with a decreasing subsequence. For example, the permutation 28364571, shown in Figure 1, is a skew-merged permutation. Stankova [16] was the first to find the basis of this set.

Theorem 1. *[2, 10, 16] The skew-merged permutations are precisely the permutations that avoid both* 2143 *and* 3412*.*

Later Kédzy, Snevily, and Wang [10] showed using permutation graphs that this result follows easily from Földes and Hammer's characterization of split graphs. Atkinson [2] gave another proof of Theorem 1 and showed that the generating function for the set of skew-merged permutations is

$$\frac{1-3x}{(1-2x)\sqrt{1-4x}}.$$

In general, the profile class of a $0/\pm 1$ matrix M contains all permutations that can be divided in a prescribed manner (dictated by M) into a finite number of monotonic blocks. For example, skew-merged permutations can be divided into four monotonic blocks, two increasing and two decreasing.

Suppose that $p \in S_n$ and $N = N_1 \times N_2$, where $N_1, N_2 \subseteq [n]$. Letting $\{i : (i, p(i)) \in N\} = \{a_1 < a_2 < \cdots < a_k\}$, we denote the sequence $p(a_1), p(a_2), \ldots, p(a_k)$ by p_N. Thus p_N corresponds to the subpermutation of p obtained by restricting the plot of p on $[n] \times [n]$ to the rectangular subregion $N_1 \times N_2$.

Now suppose that M is a $r \times s$ $0/\pm 1$ matrix. We say that the pair of sequences $\{1 = i_1 < i_2 < \ldots < i_{r+1} = n+1\}$ and $\{1 = j_1 < j_2 < \ldots < j_{s+1} = n+1\}$ *form an M-partition* for the n-permutation p (and that p *admits an M-partition*) if, for all $k \in [r]$ and $\ell \in [s]$, $p_{[i_k, i_{k+1}) \times [j_\ell, j_{\ell+1})}$ is:

- increasing if $M_{k,\ell} = 1$,
- decreasing if $M_{k,\ell} = -1$,
- empty if $M_{k,\ell} = 0$.

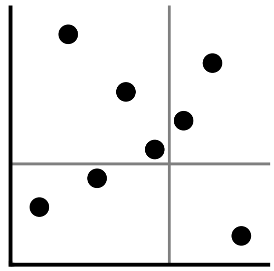

Figure 1: The plot of $p = 28364571$, showing one of the possible ways to divide it into four monotonic blocks.

Finally, we define the profile class of M, denoted by $\text{Prof}(M)$, to be the set of all finite permutations that possess an M-partition. In this terminology, the skew-merged permutations can be expressed as

$$\text{Prof}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Three different types of profile classes have proved useful in previous research. Atkinson [1] used profile classes of permutation matrices for enumeration. For example, he showed that $\mathcal{A}(132, 4321)$ is the union of the profile classes of the permutation matrices of 32415 and 42135, from which it can be easily computed that the number of $\{132, 4321\}$-avoiding n-permutations is $1 + \binom{n+1}{3} + 2\binom{n}{4}$.

Profile classes of $0/\pm1$ column vectors were applied to partial well-order (PWO)[1] questions by Atkinson, Murphy, Ruškuc [6] (who called them generalised W classes). They proved that these profile classes are always PWO.

Profile classes of $0/\pm1$ vectors are also interesting from an enumerative point of view. Albert, Atkinson, and Ruškuc [3] showed that every closed subset[2] of a profile class of a $0/\pm1$ vector is encoded by a regular language, and thus has a rational generating function. One particularly nice application of this fact is to permutations with a bounded number of descents. For example, the set of 1324-avoiding permutations with at most 3 descents is the set of 1324-avoiding permutations in the profile class of $\begin{pmatrix} 1 & -1 & 1 & -1 & 1 & -1 \end{pmatrix}^T$. Counting restricted permutations by their number of descents is examined in greater detail by Elder, Rechnitzer, and Zabrocki [8].

Murphy and Vatter [13] introduced profile classes in the greater generality they are presented here and settled the PWO problem for them. Given an $r \times s$ $0/\pm1$ matrix M,

[1] A partially ordered set (such as the set of permutations with the pattern-containment ordering) is partially well-ordered (PWO) if it contains neither an infinite properly decreasing sequence nor an infinite antichain (a set of pairwise incomparable elements). There is clearly no infinite strictly decreasing sequence of (finite) permutations, so a set of permutations is PWO if and only if it does not contain an infinite antichain. The set of permutations is not PWO, but many subsets of it are, for example the 132-avoiding permutations.

[2] A set X of permutations is said to be closed (synonymously: a down-set or an ideal) if whenever $p \in X$ contains a q-pattern, then q also lies in X.

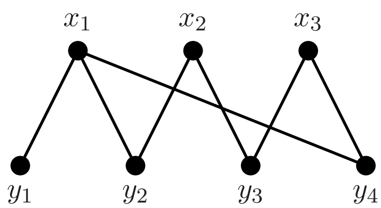

Figure 2: The bipartite graph corresponding to $\begin{pmatrix} -1 & 1 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$.

they define a bipartite graph $G(M)$ with vertices $\{x_1, \dots, x_r\} \cup \{y_1, \dots, y_s\}$ and edges $\{(x_i, y_j) : |M_{i,j}| = 1\}$. (Note that $G(M)$ ignores the sign of nonzero entries of M.) Figure 2 shows an example.

Theorem 2. *[13] Let M be a finite $0/\pm 1$ matrix, and let $G(M)$ be the bipartite graph of M. Then $(\mathrm{Prof}(M), \leq)$ is partially well-ordered if and only if $G(M)$ is a forest.*

Little is known about the enumeration of profile classes in the general case. The example of skew-merged permutations shows that profile classes can have non-rational generating functions. Moreover, as some profile classes admit infinite antichains, there are closed subsets of profile classes that do not have P-recursive generating functions,[3] so the best that could be hoped for would be a result about finitely-based subsets of profile classes.

The three types of profile classes are recounted in the following table.

Profile class of	enumeration	PWO	reference
permutation matrix	binomial coefficents	always	Atkinson [1]
$0/\pm 1$ vector (a.k.a. W class)	rational g.f.	always	Atkinson, Murphy, and Ruškuc [6], Albert, Atkinson, and Ruškuc [3]
$0/\pm 1$ matrix	?	when $G(M)$ is a forest	Murphy and Vatter [13]

Characterizing profile classes

[3]This follows from the following counting argument. Suppose that the closed set X contains an infinite antichain. Then X contains an infinite antichain A that has at most one element of each length. If $A_1 \neq A_2$ are two subsets of A then the subsets of A_1-avoiding permutations in X and A_2-avoiding permutations in X have different enumerations. Because A is infinite, this gives $2^{\aleph_0}$ different generating functions, and thus they can not all be P-recursive.

Just as important as knowing the properties of profile classes is knowing when they apply. Theorem 3, below, offers a characterization of profile classes. Before stating it, we must introduce direct sums and skew sums of permutations. If $p \in S_m$ and $p' \in S_n$, we define the direct sum of p and p', $p \oplus p'$, to be the $(m+n)$-permutation given by

$$(p \oplus p')(i) = \begin{cases} p(i) & \text{if } 1 \leq i \leq m, \\ p'(i-m)+m & \text{if } m+1 \leq i \leq m+n. \end{cases}$$

The skew sum of p and p', $p \ominus p'$, is defined by

$$(p \ominus p')(i) = \begin{cases} p(i)+n & \text{if } 1 \leq i \leq m, \\ p'(i-m) & \text{if } m+1 \leq i \leq m+n. \end{cases}$$

Theorem 3. *A closed set of permutations lies in the profile class of some finite* $0/\pm 1$ *matrix if and only if it does not contain arbitrarily long direct sums of* 21 *or arbitrarily long skew sums of* 12*.*

The harder direction of this theorem is proving that $\{\oplus^a 21, \ominus^b 12\}$-avoiding closed sets of permutations lie in profile classes. This is proved by induction. Note that the cases where $a = 1$ or $b = 1$ are trivial. The first nontrivial case is then $a = b = 2$. By Theorem 1, the $\{\oplus^2 21, \ominus^2 12\}$-avoiding permutations are skew-merged, and thus lie in the profile class of the 2×2 matrix given before.

Partial well-order for 321-avoiding sets

The set of 321-avoiding permutations is known to contain an infinite antichain, for instance, a symmetry of the antichain constructed by Spielman and Bóna [15]. Given this, one might ask what addition restrictions one can put on the set to make it PWO. Precisely, for which permutations p is the set of $\{321, p\}$-avoiding permutations PWO?

An answer to this question was first presented by the second author at the first Conference on Permutation Patterns, although the method of proof was needlessly technical. Theorem 3 leads to a much cleaner proof. Before stating the answer to this question we need to introduce the increasing oscillating sequence,

$$\ldots, 0, -3, 2, -1, 4, 1, 6, 3, 8, 5, \ldots,$$

which consists of two interleaved increasing sequences.

Theorem 4. *The set of* $\{321, p\}$*-avoiding permutations is PWO if and only if* p *embeds into the increasing oscillating sequence.*

The fact that there is a $\{321, p\}$-avoiding antichain whenever p does not embed into the increasing oscillating sequence requires two facts: first, that p embeds into the increasing oscillating sequence if and only if it is $\{321, 2341, 3412, 4123\}$-avoiding;[4] and second, that there is a $\{321, 2341, 3412, 4123\}$-avoiding antichain.[5]

The proof of the other direction depends on the characterization of profile classes offered by Theorem 3.

Permutations that contain 132 a prescribed number of times

Bóna [7], motivated by the Gessel-Noonan-Zeilberger Conjecture [9, 14], proved that the set of permutations with at most r copies of 132 has a P-recursive enumeration. Mansour and Vainshtein [11] later showed that these sets have algebraic generating functions. Albert and Atkinson [5] used at different techinique (the machinery of simple permutations) to establish that all closed 132-avoiding sets of permutations have algebraic generating functions. Utilizing the dichotomy present by Theorem 3, it is possible to give a common generalization of these results.

Theorem 5. *For any positive integer r, every closed subset of the set of permutations that contain 132 at most r times contains at most finitely many simple permutations, and thus, by the results of Albert and Atkinson [5], has an algebraic generating function.*

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[4] This is mentioned in Murphy's thesis [12], and a structural proof appears in Albert, Atkinson, and Ruškuc [4]. It can also be proved semi-automatically in the following manner. Let B denote the set of all permutations that embed into the increasing oscillating sequence. It is not difficult to check that B is a $\{321, 2341, 3412, 4123\}$-avoiding set of permutations, and that the generating function for B is $x(1 + x^2)/(1 - 2x - x^3)$. Using the FINLABEL package described in Vatter [17], it can be computed that this is also the generating function for all $\{321, 2341, 3412, 4123\}$-avoiding permutations, so the two sets must be identical.

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