

A GEOMETRIC FORM FOR THE EXTENDED PATIENCE SORTING ALGORITHM

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ABSTRACT. Patience Sorting is a combinatorial algorithm that can be viewed as an iterated, non-recursive form of the Schensted Insertion Algorithm. In recent work the authors extended Patience Sorting to a full bijection between the symmetric group and certain pairs of combinatorial objects that are most naturally defined in terms of generalized permutation pattern avoidance. This Extended Patience Sorting Algorithm is very similar to the Robinson-Schensted-Knuth (or RSK) Correspondence, which is itself built from repeated application of the Schensted Insertion Algorithm.

In this work we introduce a geometric form for the Extended Patience Sorting Algorithm that is in some sense a natural dual algorithm to G. Viennot’s celebrated Geometric RSK Algorithm.

1. INTRODUCTION

The term *Patience Sorting* was introduced in 1962 by C.L. Mallows [4, 5] as the name of a card sorting algorithm invented by A.S.C. Ross. This algorithm works by first partitioning a shuffled deck of cards (which we take to be a permutation $\sigma \in \mathfrak{S}_n$) into its left-to-right minima subsequences (called *piles* in this context), and the method used to form these piles can be viewed as an iterated, non-recursive form of the Schensted Insertion Algorithm for interposing values into the rows of a Young tableau (see [1]). For given $\sigma \in \mathfrak{S}_n$, we call the resulting collection of piles (given as part of the more general Algorithm 1.2 below) the *pile configuration* corresponding to σ and denote it by $R(\sigma)$.

Given a pile configuration R , one can form its *reverse patience word* $RPW(R)$ by listing the piles in R “from bottom to top, left to right” (see Example 1.1 below). In recent work [3] the authors used G. Viennot’s (northeast) shadow diagram construction (defined in [7] and reviewed in Section 2.1 below) to characterize these words as being the elements of the set $S_n(3\bar{1}\text{-}42)$. That is, reverse patience words are exactly those permutations that avoid the generalized permutation pattern 2-31 unless the elements corresponding to this pattern are also contained in a 3-1-42 pattern. (Recall that by default the elements matching a generalized permutation pattern are required to be contiguous unless a dash is inserted between them. See Bóna [2] for more details regarding permutation patterns.)

Example 1.1. The permutation $\sigma = 64518723 \in \mathfrak{S}_8$ has pile configuration $R(\sigma) = \{\{6 > 4 > 1\}, \{5 > 2\}, \{8 > 7 > 3\}\}$, which we represent visually as

$$\begin{array}{ccc} 1 & & 3 \\ 4 & 2 & 7 \\ 6 & 5 & 8 \end{array}$$

Furthermore, the reverse patience word for $R(\sigma)$ is $RPW(R(64518723)) = 64152873 \in S_n(3\bar{1}\text{-}42)$.

In [3] the authors also extended the process of forming piles under Patience Sorting so that it essentially becomes a full non-recursive analog of the famous Robinson-Schensted-Knuth (or RSK) Correspondence. As with RSK, this Extended Patience Sorting Algorithm (Algorithm 1.2 below) takes a simple idea (that of placing cards into piles) and uses it to build a bijection between elements of the symmetric group \mathfrak{S}_n and certain pairs of combinatorial objects. In the case of RSK, one uses the Schensted insertion algorithm to build

Date: December 31, 2004.

2000 Mathematics Subject Classification. Primary: 05A05, 05A18; Secondary: 05E10.

Key words and phrases. sorting, patience sorting, generalized permutation patterns, shadow diagrams, Robinson-Schensted-Knuth Correspondence, Geometric RSK Algorithm.

The work of the second author was supported in part by the U.S. National Science Foundation under Grants DMS-0135345 and DMS-0304414.

a bijection with pairs of standard Young tableau having the same shape (see [6]). However, in the case of Patience Sorting, one achieves a bijection between permutations and somewhat more restricted pairs of pile configurations. In particular, these pairs must not only have the same shape but their reverse patience words must also simultaneously avoid certain pairs of generalized permutation patterns. This simultaneous pattern avoidance can also be characterized geometrically using Viennot's (northeast) shadow diagram construction for the permutation implicitly defined by a pair of pile configurations (see [3] for more details).

Viennot introduced the shadow diagram of a permutation in the context of studying an important symmetry property for RSK. In particular, one can use recursively defined shadow diagrams to construct the RSK Correspondence completely geometrically. We review this process in Section 2 below. Then in Section 3 we define a natural dual to Viennot's Geometric RSK Algorithm that similarly characterizes the Extended Patience Sorting Algorithm geometrically. We conclude in Section 4 by discussing the relationship between RSK and Extended Patience Sorting as made explicit by their geometric forms.

We close this introduction by stating the Extending Patience Sorting Algorithm and giving an example.

Algorithm 1.2 (Extended Patience Sorting Algorithm). Given a shuffled deck of cards $\sigma = c_1 c_2 \cdots c_n$, inductively build *insertion piles* $R = R(\sigma) = \{r_1, r_2, \dots, r_m\}$ and *recording piles* $S = S(\sigma) = \{s_1, s_2, \dots, s_m\}$ as follows:

- Place the first card c_1 from the deck into a pile r_1 by itself, and set $s_1 = \{1\}$.
- For each remaining card c_i ($i = 2, \dots, n$), consider the cards d_1, d_2, \dots, d_k atop the piles r_1, r_2, \dots, r_k that have already been formed.
 - If $c_i > \max\{d_1, d_2, \dots, d_k\}$, then put c_i into a new pile r_{k+1} by itself and set $s_{k+1} = \{i\}$.
 - Otherwise, find the left-most card d_j that is larger than c_i and put the card c_i atop pile r_j while simultaneously putting i at the bottom of pile s_j .

Example 1.3. Let $\sigma = 64518723 \in \mathfrak{S}_8$. Then according to Algorithm 1.2 we simultaneously form the following pile configurations:

	insertion piles	recording piles		insertion piles	recording piles
Form a new pile with 6 :	6	1	Then play the 4 on it:	4 6	1 2
Form a new pile with 5 :	4 6 5	1 2 3	Add the 1 to left pile:	1 4 6 5	1 2 4 3
Form a new pile with 8 :	1 4 6 5 8	1 2 4 3 5	Then play the 7 on it:	1 4 7 6 5 8	1 2 5 4 3 6
Add the 2 to a pile:	1 4 2 7 6 5 8	1 2 3 5 4 7 6	Add the 3 to a pile:	1 3 4 2 7 6 5 8	1 5 2 3 6 4 7 9

The idea behind Algorithm 1.2 is that we are using the recording piles $S(\sigma)$ to implicitly label the order in which the elements of the permutation σ are added to the insertion piles $R(\sigma)$. It is clear that this

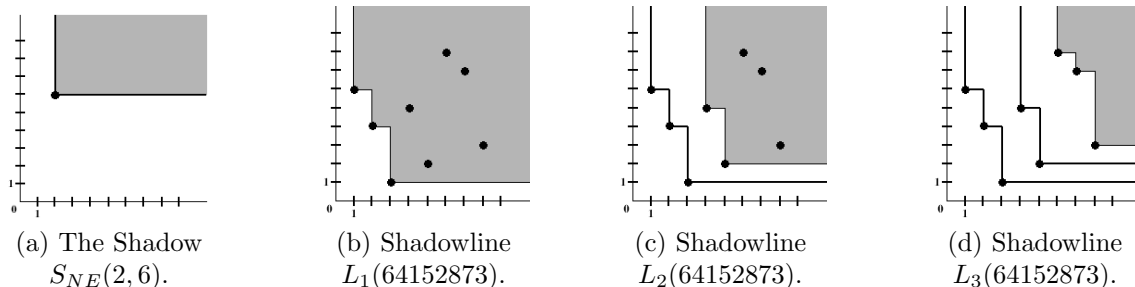


FIGURE 2.1. Examples of Northeast Shadow and Shadowline Constructions

information then allows us to uniquely reconstruct σ by reversing the order in which the cards were played. However, even though reversing the Extended Patience Sorting Algorithm is much easier than reversing the RSK Algorithm through recursive “reverse row bumping,” the trade-off is that the pairs of pile configurations that result from the Extended Patience Sorting Algorithm are not independent whereas the standard Young tableau pairs generated by RSK are completely independent (up to shape).

2. NORTHEAST SHADOW DIAGRAMS AND VIENNOT’S GEOMETRIC RSK

In this section we briefly develop Viennot’s geometric form for RSK in order to motivate the geometric form for the Extended Patience Sorting that is introduced in Section 3 below.

2.1. The Northeast Shadow Diagram of a Permutation. We begin with the following fundamental definition:

Definition 2.1. Given a lattice point $(m, n) \in \mathbb{Z}^2$, we define the *northeast shadow* of (m, n) to be the quarter space $S_{NE}(m, n) = \{(x, y) \in \mathbb{R}^2 \mid x \geq m, y \geq n\}$.

See Figure 2.1(a) for an example of a point’s northeast shadow.

The most important use of these shadows is in building so-called northeast shadowlines:

Definition 2.2. Given lattice points $(m_1, n_1), (m_2, n_2), \dots, (m_k, n_k) \in \mathbb{Z}^2$, we define their *northeast shadowline* to be the boundary of the quarter space formed by taking the unions of the northeast shadows $S_{NE}(m_1, n_1), S_{NE}(m_2, n_2), \dots, S_{NE}(m_k, n_k)$.

In particular, we wish to associate to each permutation a certain collection of northeast shadowlines (as illustrated in Figure 2.1(b)–(d)):

Definition 2.3. Given a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$, the *northeast shadow diagram* $D_{NE}(\sigma)$ of σ consists of the shadowlines $L_1(\sigma), L_2(\sigma), \dots, L_k(\sigma)$ formed as follows:

- $L_1(\sigma)$ is the northeast shadowline for the lattice points $\{(1, \sigma_1), (2, \sigma_2), \dots, (n, \sigma_n)\}$.
- While at least one of the points $(1, \sigma_1), (2, \sigma_2), \dots, (n, \sigma_n)$ is not contained in the shadowlines $L_1(\sigma), L_2(\sigma), \dots, L_j(\sigma)$, define $L_{j+1}(\sigma)$ to be the northeast shadowline for the points

$$\{(i, \sigma_i) \mid (i, \sigma_i) \notin \bigcup_{k=1}^j L_k(\sigma)\}.$$

In other words, we define the shadow diagram inductively by taking $L_1(\sigma)$ to be the shadowline for the diagram $\{(1, \sigma_1), (2, \sigma_2), \dots, (n, \sigma_n)\}$ of the permutation. Then we ignore the points whose shadows were actually used in building $L_1(\sigma)$ and define $L_2(\sigma)$ to be the shadowline of the resulting subset of the permutation’s diagram. We then build $L_3(\sigma)$ as the shadowline for the points not yet used in constructing both $L_1(\sigma)$ and $L_2(\sigma)$, and this process continues until all points in the permutation’s diagram are exhausted.

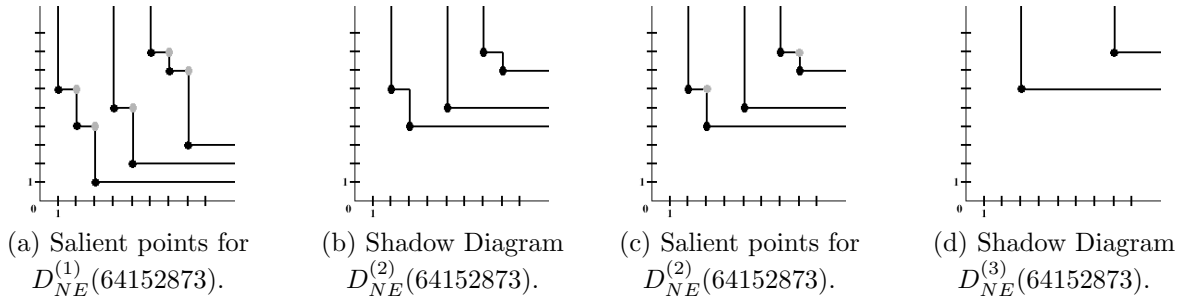


FIGURE 2.2. The northeast shadow diagrams for 64152873.

We can characterize the points whose shadows define the shadowlines at each stage as follows: they are the smallest collection of unused points whose shadows collectively contain all other remaining unused points (and hence, the shadows of those points). As a consequence of this shadow containment property, the shadowlines in a northeast shadow diagram will never cross. However, as we will see in Section 3.1 below, the dual construction to Definition 2.3 that is introduced will allow for crossing shadowlines. This distinction is caused by essentially reversing the above shadow containment property for the points used to define each shadowline.

2.2. Viennot's Geometric RSK Algorithm. As simple as northeast shadowlines were to define in the previous section, a great deal of information can still be gotten from them. One of the most basic properties of the northeast shadow diagram $D_{NE}^{(1)}(\sigma) = D_{NE}(\sigma)$ for a permutation $\sigma \in \mathfrak{S}_n$ is that it encodes the top row of the RSK insertion tableau $P(\sigma)$ (resp. recording tableau $Q(\sigma)$) as the smallest ordinates (resp. smallest abscissae) of all points belonging to the constituent shadowlines $L_1(\sigma), L_2(\sigma), \dots, L_k(\sigma)$. One proves this by comparing the use of Schensted insertion on the top row of the insertion tableau with the intersection of the vertical lines $x = a$ as a increases from 0 to n . (See Sagan [6].)

Remarkably, one can then use the northeast corners (called the *salient points*) of $D_{NE}^{(1)}(\sigma)$ to form a new shadow diagram $D_{NE}^{(2)}(\sigma)$ that similarly gives the second rows of $P(\sigma)$ and $Q(\sigma)$. Then inductively the salient points of $D_{NE}^{(2)}(\sigma)$ can be used to give the third rows of $P(\sigma)$ and $Q(\sigma)$, and so on. As such, one can view this recursive formation of shadow diagrams as a geometric form for the RSK correspondence. We illustrate this process in Figure 2.2.

3. SOUTHWEST SHADOW DIAGRAMS AND GEOMETRIC PATIENCE SORTING

In this section we introduce a very natural dual algorithm to Viennot's geometric form for RSK as given in Section 2.2 above.

3.1. The Southwest Shadow Diagram of a Permutation. As in Section 2.1 above, we begin with the following fundamental definition:

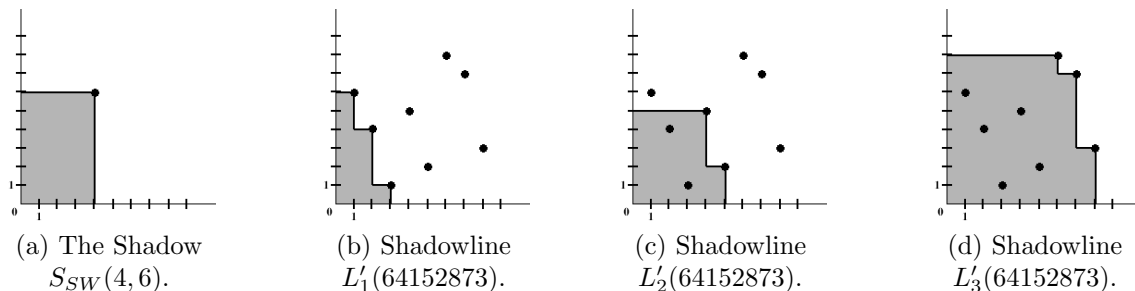


FIGURE 3.1. Examples of Southwest Shadow and Shadowline Constructions

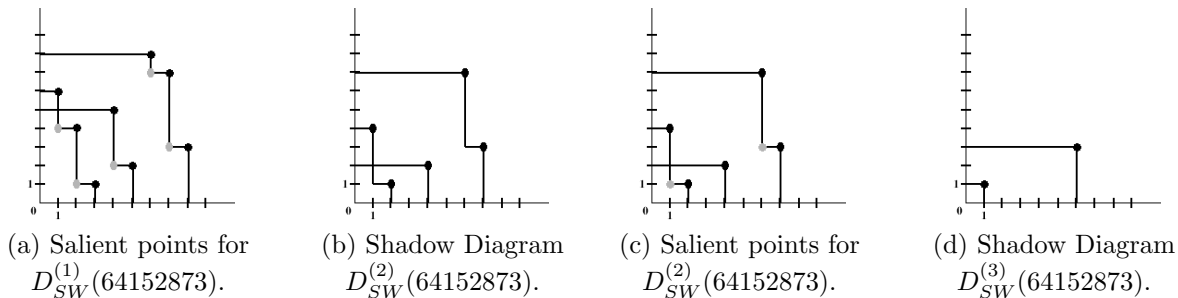


FIGURE 3.2. The southwest shadow diagrams for 64152873.

Definition 3.1. Given a lattice point $(m, n) \in \mathbb{Z}^2$, we define the *southwest shadow* of (m, n) to be the quarter space $S_{SW}(m, n) = \{(x, y) \in \mathbb{R}^2 \mid x \leq m, y \leq n\}$.

See Figure 3.1(a) for an example of a point's southwest shadow.

As with their northeast counterparts, the most important use of these shadows is in building southwest shadowlines:

Definition 3.2. Given lattice points $(m_1, n_1), (m_2, n_2), \dots, (m_k, n_k) \in \mathbb{Z}^2$, we define their *southwest shadowline* to be the boundary of the unions of the shadows $S_{SW}(m_1, n_1), S_{SW}(m_2, n_2), \dots, S_{SW}(m_k, n_k)$.

In particular, we wish to associate to each permutation a certain collection of southwest shadowlines. However, unlike the northeast case, these shadowlines sometimes cross (as illustrated in Figure 3.1(b)–(d)):

Definition 3.3. Given a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$, the *southwest shadow diagram* $D_{SW}(\sigma)$ of σ consists of the southwest shadowlines $L'_1(\sigma), L'_2(\sigma), \dots, L'_k(\sigma)$ formed as follows:

- $L'_1(\sigma)$ is the shadowline for those lattice points $(x, y) \in \{(1, \sigma_1), (2, \sigma_2), \dots, (n, \sigma_n)\}$ such that $S_{SW}(x, y)$ does not contain any other lattice points (and hence, does not completely contain the southwest shadow of any other lattice points).
- While at least one of the points $(1, \sigma_1), (2, \sigma_2), \dots, (n, \sigma_n)$ is not contained in the shadowlines $L'_1(\sigma), L'_2(\sigma), \dots, L'_j(\sigma)$, define $L'_{j+1}(\sigma)$ to be the shadowline for the points

$$(x, y) \in \{(i, \sigma_i) \mid (i, \sigma_i) \notin \bigcup_{k=1}^j L'_k(\sigma)\}$$

such that $S_{SW}(x, y)$ does not contain any other lattice points in the same set (and hence, does not completely contain the southwest shadow of any of those points).

In other words, we again define a shadow diagram by inductively eliminating certain points in the permutation's diagram until every point has been used to define a shadowline. However, we are here reversing both the direction of the shadows and the shadow containment property from the northeast case. It is in this sense that the geometric form for the Extended Patience Sorting Algorithm given in the next section can be viewed as “dual” to Viennot's geometric form for RSK.

3.2. The Geometric Patience Sorting Algorithm. As in Section 2.2, one can produce a sequence $D_{SW}(\sigma) = D_{SW}^{(1)}(\sigma), D_{SW}^{(2)}(\sigma), \dots$, of shadow diagrams for a given permutation $\sigma \in \mathfrak{S}_n$ by recursively applying Definition 3.3 to salient points. Here, however, salient points are defined to be the southwest corner points of a given set of shadowlines. See Figure 3.2 for an example of how this works.

Moreover, the resulting sequence of shadow diagrams can then be used to reproduce the pair of pile configurations given by the Extended Patience Sorting Algorithm (Algorithm 1.2). To accomplish this, index the cards in a pile configuration using the French convention for tableaux so that the row index increases from bottom to top and the column index from left to right. (I.e., we are labelling boxes as we

would lattice points in the first quadrant of \mathbb{R}^2). Then, for a given permutation $\sigma \in \mathfrak{S}_n$, the elements of the i th row of the insertion piles $R(\sigma)$ (resp. recording piles $S(\sigma)$) are given by the largest ordinates (resp. abscissae) of the shadowlines that compose $D_{SW}^{(i)}$.

The main difference between this process and Viennot’s Geometric RSK is that care must be taken to assemble each row in its proper order. Unlike the entries of a Young tableau, the elements in the rows of a pile configuration do not necessarily increase from left to right. As such, the components of each row should be recorded in the order that the shadowlines are formed. The rows can then uniquely be assembled into a legal pile configuration since the elements in the columns of a pile configuration must both decrease and appear in the left-most pile possible.

The proof that this works is along the same lines as that of Viennot’s geometric RSK. Namely, one views the shadowlines as a visual record for how cards are played atop each other by intersecting the shadow diagrams $D_{SW}^{(1)}(\sigma)$, $D_{SW}^{(2)}(\sigma)$, \dots , with vertical lines of the form $x = a$ as a increases from zero to n .

4. COMPARING GEOMETRIC PATIENCE SORTING WITH GEOMETRIC RSK

Recall that for a partial permutation $\pi = \pi_1\pi_2\cdots\pi_l$, the *left-to-right minima subsequence* (AKA basic subsequence or records) of π consists of those $\pi_j = \min\{\pi_i \mid 1 \leq i \leq j\}$. We then inductively define the *left-to-right minima subsequences* s_1, s_2, \dots, s_k of a permutation $\sigma \in \mathfrak{S}_n$ by taking s_1 to be the left-to-right minima subsequence for σ itself and then s_i to be the left-to-right minima subsequence for the partial permutation obtained by removing the elements contained in s_1, s_2, \dots, s_{i-1} from σ .

Based upon the order in which northeast and southwest shadowlines are formed for a given permutation $\sigma \in \mathfrak{S}_n$, it is relatively clear that the left-to-right minima subsequences of σ are exactly the points connected with shadowlines in both $D_{NE}^{(1)}(\sigma)$ and $D_{SW}^{(1)}(\sigma)$. Thus, the main difference between the geometric algorithms given in Sections 2.2 and 3.2 lies in how the salient points of these two shadow diagrams are formed. For example, the order of the abscissae for these salient points are reversed in permutations like $2431 \in \mathfrak{S}_4$, and such reversals seem to symbolize the inherent philosophical difference between RSK and the Extended Patience Sorting Algorithm. In particular, as playing a card atop a pre-existing pile under Patience Sorting is essentially like non-recursive Schensted Insertion, certain particularly egregious “double bumps” that occur under the Schensted insertion algorithm prove to be too complicated to be properly modelled by the “static insertions” of Patience Sorting.

At the same time, it is also easy to see that for a given $\sigma \in \mathfrak{S}_n$, the cards atop the piles in the pile configurations $R(\sigma)$ and $S(\sigma)$ (as given by Algorithm 1.2) are exactly the cards in the top rows of the RSK insertion tableau $P(\sigma)$ and recording tableau $Q(\sigma)$, respectively. Thus, this raises the question of when the remaining rows of $P(\sigma)$ and $Q(\sigma)$ can likewise be recovered from $R(\sigma)$ and $S(\sigma)$. While this appears to be related to things like the salient point abscissae order reversal for 2431 , one would ultimately hope to characterize the answer in terms of generalized pattern avoidance similar to the description of reverse patience words for pile configurations.

REFERENCES

- [1] D. Aldous and P. Diaconis. “Longest Increasing Subsequences: From Patience Sorting to the Baik-Deift-Johansson Theorem”, *Bull. Amer. Math. Soc.* **36** (1999), 413–432. Available online at <http://www.ams.org/bull/1999-36-04/>
- [2] M. Bóna. *Combinatorics of Permutations*. Chapman & Hall/CRC Press, 2004.
- [3] A. Burstein and I. Lankham. “Combinatorics of Patience Sorting Piles.” Submitted to FPSAC 2005.
- [4] C. L. Mallows. “Problem 62-2, Patience Sorting”. *SIAM Review* **4** (1962), 148–149.
- [5] C. L. Mallows. “Problem 62-2”. *SIAM Review* **5** (1963), 375–376.
- [6] B. Sagan. *The Symmetric Group, Second Edition*. Graduate Texts in Mathematics 203. Springer-Verlag, 2000.
- [7] G. Viennot. “Une forme géométrique de la correspondance de Robinson-Schensted”, in *Combinatoire et Représentation du Groupe Symétrique*, D. Foata, ed. Lecture Notes in Mathematics 579. Springer-Verlag, 1977, pp. 29–58.

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