The Number of Occurrences of Patterns in a Random Permutation

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Abstract

Let $k = 3$ and consider the patterns 123 or 132. Or, let k be arbitrary and let the pattern be $123\ldots k$ or $k(k-1)\ldots 321$. Let X denote the number of occurrences of the stated pattern in a random permutation on $[n]$. Then we prove results along the following general lines:

(i) $E(X)$ and $Var(X)$ are computed exactly for small k and the leading term is extracted for large k (of course $E(X)$ is trivial).

(ii) It is proved that the distribution of X approaches a standard normal at rate $1/\sqrt{n}$ if k is fixed and n goes to infinity.

(iii) The central limit theorem "gets into trouble" as k gets larger roughly than $\log n$. After that the situation gets quite volatile, culminating of course in the delicate behavior of the longest monotone subsequence and the Tracy-Widom distribution when k is roughly $2\sqrt{n}$.

Patterns with repetition are studied as well, with specific results being presented for 112 patterns.