PARTIALLY ORDERED GENERALIZED PATTERNS AND THEIR COMBINATORIAL INTERPRETATIONS

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Abstract. This paper is a continuation of the study of partially ordered generalized patterns (POGPs) considered in [6, 7, 8]. We provide two general approaches: one to obtain connections between restricted permutations and other combinatorial structures, and one to treat the avoidance problems for words in case of segmented patterns. The former approach is related to coding combinatorial objects in terms of restricted permutations. We provide several examples of relations of our objects to other combinatorial structures, such as labeled graphs, walks, binary vectors, and others. Also, we show how restricted permutations are related to Cartesian products of certain objects.

Keywords: pattern avoidance, segmented patterns, permutations, words, walks, labeled general graphs, binary vectors, coding

1. Introduction

We write permutations as words $\pi = a_1a_2\cdots a_n$, whose letters are distinct and usually consist of the integers $1, 2, \ldots, n$. An occurrence of a *pattern* τ in a permutation π is "classically" defined as a subsequence in π (of the same length as τ) whose letters are in the same relative order as those in τ.

Generalized permutation patterns (*GPs*) allow the requirement that two adjacent letters in a pattern must be adjacent in the permutation. In order to avoid confusion we write a "classical" pattern, say 231, as 2-3-1, and if we write, say 2-31, then we mean that if this pattern occurs in a permutation π, then the letters in π that correspond to 3 and 1 are adjacent.

In [7], a further generalization of GPs was introduced, namely *partially ordered generalized patterns* (*POGPs*). A POGP is a GP some of whose letters are incomparable. For instance, if we write $p = 1\text{-}1'2'$ then we mean that in an occurrence of p in a permutation π the letter corresponding to the 1 in p can be either larger or smaller than the letters corresponding to $1'2'$. Thus, the permutation 31254 has three occurrences of p, namely 3-12, 3-25, and 1-25.

We refer to [2, 9] for a motivation to study GPs, POGPs, and classical patterns.

A POGP with no dashes is called a *segmented POGP (SPOGP)*.

In this paper we continue the study of POGPs considered in [6], [7], and [8]. Sections 2 and 3 are devoted to POGPs in permutations, which is the main concern of the paper.

Let $\mathcal{D}_n$ and $\mathcal{M}_n$ denote the set of *Dyck paths* of length $2n$ and *Motzkin paths* of length n respectively. What could be a natural combinatorial interpretation for, say, the set $\mathcal{D}_{i_1} \times \mathcal{M}_{i_2} \times \mathcal{M}_{n-i_1-i_2}$? It turns out that such a combinatorial interpretation in these and many other cases is given by a set of (n-)permutations simultaneously avoiding certain sets of POGPs. In section 3 we explain this phenomenon as well as suggest a general approach for looking for different connections between restricted permutations and other combinatorial objects. This direction is related to coding combinatorial objects in terms of (POGP-)restricted permutations.

Based on results in [4], in section 4 we discuss a general approach for treating avoidance problems for segmented patterns in words, that is patterns whose occurrences in words form contiguous subwords of the words of the corresponding lengths.

Throughout the paper we assume that A_n (resp. $A(x)$, $G(x)$) denotes the number (resp. the exponential and ordinary generating functions for the number) of permutations that avoid a pattern or a set of patterns under consideration.

2. Multi-avoidance of POGPs in permutations

In this section we give some relations between multi-avoidance of POGPs and other combinatorial objects such as certain walks and labeled general graphs.

2.1. **4-SPOGPs and walks.**

In [7] a bijection was given between the set of $(n+1)$-permutations avoiding the SPOGP $12'21'$ and the set of walks of n unit steps between lattice points, each in a direction N, S, E or W, starting from the origin and remaining in the positive quadrant. Proposition 2.2 below establishes a connection between certain 1-dimensional walks and permutations avoiding the SPOGPs $11'22'$ and $22'11'$ simultaneously.

Proposition 2.1. *For the set of patterns* $\{11'22', 22'11'\}$ *and* $n \geq 3$, $A_n = 2\binom{n}{\lfloor n/2 \rfloor}$.

Proposition 2.2. *Let* $k \geq 0$ *be an integer. For* $n \geq 3$*, there is a bijection between the set of all* n*-permutations avoiding simultaneously the patterns* $11'22'$ *and* $22'11'$*, and the set of all* $(n+k)$*-step walks on the* x*-axis with the steps* $a = (1,0)$ *and* $\bar{a} = (-1,0)$ *starting from the origin but not returning closer than* k *units to it. That is, for* $k \geq 1$*, the points* $(0,0)$, $(\pm 1, 0)$, $(\pm 2, 0), \ldots, (\pm(k-1), 0)$ *may be visited only once by a legal walk, whereas for* $k = 0$*, a legal walk is not allowed to cross the origin, it is only allowed to touch it.*

2.2. **4-SPOGPs and semi-alternating permutations.**

In this section we consider the set of permutations avoiding the pair of patterns $(121'2', 212'1')$. Since this set of patterns is closed under reversal and complementation, the set $S_n(121'2', 212'1')$ is also closed under reversal and complementation.

The permutations $\pi \in S_n(121'2', 212'1')$ are characterized as follows: $\pi \in S_n(121'2', 212'1')$ if and only if for each $t \geq 1$, $\pi(t) > \pi(t+1)$ exactly when $\pi(t+2) < \pi(t+3)$ and $\pi(t) < \pi(t+1)$ exactly when $\pi(t+2) > \pi(t+3)$. Hence, descents and ascents of π alternate at exactly every other position starting with position 2 or 3, so we call such π *semi-alternating*.

The exponential generating function for $|S_n(121'2', 212'1')|$ is given by

$$2(u+v+2w+2x+y+z)-3t-1 = 4e^Y + \int_0^t e^{2Y(s)}ds + 2y + 2\int_0^t y(s)^2 ds - 2t - 1,$$

where $Y(t)=\int_0^t y(s)ds$, $u=\frac{1}{2}\int_0^t \sinh(2Y(s))ds$, $v=\frac{1}{2}t+\frac{1}{2}\int_0^t \cosh(2Y(s))ds$, $w=\cosh(Y)$, $x=\sinh(Y)$, $z=\int_0^t y(s)^2 ds$, $x'=v+xz$, $y'=w+yz$, $z'=2x+z^2$, and the first three derivatives of $u(t)$, $v(t)$, $w(t)$, $x(t)$, $y(t)$, and $z(t)$ at $t=0$ are $(0,0,0,1)$, $(0,1,0,0)$, $(1,0,0,0)$, $(0,0,1,0)$, $(0,1,0,0)$, and $(0,0,0,2)$ respectively.

2.3. **POGPs and general graphs.** A *general graph* (*pseudograph*) is a graph in which both graph loops and multiple edges are permitted. In proposition 2.3 we use Babson-Steingrímsson notation, where "[" in $p=[xy\cdots$ means that an occurrence of p in a permutation must begin from the leftmost letter of the permutation.

Proposition 2.3. *For $n\geq 0$, there is a bijection between the set of $(n+3)$-permutations simultaneously avoiding the SPOGPs $p_1=121'2'$, $p_2=211'2'$, $p_3=123$, $p_4=[21'1$ and $p_5=[1'21$ and the set of arbitrary general graphs on two labeled nodes and n edges.*

Let $A_{n,k}$ be the $k\times k$ adjacency matrix of a graph on k labeled nodes and n edges (multiple edges and loops are allowed). We assign labels $1,2,\ldots,k^2$ to the entries of $A_{n,k}$ by reading the matrix from left to right and from top to bottom.

We define a class $\mathcal{C}_{n,k}(a_1,a_2,\ldots,a_\ell)$ of graphs by indicating $k^2-\ell$ entries of $A_{n,k}$ that must be 0 (the other entries, having labels a_1, $a_2,\ldots,a_\ell$, may or may not be 0).

Definition 2.4. *Suppose $p=a_1a_2\cdots a_k$ is a permutation and, for fixed non-negative integers $\ell_1,\ell_2,\ldots,\ell_{k-1}$, the letters $b^{(i,j)}$, $1\leq i\leq k-1$, $1\leq j=j(i)\leq \ell_i$, are incomparable with each other and with the a_i's, $1\leq i\leq k$. We call the SPOGP*

$$a_1b^{(1,1)}b^{(1,2)}\cdots b^{(1,\ell_1)}a_2b^{(2,1)}b^{(2,2)}\cdots b^{(2,\ell_2)}a_3\cdots a_{k-1}b^{(k-1,1)}b^{(k-1,2)}\cdots b^{(k-1,\ell_{k-1})}a_k$$

separated segmented POGP (SSPOGP)*. For the SSPOGP above we use the notation*

$$\tau_k(\ell_1,\ell_2,\ldots,\ell_{k-1}) = a_1|_{\ell_1}a_2|_{\ell_2}a_3\cdots a_{k-1}|_{\ell_{k-1}}a_k.$$

We use "$|$" instead of "$|_1$".

SSPOGPs were introduced in [7]. These patterns allow us to control the distance between certain letters in permutations and we use this property in theorem 2.5. If we write, say, $p=[|_t xy$, then we mean that an occurrence of the pattern p in a permutation must start with the leftmost letter of the permutation, and the first t letters of the permutation can be arbitrary, while the relative order of x and y must be preserved.

Let P' be a set of SPOGPs (or rather SSPOGPs) $\{[|_i 2|_{\ell-i-2}1\}_{0\leq i\leq \ell-2}$. We define P to be $P'\cup\{p_1=|_{\ell-1}12,\ p_2=12\text{-}3\}$.

Theorem 2.5. *Let $\mathcal{C}_{n,k}(a_1,a_2,\ldots,a_\ell)$ and P be as defined above. There is a bijection between the set of graphs $\mathcal{C}_{n,k}(a_1,a_2,\ldots,a_\ell)$ and the set of $(n+\ell)$-permutations avoiding simultaneously the patterns from P.*

Corollary 2.6. *There is a bijection between the set of all graphs on k nodes with n edges and the set of $(n+k^2)$-permutations that avoid simultaneously all the patterns from the set* $\{[|_i 2|_{k^2-i-2}1\}_{0\leq i\leq k^2-2} \cup \{|_{k^2-1}12,\ 12\text{-}3\}$.

3. A general approach for looking for connections between restricted permutations and other combinatorial objects

3.1. **A general approach.** A standard approach to find relations between restricted permutations and other combinatorial objects seems to be as follows. First one considers a particular set of patterns P; then one either finds a formula for, say, a number of permutations avoiding P or makes a (computer) experiment to find initial values of the number of permutations avoiding P; thereafter one may check if the numbers appear in [10], which might establish relations to other combinatorial objects.

We suggest to start from consideration of a structure of permutations that are supposed to avoid some set of patterns. The idea is to consider those structures that can be "controlled," that is, for which we can find a set of patterns that force our permutations to have the prescribed structure. Moreover, in order to increase the probability of obtaining relations to other objects, we may try to make the number of permutations having our structure be of a "nice" form, like expressions containing binomial coefficients, or, say, powers of 2, etc. As the next step we can either go to [10], or we may recognize the formula involved to count objects known to us. Then we try to find a bijection between our object and the other one.

3.2. **Cartesian products of combinatorial objects and restricted permutations.** Let $\mathcal{O}_n$ be the set of all (natural) combinatorial objects of size associated with n and having interpretation in terms of restricted permutations. For instance, Motzkin paths $\mathcal{M}_n$ of length n, as well as Dyck paths $\mathcal{D}_n$ of length $2n$ are elements of $\mathcal{O}_n$ since the former is related to the n-permutations avoiding the GPs 1-23 and 13-2, whereas the latter is related to n-permutations avoiding the GP 2-13 (see [3]). There are many other combinatorial objects including different types of labeled trees appearing in $\mathcal{O}_n$. Let $\mathcal{O} = \cup_{n\geq 0}\mathcal{O}_n$.

Theorem 3.1. *Suppose $A_{n_i} \in \mathcal{O}_{n_i}$ for $i = 1, 2, \ldots, k$, and $n = \sum_i n_i$. Then*

$$A_{n_1} \times A_{n_2} \times \cdots \times A_{n_k} \in \mathcal{O}_n \subset \mathcal{O}.$$

In other words, the Cartesian product of combinatorial objects related to restricted permutations has an interpretation in terms of restricted permutations.

Corollary 3.2. *The cardinality of $\mathcal{O}$ is at least continuum.*

3.3. **Examples of applying the approach.** For each of the permutation structures considered in this subsection, there is an explicit set of prohibited patterns giving these structures.

Let $\mathcal{S}_1(i, j, n+1)$ denote the set of $(n+1)$-permutations having the structure $AxB(n+1)C$, where A, B, and C are decreasing, x is the largest letter in AxB, $|AxB| = i$, and $|A| = j$.

The set $\mathcal{S}_2(k, n)$ is obtained by prohibiting the patterns $|_k$-132 and $|_k$-231 in n-permutations.

We define the set $\mathcal{S}_3(i, j, n+2)$ to be the set of $(n+2)$-permutations of the form $A1B(n+2)C$ where $|A| = i$, $|B| = j$, A and B are decreasing, C avoids 132 and 231.

Let $\mathcal{S}_4(k, n+k)$ be the set of $(n+k)$-permutations having the structure AB, where $|A| = k$, any letter in A is greater than any letter in B, A avoids 123, 132, and 231, and B avoids 1-2-3 and 2-3-1.

Set of permutations	Formula	Sequence in [10]	Combinatorial interpretation
$\mathcal{S}_1(2,0,n+1)$	$\binom{n}{2}$	A000292	$(n+3)$-permutations avoiding 1-3-2-4 and having exactly one descent (a descent in a permutation π is an i such that $\pi(i) > \pi(i+1)$).
$\mathcal{S}_1(3,1,n+1)$	$2\binom{n}{3}$	A007290	Acute triangles made from the vertices of a regular n-polygon.
$\mathcal{S}_2(1,n)$	$n2^{n-2}$	A057711	Number of 1's in all palindromic compositions of $N = 2(n-1)$. E.g., there are 5 palindromic compositions of 6, namely 111111, 11211, 2112, 1221, and 141, containing a total of 16 1's.
$\mathcal{S}_3(1,1,n+2)$	$\binom{n}{2}2^{n-1}$	A001815	$(n+3)$-permutations containing 1-3-2 and 1-2-3 exactly once.
$S_4(2,n+2)$	n^2-n+2	A014206	Binary bitonic sequences of length n (a bitonic sequence is $a_1 \leq a_2 \leq \cdots \leq a_h \geq a_{h+1} \geq \cdots \geq a_{n-1} \geq a_n$ or $a_1 \geq a_2 \geq \cdots \geq a_h \leq a_{h+1} \leq \cdots \leq a_{n-1} \leq a_n$).

Table 1. Examples of relations between restricted permutations and other combinatorial objects

4. Avoidance of SPOGPs in words – a general approach

Any problem on (multi-)avoidance of any set of segmented GPs or, more generally, POGPs in words over $[k]$ is algorithmically solvable. This observation does not seem to be made in the papers in this direction, although it is natural due to work in [4].

Indeed, suppose we count the number of words over $[k]$ that avoid SPOGPs (in particular, segmented GPs) from a finite set $P = \{P_1, P_2, \ldots\}$. Clearly, since k is a fixed number, we can write explicitly the (finite) set of prohibited words $S = \{S_1, S_2, \ldots\}$ corresponding to P. Then we apply a result from [4] (see theorem 4.1 below) to answer the original question.

4.1. **Review of a fundamental result.** If $X_1 = a_0a_1\ldots a_{m-1}$ and $X_2 = b_0b_1\ldots b_{\ell-1}$ are two words, over some alphabet, of length m and ℓ respectively, then the *correlation* $c_{12} = c_0c_1\ldots c_{m-1}$ is the binary string obtained by setting $c_j = 1$ if $a_i = b_{i+j}$ for every $1 \leq i \leq m-1-j$, and $c_j = 0$ otherwise. The *autocorrelation* of a word X_1 is just c_{11}, the correlation of X_1 with itself. This is convenient to interpret a correlation $c_{ij} = c_0c_1\ldots c_{m-1}$ as a polynomial $c_{ij}(x) = c_0 + c_1x + \cdots + c_{m-1}x^{m-1}$.

The following theorem can be obtained from the considerations in [4] and is similar to that in [11, Th. 24] in binary case.

Theorem 4.1. *The generating function for the number of words over* $[k]$ *that avoid the subwords* $b_1, b_2, \ldots, b_n$, *of length* $\ell_1, \ell_2, \ldots, \ell_n$ *respectively, none included in any other, is given by the formula*

$$S(x) = \begin{vmatrix} c_{11}(x) & \cdots & c_{1n}(x) \\ \vdots & \ddots & \vdots \\ c_{n1}(x) & \cdots & c_{nn}(x) \end{vmatrix} \Bigg/ \begin{vmatrix} 1-kx & 1 & \cdots & 1 \\ -x^{\ell_1} & c_{11}(x) & \cdots & c_{1n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ -x^{\ell_n} & c_{n1}(x) & \cdots & c_{nn}(x) \end{vmatrix}. \qquad (1)$$

4.2. **Corollaries to Theorem 4.1.** Recall definition 2.4.

Proposition 4.2. *Let* $\tau \in S_k$ *be a segmented GP and* $\tau = \tau_1\tau_2$, *where* τ_1 *and* τ_2 *are non-empty. Assuming* $m < \max\{|\tau_1|, |\tau_2|\}$, *for SSPOGP* $\sigma = \tau_1|_m\tau_2$ *and the words over* $[k]$, *we have* $G(x) = (1 - kx + k^m x^{k+m})^{-1}$.

Proposition 4.3. *Suppose* $p = 1^{\ell_1}(1')^{\ell_2}2^{\ell_3}$ *and* $\ell_1, \ell_2, \ell_3 > 0$. *Let* $\ell = \ell_1 + \ell_2 + \ell_3$. *For binary alphabet, we have that* $G(x) = (1 - 2x + 2x^\ell)^{-1}$.

References

[1] E. Babson and E. Steingrímsson, Generalized permutation patterns and a classification of the Mahonian statistics, *Séminaire Lotharingien de Combinatoire*, B44b:18pp, 2000.
[2] M. Bóna, *Combinatorics of Permutations*, Chapman & Hall/CRC, NY, 2004.
[3] A. Claesson, Generalised pattern avoidance, *Europ. J. Combin.* **22** (2001), 961–971.
[4] L. J. Guibas and A. M. Odlyzko, String overlaps, patterns matching, and nontransitive games, *J. Comb. Theory Ser. A* **30** (1981), 19–42.
[5] S. Kitaev, Multi-avoidance of generalised patterns, *Discrete Math.* **260** (2003), 89–100.
[6] S. Kitaev, Partially ordered generalized patterns, *Discrete Math.*, to appear.
[7] S. Kitaev, Segmented partially ordered generalized patterns, preprint.
[8] S. Kitaev and T. Mansour, Partially ordered generalized patterns and k-ary words, *Ann. Combin.* **7** (2003), 191–200.
[9] S. Kitaev and T. Mansour, A survey on certain pattern problems, University of Kentucky, research report 2003–09.
[10] N.J.A. Sloane and S. Plouffe, *The Encyclopedia of Integer Sequences*, available online at `http://www.research.att.com/~njas/sequences/`.
[11] B. Winterfjord, Binary strings and substring avoidance, Master thesis, CTH and Göteborg University, 2002.