ASYMPTOTIC ENUMERATION OF PERMUTATIONS AVOIDING GENERALIZED PATTERNS

SERGI ELIZALDE

Abstract. Motivated by the recent proof of the Stanley-Wilf conjecture, we study the asymptotic behavior of the number of permutations avoiding a generalized pattern. Generalized patterns allow the requirement that some pairs of letters must be adjacent in an occurrence of the pattern in the permutation, and consecutive patterns are a particular case of them.

We determine the asymptotic behavior of the number of permutations avoiding a consecutive pattern, showing that they are an exponentially small proportion of the total number of permutations. For some other generalized patterns we give partial results, showing that the number of permutations avoiding them grows faster than for classical patterns but more slowly than for consecutive patterns.

1. Introduction

One of the most important breakthroughs in the subject of pattern-avoiding permutations has been the proof by Marcus and Tardos [17] of the so-called Stanley-Wilf conjecture, which had been open for over a decade. This is a basic result regarding the asymptotic behavior of the number of permutations that avoid a given pattern. It states that for any pattern σ there exists a constant λ such that, if $\alpha_n(\sigma)$ denotes the number of σ-avoiding permutations of size n, then $\alpha_n(\sigma) < \lambda^n$. The notion of pattern avoidance that this result is concerned with is the standard one, namely, where a permutation is said to avoid a pattern if it does not contain any subsequence which is order-isomorphic to it.

In [3], Babson and Steingrímsson introduced the notion of *generalized patterns*, which allows the requirement that certain pairs of letters of the pattern must be adjacent in any occurrence of it in the permutation. One particular case of these are *consecutive patterns*, which were independently studied by Elizalde and Noy [11]. For a subsequence of a permutation to be an occurrence of a consecutive pattern, its elements have to appear in adjacent positions of the permutation.

Analogously to the case of classical patterns, it is a natural to study the asymptotic behavior of the number of permutations avoiding a generalized pattern. This problem is far from being understood. It follows from our work that for most generalized patterns the number of permutations avoiding them behaves very differently than in the case of classical patterns. In this paper we determine the asymptotic behavior for the case of consecutive patterns, showing that if σ is a consecutive pattern and $\alpha_n(\sigma)$ denotes the number of permutations of size n avoiding it, then $\lim_{n\to\infty} \sqrt[n]{\alpha_n(\sigma)/n!}$ is a positive constant. For some particular generalized patterns we obtain the same asymptotic behavior, and for patterns of length 3 the problem is solved as well. However, the general case remains open, and it seems from our investigation that there is a big range of possible asymptotic behaviors. For some generalized patterns σ of length 4 we give asymptotic upper and lower bounds on $\alpha_n(\sigma)$.

The paper is structured as follows. In Section 2 we introduce the definitions and notation for generalized pattern avoidance, and we mention previous results regarding consecutive patterns. In Section 3 we give the exponential generating functions for permutations avoiding a special kind of generalized patterns, extending the results from [11]. In Section 4 we study the asymptotic behavior as n goes to infinity of the number of permutations of size n avoiding a generalized pattern, solving the problem only in some cases. In Section 5 we give lower and upper bounds on the number of 12-34-avoiding permutations, and in Section 6 we obtain a similar result for the pattern 1-23-4. Finally, in Section 7 we discuss some open problems and further research.

2. Preliminaries

In this section we define most of the notation that will be used later on. We start introducing the notion of generalized pattern avoidance.

2.1. **Generalized patterns.** These patterns were introduced by Babson and Steingrímsson [3] to extend the classical notion of pattern avoidance. We will denote by $\mathcal{S}_n$ the symmetric group on $\{1, 2, \dots, n\}$. Let n, m be two positive integers with $m \le n$, and let $\pi = \pi_1\pi_2 \cdots \pi_n \in \mathcal{S}_n$ be a permutation. A generalized pattern σ is obtained from a permutation $\sigma_1\sigma_2 \cdots \sigma_m \in \mathcal{S}_m$ by choosing, for each $j = 1, \dots, m-1$, either to insert a dash - between σ_j and σ_{j+1} or not. More formally, $\sigma = \sigma_1\varepsilon_1\sigma_2\varepsilon_2 \cdots \varepsilon_{m-1}\sigma_m$, where each ε_j is either the symbol - or the empty string.

With this notation, we say that π *contains* (the generalized pattern) σ if there exist indices $i_1 < i_2 < \dots < i_m$ such that

(i) for each $j = 1, \dots, m-1$, if ε_j is empty then $i_{j+1} = i_j + 1$, and
(ii) $\rho(\pi_{i_1}\pi_{i_2} \cdots \pi_{i_m}) = \sigma_1\sigma_2 \cdots \sigma_m$, where ρ is the reduction consisting in relabeling the elements with $\{1, \dots, m\}$ so that they keep the same order relationships they had in π. (Equivalently, this means that for all indices a and b, $\pi_{i_a} < \pi_{i_b}$ if and only if $\sigma_a < \sigma_b$.)

In this case, $\pi_{i_1}\pi_{i_2} \cdots \pi_{i_m}$ is called an *occurrence* of σ in π.

If π does not contain σ, we say that π *avoids* σ, or that it is σ*-avoiding*. For example, the permutation $\pi = 3542716$ contains the pattern 12-4-3, and it has exactly one occurrence of it, namely the subsequence 3576. On the other hand, π avoids the pattern 12-43.

Observe that in the case where σ has dashes in all $m-1$ positions, we recover the classical definition of pattern avoidance, because in this case condition (i) holds trivially. On the other end, the case in which σ has no dashes corresponds to consecutive patterns. In this situation, an occurrence of σ in π has to be a consecutive subsequence. Consecutive patterns were introduced independently in [11], where the authors give generating functions for the number of occurrences of certain consecutive patterns in permutations. Several papers deal with the enumeration of permutations avoiding generalized patterns. In [7], Claesson presented a complete solution for the number of permutations avoiding any single 3-letter generalized pattern with exactly one adjacent pair of letters. Claesson and Mansour [8] (see also [16]) did the same for any pair of such patterns. In [10], Elizalde and Mansour studied the distribution of several statistics on permutations avoiding 1-3-2 and 1-23 simultaneously. On the other hand, Kitaev [14] investigated simultaneous avoidance of two or more 3-letter generalized patterns without dashes.

All the patterns that appear in this paper will be represented by the notation just described. In particular, a pattern $\sigma = \sigma_1\sigma_2 \cdots \sigma_m$ without dashes will denote a consecutive pattern. We will represent classical patterns by writing dashes between any two adjacent elements, namely, as σ_1-σ_2-$\cdots$-σ_m.

If σ is a generalized pattern, let $\mathcal{S}_n(\sigma)$ denote the set of permutations in $\mathcal{S}_n$ that avoid σ. Let $\alpha_n(\sigma) = |\mathcal{S}_n(\sigma)|$ be the number of such permutations, and let

$$A_\sigma(z) = \sum_{n\ge 0} \alpha_n(\sigma)\frac{z^n}{n!}$$

be the exponential generating function counting σ-avoiding permutations.

2.2. **Consecutive patterns of length 3.** For patterns of length 3 with no dashes, it follows from the trivial reversal and complementation operations that $\alpha_n(123) = \alpha_n(321)$ and $\alpha_n(132) = \alpha_n(231) = \alpha_n(312) = \alpha_n(213)$. The EGFs for these numbers are given in the following theorem of Elizalde and Noy [11], which we will use later in the paper.

Theorem 2.1 ([11]). *We have*

$$A_{123}(z) = \frac{\sqrt{3}}{2}\frac{e^{z/2}}{\cos(\frac{\sqrt{3}}{2}z + \frac{\pi}{6})}, \qquad A_{132}(z) = \frac{1}{1 - \int_0^z e^{-t^2/2}dt}.$$

Their coefficients satisfy

$$\alpha_n(123) \sim \gamma_1 \cdot (\rho_1)^n \cdot n!, \qquad \alpha_n(132) \sim \gamma_2 \cdot (\rho_2)^n \cdot n!,$$

where $\rho_1 = \frac{3\sqrt{3}}{2\pi}$, $\gamma_1 = e^{3\sqrt{3}\pi}$, $(\rho_2)^{-1}$ *is the unique positive root of* $\int_0^z e^{-t^2/2}dt = 1$, *and* $\gamma_2 = \exp((\rho_2)^{-2}/2)$, *the approximate values being*

$$\rho_1 = 0.8269933, \quad \gamma_1 = 1.8305194, \quad \rho_2 = 0.7839769, \quad \gamma_2 = 2.2558142.$$

Furthermore, for every $n \ge 4$, *we have* $\alpha_n(123) > \alpha_n(132)$.

3. Patterns of the form 1-σ

In this section we study a very particular class of generalized patterns, namely those that start with 1-, followed by a consecutive pattern (i.e., without dashes).

Proposition 3.1. *Let $\sigma = \sigma_1\sigma_2\cdots\sigma_k \in \mathcal{S}_k$ be a consecutive pattern, and let 1-σ denote the generalized pattern $1\text{-}(\sigma_1+1)(\sigma_2+1)\cdots(\sigma_k+1)$. Then,*

$$A_{1\text{-}\sigma}(z) = \exp\left(\int_0^z A_\sigma(t)\,dt\right).$$

Proof. Given a permutation π, let $m_1 > m_2 > \cdots > m_r$ be the values of its left-to-right minima (recall that π_i is a left-to-right minimum of π if $\pi_j > \pi_i$ for all $j < i$). We can write $\pi = m_1w_1m_2w_2\cdots m_rw_r$, where each w_i is a (possibly empty) subword of π, each of whose elements is greater than m_i. We claim that π avoids 1-σ if and only if each of the blocks w_i (more precisely, its reduction $\rho(w_i)$) avoids the consecutive pattern σ. Indeed, it is clear that if one of the blocks w_i contains σ, then m_i together with the occurrence of σ forms an occurrence of 1-σ. Conversely, if π contains 1-σ, then the elements of π corresponding to σ have to be adjacent, and none of them can be a left-to-right minimum (since the element corresponding to '1' has to be to their left), therefore they must be all inside the same block w_i for some i.

The generating function for each block m_iw_i is $\int_0^z A_\sigma(t)\,dt$, where the integral comes from the fact that m_i has the smallest label. Given a set of blocks m_iw_i, there is a unique way to order them, namely with the left-to-right minima in decreasing order. The expression $A_{1\text{-}\sigma}(z) = \exp(\int_0^z A_\sigma(t)\,dt)$ follows now from this construction. □

Example. The exponential generating function (EGF in what follows) of permutations avoiding 123 or 321 is given in Theorem 2.1. Proposition 3.1 implies now that

$$A_{1\text{-}234}(z) = A_{1\text{-}432}(z) = \exp\left(\frac{\sqrt{3}}{2}\int_0^z \frac{e^{t/2}\,dt}{\cos(\frac{\sqrt{3}}{2}t + \frac{\pi}{6})}\right).$$

Combined with the results of [11], Proposition 3.1 gives expressions for the EGFs $A_{1\text{-}\sigma}(z)$ where σ has one of the following forms: $\sigma = 123\cdots k$, $\sigma = k(k-1)\cdots 21$, $\sigma = 12\cdots a\,\tau\,(a+1)$, $\sigma = (a+1)\,\tau\,a(a-1)\cdots 21$, $\sigma = k(k-1)\cdots(k+1-a)\,\tau'\,(k-a)$, $\sigma = (k-a)\,\tau'\,(k+1-a)(k+2-a)\cdots k$, where k, a are positive integers with $a \le k-2$, τ is any permutation of $\{a+2, a+3, \cdots, k\}$ and τ' is any permutation of $\{1, 2, \cdots, k-a-1\}$.

4. Asymptotic enumeration

Here we discuss the behavior of the numbers $\alpha_n(\sigma)$ as n goes to infinity, for a given generalized pattern σ. We use the symbol $\sim$ to indicate that two sequences of numbers have the same asymptotic behavior (i.e., we write $a_n \sim b_n$ if $\lim_{n\to\infty}\frac{a_n}{b_n} = 1$), and we use the symbol $\ll$ to indicate that a sequence is asymptotically smaller than another one (i.e., we write $a_n \ll b_n$ if $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$).

Let us first consider the case of consecutive patterns.

Theorem 4.1. *Let $k \ge 3$ and let $\sigma \in \mathcal{S}_k$ be a consecutive pattern.*

(i) *There exist constants $0 < c, d < 1$ such that $c^n n! \le \alpha_n(\sigma) \le d^n n!$ for all $n \ge k$.*
(ii) *There exists a constant $0 < w \le 1$ such that*

$$\lim_{n\to\infty}\left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n} = w.$$

Note that c, d and w depend only on σ.

Proof. The key observation is that, for any consecutive pattern σ,

$$\alpha_{m+n}(\sigma) \le \alpha_m(\sigma)\alpha_n(\sigma)\binom{m+n}{n}. \tag{1}$$

To see this, just observe that a σ-avoiding permutation of length $m+n$ induces two juxtaposed σ-avoiding permutations of lengths m and n.

By induction on $n \geq k$ one gets

$$\alpha_{m+n}(\sigma) \leq d^m m!\, d^n n! \binom{m+n}{n} = d^{m+n}(m+n)!$$

for some positive $d < 1$.

For the lower bound, let $\tau = \rho(\sigma_1\sigma_2\sigma_3)$ be the reduction of the first three elements of σ. Clearly $\mathcal{S}_n(\tau) \subseteq \mathcal{S}_n(\sigma)$ for all n, since an occurrence of σ in a permutation produces also an occurrence of τ, hence $\alpha_n(\tau) \leq \alpha_n(\sigma)$. But the fact that $\sigma \in \mathcal{S}_3$ implies that $\alpha_n(\sigma)$ equals either $\alpha_n(123)$ or $\alpha_n(132)$. In any case, by Theorem 2.1 we have that $\alpha_n(\sigma) \geq \alpha_n(132) \geq c^n n!$ for some $c > 0$.

To prove part (ii), we can express (1) as

$$\frac{\alpha_{m+n}(\sigma)}{(m+n)!} \leq \frac{\alpha_m(\sigma)}{m!}\frac{\alpha_n(\sigma)}{n!}$$

and apply *Fekete's lemma* (see [19, Lemma 11.6] or [12]) to the function $n!/\alpha_n(\sigma)$ to conclude that $\lim_{n\to\infty}\left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n}$ exists. Calling it w, then part (i) implies that $w \leq 1$ and $w \geq \lim_{n\to\infty}\left(\frac{\alpha_n(132)}{n!}\right)^{1/n} = 0.7839769$. □

In order to study the asymptotic behavior of $\alpha_n(\sigma)$ for a generalized pattern σ we separate the problem into the following three cases. Assume from now on that $k \geq 3$ and that σ is a generalized pattern of length k. We use the word *slot* to refer to the place between two adjacent elements of σ, where there can be a dash or not.

- **Case 1.** *The pattern σ has dashes between any two adjacent elements, i.e., $\sigma = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$.*

These are just the classical patterns, which have been widely studied in the literature. The asymptotic behavior of the number of permutations avoiding them is given by the Stanley-Wilf conjecture, which has been recently proved by Marcus and Tardos [17], after several authors had given partial results over the last few years [1, 2, 5, 15].

Theorem 4.2 (*Stanley-Wilf conjecture*, proved in [17])**.** *For every classical pattern $\sigma = \sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k$, there is a constant λ (which depends only on σ) such that $\alpha_n(\sigma) < \lambda^n$ for all $n \geq 1$.*

On the other hand, it is clear that $\alpha_n(\sigma) > \alpha_n(\rho(\sigma_1\text{-}\sigma_2\text{-}\sigma_3)) = \mathbf{C}_n \sim \frac{1}{\sqrt{\pi n}}4^n$, where $\mathbf{C}_n$ denotes the n-th *Catalan number*. As shown by Arratia [2], Theorem 4.2 is equivalent to the statement that $\lim_{n\to\infty}\sqrt[n]{\alpha_n(\sigma)}$ exists. The value of this limit has been computed for several classical patterns: it is clearly 4 for patterns of length 3, it is known [18] to be $(k-1)^2$ for $\sigma = 1\text{-}2\text{-}\cdots\text{-}k$, it has been shown [4] to be 8 for $\sigma = 1\text{-}3\text{-}4\text{-}2$, and it has recently been proved by Bóna [6] to be nonrational for certain patterns.

- **Case 2.** *The pattern σ has two consecutive slots without a dash (equivalently, three consecutive elements without a dash between them), i.e., $\sigma = \cdots\sigma_i\sigma_{i+1}\sigma_{i+2}\cdots$.*

Proposition 4.3. *Let σ be a generalized pattern having three consecutive elements without a dash. Then there exist constants $0 < c, d < 1$ such that $c^n n! \leq \alpha_n(\sigma) \leq d^n n!$ for all $n \geq k$.*

Proof. For the upper bound, notice that if a permutation contains the consecutive pattern $\sigma_1\sigma_2\sigma_3\cdots\sigma_k$ obtained by removing all the dashes in σ, then it also contains σ. Therefore, $\alpha_n(\sigma) \leq \alpha_n(\sigma_1\sigma_2\sigma_3\cdots\sigma_k)$ for all n, and now the upper bound follows from part (i) of Theorem 4.1.

For the lower bound, we use that $\alpha_n(\sigma) \geq \alpha_n(\rho(\sigma_i\sigma_{i+1}\sigma_{i+2})) \geq \alpha_n(132) \geq c^n n!$, where $\sigma_i\sigma_{i+1}\sigma_{i+2}$ are three consecutive elements in σ without a dash. □

- **Case 3.** *The pattern σ has at least a slot without a dash, but not two consecutive slots without dashes.*

This case includes all the patterns not considered in Cases 1 and 2. The asymptotic behavior of $\alpha_n(\sigma)$ for these patterns is not known in general. The case of patterns of length 3 is covered by the following result, due to Claesson [7]. Let $\mathbf{B}_n$ denote the n-th *Bell number*, which counts the number of partitions of an n-element set.

Proposition 4.4 ([7])**.** *Let σ be a generalized pattern of length 3 with one dash.*

(i) *If* $\sigma \in \{1\text{-}23, 3\text{-}21, 32\text{-}1, 12\text{-}3, 1\text{-}32, 23\text{-}1, 3\text{-}12, 21\text{-}3\}$, *then* $\alpha_n(\sigma) = \mathbf{B}_n$.
(ii) *If* $\sigma \in \{2\text{-}13, 2\text{-}31, 31\text{-}2, 13\text{-}2\}$, *then* $\alpha_n(\sigma) = \mathbf{C}_n$.

It is known that the asymptotic behavior of the Catalan numbers is given by $\mathbf{C}_n \sim \frac{1}{\sqrt{\pi n}} 4^n$. For the Bell numbers, one has the formula

$$\mathbf{B}_n \sim \frac{1}{\sqrt{n}} \lambda(n)^{n+1/2} e^{\lambda(n)-n-1},$$

where $\lambda(n)$ is defined by $\lambda(n)\ln(\lambda(n)) = n$. Another useful description of the asymptotic behavior of $\mathbf{B}_n$ is the following:

$$\frac{\ln \mathbf{B}_n}{n} = \ln n - \ln\ln n + O\left(\frac{\ln\ln n}{\ln n}\right).$$

This shows in particular that $\delta^n \ll \mathbf{B}_n \ll c^n n!$ for any constants $\delta, c > 0$.

For patterns σ of length $k \geq 4$ that have slots without a dash, but not two consecutive slots without dashes, not much is known in general about the number of permutations avoiding them. It follows from Cases 1 and 2 that $\delta^n < \alpha(\sigma_1\text{-}\sigma_2\text{-}\cdots\text{-}\sigma_k) < \alpha_n(\sigma) < \alpha(\sigma_1\sigma_2\cdots\sigma_k) < d^n n!$ for some constants $\delta > 0$ and $d < 1$. Clearly, if σ contains one of the patterns in part (i) of Proposition 4.4, then the lower bound can be improved to $\mathbf{B}_n$. However, determining the precise asymptotic behavior of $\alpha_n(\sigma)$ seems to be a difficult problem. In the rest of the paper we discuss a few partial results in this direction.

The next statement is about permutations of the form 1-σ.

Corollary 4.5. *Let σ be a consecutive pattern, and let 1-σ be defined as in Proposition 3.1. Then,*

$$\lim_{n\to\infty}\left(\frac{\alpha_n(1\text{-}\sigma)}{n!}\right)^{1/n} = \lim_{n\to\infty}\left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n}.$$

Proof. By Proposition 3.1 we know that $A_{1\text{-}\sigma}(z) = \exp\left(\int_0^z A_\sigma(t)\,dt\right)$. Since the exponential is an analytic function over the whole complex plane, we obtain that $A_{1\text{-}\sigma}(z)$ has the same radius of convergence as $A_\sigma(z)$, from where the result follows. □

5. The pattern 12-34

The next proposition gives an upper and a lower bound for the numbers $\alpha_n(12\text{-}34)$. Given two formal power series $F(z) = \sum_{n\geq 0} f_n z^n$ and $G(z) = \sum_{n\geq 0} g_n z^n$, we use the notation $F(z) < G(z)$ to indicate that $f_n < g_n$ for all n, and $F(z) \ll G(z)$ to indicate that $f_n \ll g_n$.

Proposition 5.1. *For $k \geq 1$, let*

$$h_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}, \qquad b_k(z) = \sum_{i=0}^{k}\binom{k}{i}^2 [z + 2(h_{k-i} - h_i)]\, e^{iz},$$

$$c_k(z) = \frac{e^{(k+1)z}}{k+1} - \sum_{i=0}^{k}\binom{k}{i}\binom{k+1}{i}\left[z + 2(h_{k-i} - h_i) + \frac{1}{k+1-i}\right] e^{iz}, \qquad S(z) = \sum_{k\geq 1} b_k(z) + \sum_{k\geq 1} c_k(z).$$

Then

$$e^{S(z)} < A_{12\text{-}34}(z) < e^{S(z)+e^z+z-1}.$$

The idea of the proof is to study the structure of 12-34-avoiding permutations by decomposing them in a unique way. Since this structure is too complicated to find an exact formula, we add and remove restrictions to simplify the description, obtaining the lower and upper bounds. Due to lack of space, the proof is omitted in this extended abstract.

If we write $e^{S(z)} = \sum l_n \frac{z^n}{n!}$ and $e^{S(z)+e^z+z-1} = \sum u_n \frac{z^n}{n!}$ to denote the coefficients of the series giving the lower and the upper bound respectively, then the graph in Figure 1 shows the values of $\sqrt[n]{\alpha_n(12\text{-}34)/n!}$ for $n \leq 13$, bounded between the values $\sqrt[n]{l_n/n!}$ and $\sqrt[n]{u_n/n!}$ for $n \leq 120$. The two horizontal dotted lines are at height 0.7839769 and 0.8269933, which are $\lim_{n\to\infty}\sqrt[n]{\alpha_n(\sigma)/n!}$ for $\sigma = 132$ and $\sigma = 123$ respectively, given by Theorem 2.1. From this plot it seems plausible that $\lim_{n\to\infty}\sqrt[n]{\alpha_n(12\text{-}34)/n!} = 0$, although we have not succeeded in proving this.

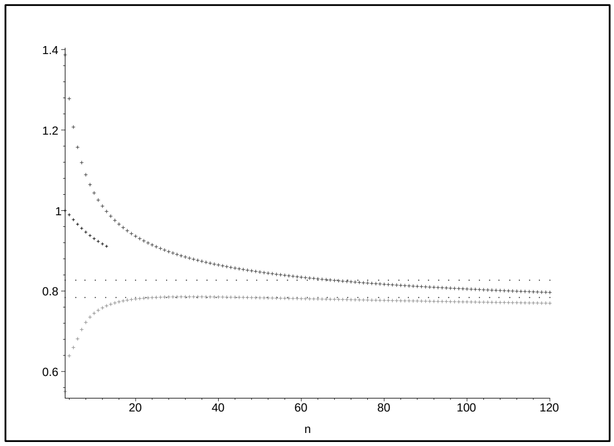

FIGURE 1. The first values of $\sqrt[n]{\alpha_n(12\text{-}34)/n!}$ between the lower and the upper bound given by Proposition 5.1.

Note that the lower bound, together with the fact that $S(z) \gg e^z - 1$ (which follows from the definition), shows that $A_{12\text{-}34}(z) > e^{S(z)} \gg e^{e^z-1}$, which means that $\alpha_n(12\text{-}34) \gg \mathbf{B}_n$, that is, the number of 12-34-avoiding permutations is asymptotically larger than the Bell numbers.

The decomposition of 12-34-avoiding permutations used to proove Proposition 5.1 can be generalized to permutations avoiding a pattern of the form 12-σ. If $\sigma = \sigma_1\sigma_2\cdots\sigma_k \in \mathcal{S}_k$ is a consecutive pattern, 12-σ denotes the generalized pattern 12-$(\sigma_1+2)(\sigma_2+2)\cdots(\sigma_k+2)$.

Any permutation π that avoids 12-σ can be uniquely decomposed as $\pi = B_0a_1B_1a_2B_2a_3B_3\cdots$, where a_1 and the element preceding it form the first ascent of π, a_2 and the element preceding it form the first ascent such that $a_2 < a_1$, a_3 and the element preceding it form the first ascent such that $a_3 < a_2$, and so on. Then, by definition, B_0 is a non-empty decreasing word whose last element is less than a_1, and each B_i with $i \geq 1$ can be written uniquely as a sequence $B_i = w_{i,0}U_{i,1}w_{i,1}U_{i,2}w_{i,2}\cdots U_{i,r_i}w_{i,r_i}$ for some $r_i \geq 1$ (r_i can be 0 if $w_{i,0}$ is nonempty) with the following properties:

(i) each $w_{i,j}$ is a decreasing word all of whose elements are less than a_i,
(ii) each $U_{i,j}$ is a nonempty permutation avoiding σ, all of whose elements are greater than a_i,
(iii) $w_{i,j}$ is nonempty for $j \geq 1$,
(iv) the last element of B_i is less than a_{i+1}.

From this decomposition the following result follows immediately.

Proposition 5.2. *If $\sigma \sim \tau$ are two consecutive patterns, then* 12-$\sigma \sim$ 12-τ.

6. THE PATTERN 1-23-4

Similarly to what we did for the pattern 12-34, analyzing the structure of permutations avoiding 1-23-4 we can give lower and upper bounds for the numbers $\alpha_n(1\text{-}23\text{-}4)$. Let $\mathbf{C}^{\exp}(z) := \sum_{n\geq 0} \mathbf{C}_n \frac{z^n}{n!}$ be the EGF for the Catalan numbers.

Proposition 6.1. *We have that*

$$\frac{1}{2}\int_0^z e^{2e^y-2}\,dy - \frac{z}{2} < A_{1\text{-}23\text{-}4}(z) < \mathbf{C}^{\exp}(e^z-1).$$

Again, due to lack of space we omit the proof of this proposition.

Writing $\frac{1}{2}\int_0^z e^{2e^y-2}\,dy - \frac{z}{2} = \sum l_n \frac{z^n}{n!}$ and $\mathbf{C}^{\exp}(e^z-1) = \sum u_n \frac{z^n}{n!}$ to denote the coefficients of the series giving the lower and the upper bound respectively, then the values of $\sqrt[n]{l_n/n!}$ and $\sqrt[n]{u_n/n!}$ for $n \leq 90$ are plotted in Figure 2, bounding the values of $\sqrt[n]{\alpha_n(1\text{-}23\text{-}4)/n!}$ for $n \leq 11$.

Note that the lower bound implies that $\alpha_n(1\text{-}23\text{-}4) \gg \mathbf{B}_n$, since $e^{2e^z-2} \gg e^{e^z-1}$. The upper bound given in the above proposition yields the following corollary.

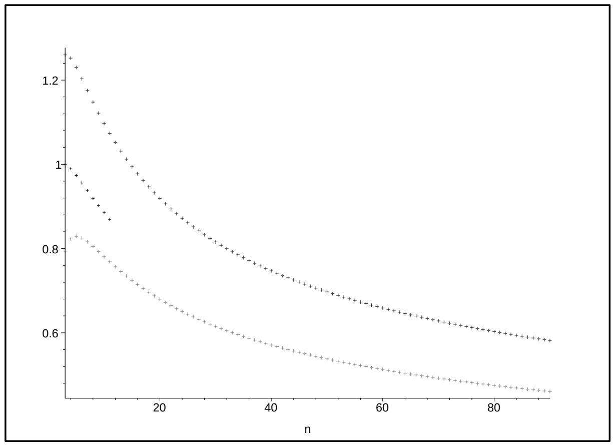

FIGURE 2. The first values of $\sqrt[n]{\alpha_n(\text{1-23-4})/n!}$ between the lower and the upper bound given by Proposition 6.1.

Corollary 6.2. *We have that*

$$\lim_{n\to\infty}\left(\frac{\alpha_n(\text{1-23-4})}{n!}\right)^{1/n}=0.$$

Proof. The power series $\mathbf{C}^{\exp}(z)$ can be bounded by

$$\mathbf{C}^{\exp}(z)<\sum_{n\geq 0}4^n\frac{z^n}{n!}=e^{4z},$$

which converges for all z. Therefore, so does $\mathbf{C}^{\exp}(e^z-1)$, which is an upper bound for $A_{\text{1-23-4}}(z)$. The result follows now from the fact that if $\sum_n f_n z^n$ is an analytic function in the whole complex plane, then $\sqrt[n]{f_n}=0$ (see [13, Chapter 4] for a discussion). □

If $\sigma=\sigma_1\sigma_2\cdots\sigma_{k-2}\in\mathcal{S}_{k-2}$ is a consecutive pattern, let 1-σ-k denote the generalized pattern 1-$(\sigma_1+1)(\sigma_2+1)\cdots(\sigma_{k-2}+1)$-$k$. The decomposition of 1-23-4-avoiding permutations that is used to prove the above proposition can be generalized to permutations avoiding any pattern of the form 1-σ-k.

Any permutation π that avoids 1-σ-k can be uniquely decomposed as $\pi=c_1w_1c_2w_2\cdots c_{m-1}w_{m-1}c_m$, where the c_i are all the left-to-right minima and right-to-left maxima of π, and each w_i is a permutation that avoids σ, all of whose elements are bigger than the closest left-to-right minimum to its left and smaller than the closest right-to-left maximum to its right.

Using exactly the same reasoning as in the proof of Proposition 6.1, we obtain the following lower and upper bounds for the numbers $\alpha_n(\text{1-}\sigma\text{-}k)$.

Proposition 6.3. *Let $\sigma\in\mathcal{S}_{k-2}$ be a consecutive pattern, and let 1-σ-k be defined as above. Then,*

$$\int_0^z\int_0^u e^{2\int_0^y A_\sigma(t)dt+y}\,dy\,du<A_{\text{1-}\sigma\text{-}k}(z)<\mathbf{C}^{\exp}\left(\int_0^z A_\sigma(t)\,dt\right).$$

Corollary 6.4. *With the same definitions as in the above proposition,*

$$\lim_{n\to\infty}\left(\frac{\alpha_n(\text{1-}\sigma\text{-}k)}{n!}\right)^{1/n}=\lim_{n\to\infty}\left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n}.$$

Proof. The upper and lower bounds for $A_{\text{1-}\sigma\text{-}k}(z)$ given in Proposition 6.3 are analytic functions of $A_\sigma(z)$, since essentially they only involve exponentials and integrals. Therefore, $A_{\text{1-}\sigma\text{-}k}(z)$ and $A_\sigma(z)$ have the same radius of convergence, hence the limits above coincide. □

Finally, the following proposition is an immediate consequence of the structure of 1-σ-k-avoiding permutations discussed above. In particular, it implies that 1-23-4 $\sim$ 1-32-4.

Proposition 6.5. *If $\sigma\sim\tau$ are two consecutive patterns in $\mathcal{S}_{k-2}$, then 1-σ-k $\sim$ 1-τ-k.*

7. Final remarks

In Section 6 we have proved that $\alpha_n(1\text{-}23\text{-}4) \gg \mathbf{B}_n$ and that $\alpha_n(1\text{-}23\text{-}4) \ll c^n n!$ for any constant $c > 0$. For the pattern 12-34, we showed in Section 5 that the analogue to the first statement holds as well, and the second one seems to be true from numerical computations. It remains as an open problem to describe precisely the asymptotic behavior of $\alpha_n(\sigma)$ for these two patterns, and for several remaining generalized patterns of length 4.

This paper is the first attempt to study the asymptotic behavior of the numbers $\alpha_n(\sigma)$ where σ is an arbitrary generalized pattern. Despite the fact that we have been unable to provide a precise description of this behavior in most cases, we hope that our work shows the intricateness of the problem and the amount of questions that it opens. The main goal of further research in this direction would be to give a complete classification of all generalized patterns according to the asymptotic behavior of $\alpha_n(\sigma)$ as n goes to infinity.

Another interesting open problem is to find the value of $\lim_{n\to\infty} \sqrt[n]{\alpha_n(\sigma)/n!}$ for patterns σ in Case 2, for which this limit is known to be a constant. The analogous problem for patterns in Case 1, namely finding $\lim_{n\to\infty} \sqrt[n]{\alpha_n(\sigma)}$ for classical patterns σ, is a current direction of research as it remains open for most patterns as well (see [4, 6]).

References

[1] N. Alon, E. Friedgut, On the number of permutations avoiding a given pattern, *J. Combin. Theory Ser. A* 89 (2000), 133–140.

[2] R. Arratia, On the Stanley-Wilf conjecture for the number of permutations avoiding a given pattern, *Electron. J. Combin.* 6 (1999), no. 1, Note, N1.

[3] E. Babson, E. Steingrímsson, Generalized permutation patterns and a classification of the Mahonian statistics, *Sém. Lothar. Combin.* 44, Art. B44b (2000).

[4] M. Bóna, Exact Enumeration of 1342-Avoiding Permutations: A Close Link with Labeled Trees and Planar Maps, *J. Combin. Theory Ser. A* 80 (1997), 257–272.

[5] M. Bóna, The solution of a conjecture of Stanley and Wilf for all layered patterns, *J. Combin. Theory Ser. A* 85 (1999), 96–104.

[6] M. Bóna, The limit of a Stanley-Wilf sequence is not always rational, and layered patterns beat monotone patterns, preprint, arxiv:math.CO/0403502.

[7] A. Claesson, Generalised pattern avoidance, *European J. Combin.* 22 (2001), 961–973.

[8] A. Claesson, T. Mansour, Enumerating permutations avoiding a pair of Babson-Steingrímsson patterns, *Ars Combin.*, to appear, math.CO/0107044.

[9] S. Elizalde, *Consecutive patterns and statistics on restricted permutations*, Ph.D. Thesis, Universitat Politècnica de Catalunya.

[10] S. Elizalde, T. Mansour, Restricted Motzkin permutations, Motzkin paths, continued fractions, and Chebyshev polynomials, *Proceedings of FPSAC 2004*.

[11] S. Elizalde, M. Noy, Consecutive patterns in permutations, *Adv. in Appl. Math.* 30 (2003), 110–125.

[12] M. Fekete, Über die Verteilung der Wurzeln bei gewissen algebraishen Gleichungen mit ganzzahligen Koeffizienten, *Math. Zeitschr.* 17 (1923), 228–249.

[13] P. Flajolet, R. Sedgewick, *Analytic combinatorics* (book in preparation). (Individual chapters are available as INRIA Research Reports 1888, 2026, 2376, 2956, 3162.)

[14] S. Kitaev, Multi-Avoidance of generalised patterns, *Discrete Math.* 260 (2003), 89–100.

[15] M. Klazar, The Füredi-Hajnal conjecture implies the Stanley-Wilf conjecture, *Proceedings Formal power series and algebraic combinatoircs 2000*, 250–255, Springer, Berlin, 2000.

[16] T. Mansour, Continued fractions and generalized patterns, *European J. Combin.* 23 (2002), 329–344.

[17] A. Marcus, G. Tardos, Excluded permutation matrices and the Stanley-Wilf conjecture, *J. Combin. Theory Ser. A* 107 (2004) 1, 153–160.

[18] A. Regev, Asymptotic values for degrees associated with strips of Young diagrams, *Adv. in Math.* 41 (1981), 115–136.

[19] J.H. van Lint, R.M. Wilson, A course in combinatorics, Cambridge University Press, Cambridge, 1992; second edition, 2001.

Mathematical Sciences Research Institute, 17 Gauss Way, Berkeley, CA 94720
E-mail address: elizalde@msri.org