RESTRICTED PERMUTATIONS AND POLYGONS

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ABSTRACT

Several authors have examined connections among restricted permutations and different combinatorial structures. In this paper we establish a bijection between the set of permutations π which avoid $13\square2$ and the set of *odd-dissection convex polygons*, where a permutation avoids $ab\square c$ if there are no $i<j<k-1$ such that $\pi_i\pi_j\pi_k$ is order-isomorphic to abc. We also exhibit bijections between the set of permutations that avoid $12\square3$ (or $21\square3$) and the set of odd-dissection convex polygons. Using tools developed to prove these results, we give enumerations and generating functions for permutations which avoid $13\square2$ and certain additional patterns.

1. Extended abstract

1.1. **Classical patterns.** Let $[n]=\{1,2,\dots,n\}$ and denote by $\mathfrak{S}_n$ the set of permutations of $[n]$. We shall view permutations in $\mathfrak{S}_n$ as words $\pi=\pi_1\pi_2\dots\pi_n$. We denote by $\mathfrak{S}$ the set of all permutations of all sizes (including the empty permutation ϵ, that is, the permutation of length 0), that is, $\mathfrak{S}=\cup_{n\geq0}\mathfrak{S}_n$. The *reduced form* of a permutation σ on a set $\{j_1,j_2,\dots,j_k\}$, where $j_1<j_2<\cdots<j_k$ is a permutation obtained by renaming the letters of the permutation σ so that j_i is renamed i for all $i\in\{1,\dots,k\}$. For example, the reduced forms of the permutations 4973 and 1974 are 2431 and 1432, respectively.

Definition 1. For $k\leq n$, we say that a permutation $\sigma\in\mathfrak{S}_n$ has an occurrence of the pattern $\phi\in\mathfrak{S}_k$ if there exist $1\leq i_1<i_2<\cdots<i_k\leq n$ such that the reduced form of $\sigma(i_1)\sigma(i_2)\dots\sigma(i_k)$ is ϕ. We denote the number of occurrences of the pattern ϕ in the permutation σ by $\phi(\sigma)$.

We say that a permutation π *avoids* a pattern ϕ, or is ϕ*-avoiding*, if $\phi(\pi)=0$. For example, let $\pi=83176254$, $\phi=1234$ and $\theta=1243$. Then it is easy to see that π avoids ϕ, and contains exactly one occurrence of θ, that is π does not avoid θ. The set of all ϕ-avoiding permutations in $\mathfrak{S}_n$ is denoted by $\mathfrak{S}_n(\phi)$. For any set T of patterns, we let $\mathfrak{S}_n(T)=\cap_{\phi\in T}\mathfrak{S}_n$.

The first explicit result seems to be Hammersley's enumeration of $\mathfrak{S}_n(321)$ in [5]. In [11, Ch. 2.2.1] and [12, Ch. 5.1.4] Knuth shows that for any $\tau\in\mathfrak{S}_3$, we have $|S_n(\tau)|=C_n$, where C_n is the nth Catalan number given by $C_n=\frac{1}{n+1}\binom{2n}{n}$ (see [19, Sequence A000108]). Other authors considered restricted permutations in the 1970s and early 1980s (see, for example, [14], [15], and [16]), but the first systematic study was not undertaken until 1985, when Simion and Schmidt [17] solved the enumeration problem for every subset of S_3. Currently, there exist more than two hundred papers on this subject (see [8]).

1.2. **Generalized patterns.** In [1] Babson and Steingrímsson introduced *generalized permutation patterns* that add the requirement that two adjacent letters in a pattern must be adjacent in the permutation. In order to avoid confusion we write a classical pattern, say 231, as 2-3-1, and if we write, say 2-31, then we mean that if this pattern occurs in the permutation, then the letters in the permutation that correspond to 3 and 1 are adjacent. Let us give a formal definition of a generalized pattern.

Definition 2. A *generalized pattern* of length k is a word $\phi = \phi_1 x_1 \phi_2 \dots x_{k-1}\phi_k$, where $\phi_1\phi_2\dots\phi_k \in \mathfrak{S}_k$, and for $j = 1, 2, \dots, k-1$, x_j is either the empty string ϵ or a dash "-". If $x_j =$"-" then in the definition of an occurrence of a classical pattern we require $i_j \geq i_{j-1}+1$, otherwise we require $i_j = i_{j-1}+1$.

For example, the permutation $\pi = 314265$ has two occurrences of the pattern 2-31-4, namely 3-42-6 and 3-42-5. A number of interesting results on generalized patterns were obtained in [2]. Relations to several well studied combinatorial structures, such as set partitions (see [6]), Dyck paths (see [13]), Motzkin paths (see [4]) and involutions (see [18]) were shown there. As in the paper by Simion and Schmidt [17] dealing with the classical patterns, Claesson [2], Claesson and Mansour [3] considered a number of cases where permutations have to avoid two or more generalized patterns simultaneously. In [7] Kitaev gave either an explicit formula or a recursive formula for almost all cases of simultaneous avoidance of more than two generalized patterns of length three with no dashes (see also [9, 10]).

1.3. **Distanced patterns.** In this section we give a uniform language to studying the classical pattern problem (see Definition 1) and generalized pattern problem (see Definition 2) in terms of the d-pattern problem.

Definition 3. A *distanced-pattern* (or *d-pattern*) of length k is a pair $(\phi, \mathbf{d})$ where $\phi \in \mathfrak{S}_k$ and $\mathbf{d}$ is a word $\mathbf{d} = d_1^{x_1} d_2^{x_2} \dots d_{k-1}^{x_{k-1}}$ such that $d_j \geq 0$ for $j = 1, 2, \dots, k-1$, and x_j is either the empty string ϵ, a minus "-" sign, or a plus "+" sign. If $x_j = \epsilon$ (resp. x_j="+", x_j ="-") then in the definition of an occurrence of a classical pattern we require $i_j - i_{j-1} - 1 = d_j$ (resp. $i_j - i_{j-1} - 1 \geq d_j$, $i_j - i_{j-1} - 1 \leq d_j$).

For example, if $\pi = 41578362 \in S_8$ then it contains $\Phi = (132, 31)$, e.g. $\pi_1\pi_5\pi_7 = 486$ with distance $d = 31$, it contains $\Theta = (312, 03^+)$, e.g. $\pi_1\pi_2\pi_6 = 413$ and $\pi_1\pi_2\pi_8 = 412$, and it contains $\Gamma = (312, 03^-)$, e.g. $\pi_1\pi_2\pi_6 = 413$ and $\pi_5\pi_6\pi_7 = 836$.

As a remark, our Definition 3 generalizes the classical and generalized definitions of patterns. For example, avoiding the classical pattern 3421 is the same as avoiding the d-pattern $(3421, 0^+0^+0^+)$ and avoiding the generalized pattern 3-42-1 is the same as avoiding the pattern $(3421, 0^+00^+)$. The following two examples connect the d-pattern avoidance problem to binomial coefficients and Fibonacci numbers.

Example 4. Let d be any nonnegative integer number. Then it can shown that

$$\#\mathfrak{S}_{(d+1)n+\ell}((12,d)) = \prod_{j=0}^{\ell-1} \binom{(d+1-j)n+\ell-j}{n+1} \prod_{j=\ell}^{d} \binom{(d+1-j)n}{n} = \frac{((d+1)n+\ell)!}{(n+1)!^{\ell} n!^{d+1-\ell}},$$

for all $n \geq 0$ and $0 \leq \ell \leq d$.

Example 5. For any $n \geq 0$, $\#\mathfrak{S}_n(12, 1^+) = F_{n+1}$, where F_{n+1} is the $(n+1)$-st Fibonacci number. To see that, let $a_n = \#\mathfrak{S}_n(12, 1^+)$. For every permutation π in $\mathfrak{S}_n(12, 1^+)$ there are two possibilities: the entry 1 can be either the last (the n-th) element of π, or the $(n-1)$-st element of π. In the later case the entry 2 must be the last element of π. Therefore, in the first case we have a_{n-1} permutations, and in the second case we have a_{n-2} permutations, hence $a_n = a_{n-1} + a_{n-2}$. Observing that $a_0 = a_1 = 1$, we conclude that $a_n = F_{n+1}$, as claimed.

Define an *odd-dissection convex polygon permutation* or *odd-dissection gon permutation* (or *ODP-permutation*) π to be a permutation in S_n that avoid the d-pattern $13\Box 2$, where we denote the d-pattern $(abc, 0^+1^+)$ by $ab\Box c$. For example, there are exactly twenty ODP-permutations of length 4. We denote the set of all ODP-permutations in $\mathfrak{S}_n$ by $\mathcal{O}_n$. The main reason for the term "ODP-permutation" is that the cardinality of the set $\mathcal{O}_n$ is given by number of *odd-dissections of a $(n+2)$-gon*.

The main results of this paper can be formulated as follows. Let G_n be a convex n-gon in the plane $\mathbb{R}^2$ with vertices labeled $1, 2, \ldots, n$ and edges $12, 23, \ldots, (n-1)n, n1$.

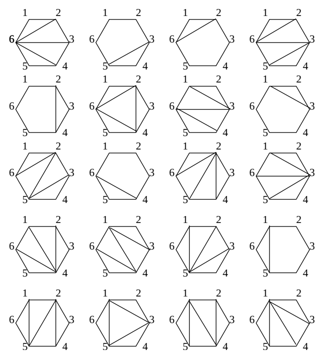

FIGURE 1. The set $\wp_4$

A *dissection* of G_n is a partition of connecting vertices of G_n into k polygons $G^1, \ldots, G^k$ by non-crossing diagonals of G_n. An *odd-dissection* of G_n is a dissection $G^1, \ldots, G^k$ of G_n such that G^i is not a $2m$-gon $(m > 1)$ for all $i = 1, \ldots, k$. We denote the set of all odd-dissections of a given convex $(n+2)$-gon by $\wp_n$ (see Figure 1 for the case $n = 4$). Observe that every odd-dissection $G \in \wp_n$ has one of two forms: (1) The vertices 1 and $n+2$ are connected by straight line segments to the same vertex i, and (2) The two vertices 1 and $n+2$ of G are not connected by a straight line segment to the same vertex.

Theorem 6. (i) *There exists a bijection Θ between $\mathcal{O}_n$ and $\wp_n$.*

(ii) *There exists a bijection between the set of $12\Box 3$-avoiding permutations in $\mathfrak{S}_n$ and $\wp_n$.*

(iii) *There exists a bijection between the set of $21\Box 3$-avoiding permutations in $\mathfrak{S}_n$ and $\wp_n$.*

Let $F(x) = \sum_{n\geq 0} \#\mathcal{O}_n x^n$, then Theorem 6(i) gives

$$F(x) = 1 + xF^2(x) + x^2F^2(x)(F(x) - 1),$$

and the values of the corresponding sequence are 1, 1, 2, 6, 20, 71, 264, 1015, 4002, 16094, 65758, 272208, 1139182 for $n = 0, 1, \ldots, 12$ (see [19, Sequence A049124]). To find an exact formula for the number of ODP-permutations on $[n]$, let $p(x;\alpha) = \alpha x(p(x;\alpha) + 1)^2(1 + xp(x;\alpha))$. Clearly, $p(x;1) = F(x) - 1$. On the other hand, by using the Lagrange inversion formula we get that

$$p(x;\alpha) = \sum_{n\geq 1}\left(\sum_{j=0}^{n-1} \frac{1}{n}\binom{2n}{j}\binom{n}{j+1} x^{2n-1-j}\right)\alpha^n.$$

Therefore, the generating function $F(x)$ can be presented as

$$F(x) = 1 + \sum_{n\geq 1}\left(\sum_{k\geq 0} \frac{1}{n-k}\binom{2n-2k}{n-1-2k}\binom{n-k}{k}\right)x^n.$$

Hence, we have the following result.

Corollary 7. *For all $n \geq 1$, the number of ODP-permutations, $12\square3$-avoiding permutations, $21\square3$-avoiding permutations in $\mathfrak{S}_n$ is given by*

$$\sum_{k\geq 0} \frac{1}{n-k}\binom{2n-2k}{n-1-2k}\binom{n-k}{k}.$$

An another application of the bijection Θ to give the generating functions for several statistics in ODP-permutations. For a permutation π, denote by $\tau_k(\pi)$ the number of occurrences of the classical pattern $\tau_k = 132\dots(2k+1)(2k)$ (in other words, $\tau_k = (132\dots(2k+1)(2k), 0^+\dots 0^+)$), for any $k \geq 1$. For an odd-dissection n-gon G with partition into k-polygons $G^1, G^2, \dots, G^k$, denote by $p_k(G)$ the number of polygons G^i with $2k+1$ vertices. These statistics can be characterized in terms of pattern avoidance as follows.

Lemma 8. *Let $\pi \in \mathcal{O}_n$ and $G = \Theta(\pi)$. Then $\tau_\ell(\pi) = 0$ if and only if $p_{\ell+1}(G) = 0$, for any $\ell \geq 1$.*

Let $F(t; x_1, x_2, \dots)$ be the generating function $\sum_{n\geq 0}\left(t^n \sum_{\pi\in\mathcal{O}_n}\prod_{\ell\geq 1} x_\ell^{\tau_\ell(\pi)}\right)$. By Lemma 8 together with Theorem 6 we have that the generating function $F(t; x_1, x_2, \dots)$ satisfies

$$F(t; x_1, x_2, \dots) = 1 + tF^2(t; x_1, x_2, \dots) + t^2F^2(t; x_1, x_2, \dots)(F(tx_1; x_1x_2, x_2x_3, \dots) - 1),$$

which is equivalent to

$$F(t; x_1, x_2, \dots) = \frac{2}{1 + \sqrt{(1-2t)^2 - 4t^2F(tx_1; x_1x_2, x_2x_3, \dots)}}. \tag{1}$$

By applying (1) repeatedly and in each step performing some rather tedious algebraic manipulations we get

Corollary 9. *The generating function for the number of ODP-permutations in $\mathfrak{S}_n$ is given by*

$$F(x; 1, 1, \dots) = \frac{2}{1 + \sqrt{(1-2x)^2 - \frac{8x^2}{1+\sqrt{(1-2x)^2 - \frac{8x^2}{\ddots}}}}}.$$

As an another application of (1), we have the following result.

Corollary 10. *The generating function for the number of ODP-permutations that avoid τ_ℓ is given by $H_\ell(x)$ where*

$$H_\ell(x) = \frac{2}{1 + \sqrt{(1-2x)^2 - 4x^2H_{\ell-1}(x)}}$$

with $H_0(x) = 1$.

For example, the generating function for the number of ODP-permutations that avoid the classical pattern 132 (which equals the number of 132-avoiding permutations; see [11]) is given by $H_1(x) = \frac{2}{1+\sqrt{1-4x}}$, the generating function for the Catalan numbers. Also, the generating function for the number of ODP-permutations that avoid the classical pattern 13254 is given by

$$H_2(x) = \frac{2}{1 + \sqrt{(1-2x)^2 - \frac{8x^2}{1+\sqrt{1-4x}}}}.$$

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