Sorting classes – Extended Abstract

M. H. Albert* R. E. L. Aldred† M. D. Atkinson*
H. P. van Ditmarsch* C. C. Handley* D. A. Holton†
D. J. McCaughan†

Abstract

Weak and strong sorting classes are pattern-closed classes that are also closed downwards under the weak and strong orders on permutations. They are studied using partial orders that capture both the subpermutation order and the weak or strong order. In both cases they can be characterised by forbidden permutations in the appropriate order. The connection with the corresponding forbidden permutations in pattern-closed classes is explored. Enumerative results are found in both cases.

1 Introduction

The theory of pattern-closed classes was originally motivated by the study of the sortable permutations associated with various computing devices (abstract data types such as stacks and deques [5], token passing networks [3], or hardware switches [2]). All these devices have the property that, if they are able to sort a sequence σ, then they are able to sort any subsequence of σ.

This *subsequence* property (that subsequences of sortable sequences are themselves sortable) is a very natural one to postulate of a sorting device. It is exactly this property that guarantees that the set of sortable permutations is closed under taking subpermutations. But there are other natural properties that a sorting device might have. We are particularly interested in the following two. Both of them reflect the idea that "more sorted" versions of sortable sequences should themselves be sortable.

1. If $s_1 s_2 \ldots s_n$ is sortable and $s_i > s_{i+1}$ then $s_1 s_2 \ldots s_{i-1} s_{i+1} s_i \ldots s_n$ is sortable, and

*Department of Computer Science, University of Otago
†Department of Mathematics and Statistics, University of Otago

2. If $s_1 s_2 \ldots s_n$ is sortable and $s_i > s_j$ where $i < j$ then so is

$$s_1 s_2 \ldots s_{i-1} s_j s_{i+1} \ldots s_{j-1} s_i s_{j+1} \ldots s_n$$

For the moment we call these the *weak* and *strong exchange* properties (the second obviously implies the first). The weak exchange property would hold for sorting devices that operated by exchanging adjacent out of order pairs while the strong exchange property would hold if arbitrary out of order pairs could be exchanged. Our paper is about the interaction between each of these properties and the subsequence property.

We shall study this interaction using various (partial) orders on the set Ω of all (finite) permutations. Since we shall be considering several partial orders on Ω we shall write $\sigma \ \mathcal{P} \ \tau$ when we mean that $\sigma \leq \tau$ in the partial order $\mathcal{P}$; this avoids the confusion of the symbol "$\leq$" being adorned by various subscripts. In the same vein we write $\sigma \ \overline{\mathcal{P}} \ \tau$ to mean $\sigma \not\leq \tau$ in $\mathcal{P}$.

If $\mathcal{P}$ is a partial order on Ω the *lower ideals* of $\mathcal{P}$ are those subsets X of Ω with the property

$$\beta \in X \text{ and } \alpha \ \mathcal{P} \ \beta \Longrightarrow \alpha \in X$$

Such a lower ideal can be studied through the set $b(X)$ of minimal permutations of $\Omega \setminus X$. Obviously $b(X)$ determines X uniquely since

$$X = \{\beta \mid \alpha \ \overline{\mathcal{P}} \ \beta \text{ for all } \alpha \in b(X)\}$$

In the classical study of permutation patterns we use the subpermutation order that we denote by $\mathcal{I}$ (standing for involvement). The lower ideals of $\mathcal{I}$ are generally the central objects of study and are called *closed classes*. If X is a closed class then $b(X)$ is called the *basis* of X. Indeed the most common way of describing a closed class is by giving its basis (and therefore defining it by avoided patterns). We write $av(B)$ to denote the set of permutations which avoid all the permutations of the set B. If a closed class is not given in this way then, often, the first question is to determine the basis. A second question, perhaps of even greater interest, is to enumerate the class; in other words, to determine by formula, recurrence or generating function how many permutations it has of each length.

However, these questions can be posed for *any* partial order on Ω and much of our paper is devoted to answering them for orders that capture the subsequence property *and* the weak or strong exchange properties.

A closed class is called a *weak sorting class* if it has the weak exchange property and a *strong sorting class* if it has the strong exchange property.

Our aim is to set up a framework within which these two notions can be investigated and to exploit this framework by proving some initial results about them. We shall begin by investigating the two natural analogues of the subpermutation order that are appropriate for these two concepts. In particular there are natural notions of a basis for each type of sorting class; we shall explore how the basis of a sorting class is related to the ordinary basis and use this to derive enumerative results.

The terms 'weak' and 'strong' have been chosen to recall two important orders on the set of permutations of length n: the weak and strong orders. For completeness we shall give their definitions below. In these definitions and elsewhere in the paper we use Roman letters for the individual symbols within a permutation and Greek letters for sequences of zero or more symbols.

The *weak order* $\mathcal{W}$ on the set of permutations of length n can be defined as the transitive closure of the set of pairs

$$\mathcal{W}_0 = \{(\lambda rs\mu, \lambda sr\mu) \mid r < s\}$$

The *strong order* $\mathcal{S}$ on the set of permutations of length n can be defined as the transitive closure of the set of pairs

$$\mathcal{S}_0 = \{(\lambda r\mu s\nu, \lambda s\mu r\nu) \mid r < s\}$$

The subpermutation order $\mathcal{I}$ is, of course, defined on the set of all permutations. It is the transitive closure of the set of pairs

$$\mathcal{I}_0 = \{(\lambda\mu, \lambda r\mu)\}$$

where, as usual, $\lambda r\mu$ means the result of inserting r in $\lambda\mu$ with suitable renumbering of symbols larger than r.

Weak (respectively, strong) sorting classes are the lower ideals in the partial order defined by the transitive closure of $\mathcal{I} \cup \mathcal{W}$ (respectively $\mathcal{I} \cup \mathcal{S}$) and so can be studied using the same machinery that has been used for arbitrary closed classes, adapted to the appropriate order.

We begin by giving a simple description of these transitive closures. In this description we denote the relational composition of two partial orders by juxtaposition.

Lemma 1 *The transitive closure of $\mathcal{I} \cup \mathcal{W}$ is $\mathcal{IW}$ while that of $\mathcal{I} \cup \mathcal{S}$ is $\mathcal{IS}$. In fact $\mathcal{WI} = \mathcal{IW}$ while $\mathcal{SI}$ is strictly included in $\mathcal{IS}$.*

Proof: In this extended abstract we prove only that $\mathcal{SI}$ is strictly included in $\mathcal{IS}$. To do this, observe that $321\ \mathcal{I}\ 1432\ \mathcal{S}\ 3412$ yet there exists no permutation θ with $321\ \mathcal{S}\ \theta\ \mathcal{I}\ 3412$; therefore the inclusion is strict. ∎

We have already noted that every closed class X can be described by a forbidden pattern set B as

$$\text{av}(B) = \{\sigma \mid \beta\ \overline{\mathcal{I}}\ \sigma \text{ for all } \beta \in B\}$$

We can describe weak and strong sorting classes in a similar way using the orders $\mathcal{IW}$ and $\mathcal{IS}$. In other words, given a set B of permutations we define

$$\begin{aligned} \mathrm{av}(B, \mathcal{IW}) &= \{\sigma \mid \beta\ \overline{\mathcal{IW}}\ \sigma \text{ for all } \beta \in B\} \\ \mathrm{av}(B, \mathcal{IS}) &= \{\sigma \mid \beta\ \overline{\mathcal{IS}}\ \sigma \text{ for all } \beta \in B\} \end{aligned}$$

which are weak and strong sorting classes respectively. Every weak and strong sorting class X can be defined in this way taking for B that set of permutations minimal with respect to $\mathcal{IW}$ or $\mathcal{IS}$ not belonging to X. If B is the minimal avoided set then it is tempting to call it the basis of the class it defines. Unfortunately that leads to a terminological ambiguity since both $av(B, \mathcal{IW})$ and $av(B, \mathcal{IS})$ are pattern closed classes and so have bases in the ordinary sense. To avoid such confusion we shall use the terms weak basis and strong basis. However, two significant questions now arise. If we have defined a weak sorting class by its weak basis, what is its basis in the ordinary sense? Similarly for strong sorting classes, what is the connection between the strong basis and the ordinary basis?

In the next section, on weak sorting classes, we shall see that the first of these questions has a relatively simple answer. In that section we also give a general result about the weak sorting class defined by a basis that is the direct sum of two sets. We go on to enumerate weak sorting classes whose weak basis is a single permutation of length at most 4.

In the final section, on strong sorting classes, we shall see that the ordinary basis is not easily found from the strong basis. Nevertheless we can define a process that constructs the ordinary basis from the strong basis; and we prove that the ordinary basis is finite if the strong basis is finite. We have used this process as a first step in enumerating strong sorting classes defined by a single strong basis element of length at most 4.

We also introduce a 2-parameter family of strong sorting classes denoted by $\mathcal{B}(r, s)$. These classes are important because every (proper) strong sorting class is contained in one (indeed infinitely many) of them. We shall show how the $\mathcal{B}(r, s)$ can be enumerated and give a structure theorem that expresses $\mathcal{B}(r, s)$ as a composition of very simple strong sorting classes.

2 Weak sorting classes

We begin by giving the connection between the weak basis of a weak sorting class and its ordinary basis.

Proposition 2 $\mathrm{av}(T, \mathcal{IW}) = \mathrm{av}(T, \mathcal{WI}) = \mathrm{av}(T')$ *where T' is the set of permutations*

$$\{\sigma \mid \tau\ \mathcal{W}\ \sigma \text{ for some } \tau \in T\}$$

(the upward weak closure of T).

Corollary 3 av(T) *is a weak sorting class if and only if every permutation in the upward weak closure of T involves a permutation of T.*

Corollary 4 *If a weak sorting class has a finite weak basis then its ordinary basis is also finite.*

The next result is very useful for enumerating weak sorting classes.

Theorem 5 *Let R, S be the weak bases of weak sorting classes $\mathcal{A}, \mathcal{B}$ and let $\mathcal{C}$ be the weak sorting class whose weak basis is $T = R \oplus S$. Let $(a_n), (b_n), (c_n)$ be the enumeration sequences for $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and let $a(t), b(t), c(t)$ be the associated exponential generating functions. Then*

$$c(t) = (t-1)a(t)b(t) + a(t) + b(t)$$

Proposition 2 shows that we can enumerate weak sorting classes using the various techniques that have been developed for ordinary closed classes. In the full paper we give enumerations for all weak sorting classes classes with a single weak basis permutation of length 3 or 4.

3 Strong sorting classes

For weak sorting classes Proposition 2, Corollary 3 and Corollary 4 describe how the weak basis is related to the ordinary basis. The situation for strong sorting classes is considerably more complex largely because the direct analogue of Corollary 3 is false. Despite this we can prove that a strong sorting class with a finite strong basis has a finite ordinary basis and our proof shows how this ordinary basis may be computed from the strong basis.

We begin these investigations by defining three types of operation on permutations:

Switch. Exchange two symbols of τ that are currently correctly ordered.

Left. Move a symbol t of τ to the left and insert some new symbol smaller than t in the original position of t (with appropriate renumbering of symbols).

Right. Move a symbol t of τ to the right and insert some new symbol larger than t in the original position of t (also with appropriate renumbering of symbols).

Suppose that T is some set of permutations. Then T is said to be *complete* if, for any $\tau \in T$, applying any of the types of operation switch, left, or right to τ results in a permutation that contains some permutation in T as a subpermutation.

Proposition 6 *T is complete if and only if* av(T) *is a strong sorting class.*

Now suppose that X is a strong sorting class with strong basis R. Let $c(R)$ denote the ordinary basis of X. Our aim is to describe $c(R)$ in terms of R. Let $\bar{X}$ denote the complement of X. Then, by definition

$$\bar{X} = \{\theta \mid \rho \ \mathcal{IS} \ \theta \text{ for some } \rho \in R\}$$

Also, by definition, $c(R)$ is the set of minimal permutations in $\bar{X}$ (minimal with respect to $\mathcal{I}$). The following result shows that $c(R)$ can be constructed from R by using switch, left, and right operations.

Lemma 7 *Let $\theta \in c(R)$. Then there exists a sequence of permutations*

$$\theta_0, \theta_1, \ldots, \theta_k = \theta$$

where $\theta_0 \in R$, each $\theta_i \in c(R)$, and each θ_i is obtained from θ_{i-1} by a switch, left, or right operation. Furthermore, in any sequence beginning at a permutation of R and ending at θ where each term arises from the previous one by a switch, left, or right operation, all permutations in the sequence are in $c(R)$.

This lemma indicates how $c(R)$ can be computed from R using a breadth-first search strategy. We begin from R itself and apply switch, left, and right operations discarding any results that contain previously found permutations as subpermutations; and we continue using any new permutations found. Our next result shows that this process terminates if R is finite.

Theorem 8 *Let X be a strong sorting class with strong basis R and suppose that R is finite. Then $c(R)$ is also finite.*

We turn now to the enumeration problem for strong sorting classes whose strong basis is finite. As a consequence of the previous result and work in [1] we have

Theorem 9 *Every strong sorting class whose strong basis is finite has a rational generating function.*

In the full paper we give the linear recurrences that enumerate strong sorting classes whose strong basis consists of a single permutation of length at most 4. Our general method is to first determine the ordinary basis of the class by the process described above and use our experience in closed class enumeration.

As an example of a fairly typical situation we note that

$$\begin{aligned} c(\{4231\}) &= \{4231, 4231, 4321, 35142, 45312, 42513, 45132, 35412, \\ &\quad 45213, 43512, 456123, 351624, 451623, 356124\} \end{aligned}$$

and the enumeration of $av(4231, \mathcal{IS})$ is governed by the recurrence

$$a_n = 4a_{n-1} - 2a_{n-2} + 4a_{n-3} - a_{n-5}$$

The above results suggest that the theory of strong sorting classes is more complex than that for weak sorting classes. However, the following results go some way to proving that it may actually be *less* complex.

Consider the following family of closed classes. The closed class $\mathcal{B}(r, s)$ is defined by the $r!s!$ (ordinary) basis permutations $\beta\alpha$ where $|\beta| = r, |\alpha| = s$ and every symbol of β is greater than every symbol of α. It follows easily from Proposition 6 that $\mathcal{B}(r, s)$ is a strong sorting class. The importance of the strong sorting classes $\mathcal{B}(r, s)$ stems from

Proposition 10 *Every proper strong sorting class is contained in some* $\mathcal{B}(r, r)$.

This proposition indicates that the classes $\mathcal{B}(r, s)$ are fundamental in the understanding of strong sorting classes. Our final results describe their structure.

Lemma 11 $\mathcal{B}(1, 2)$ *consists of permutations whose cycle structure is*

$$(1, 2, \ldots, k_1)(k_1 + 1, k_1 + 2, \ldots, k_2)(k_2 + 1 \ldots) \ldots$$

and $\mathcal{B}(2, 1)$ *is the class of their inverses.*

Theorem 12

$$\mathcal{B}(r, s) = \mathcal{B}(2, 1)^{r-1} \circ \mathcal{B}(1, 2)^{s-1}$$

References

[1] M. H. Albert, S. A. Linton, N. Ruškuc: The insertion encoding, in preparation.

[2] M. H. Albert, R. E. L. Aldred, M. D. Atkinson, H. van Ditmarsch, C. C. Handley, D. A. Holton, D. J. McCaughan: Compositions of pattern restricted sets of permutations, in preparation.

[3] M. D. Atkinson, D. Tulley, M. J. Livesey: Permutations generated by token passing in graphs, Theoretical Computer Science 178 (1997), 103–118.

[4] M. D. Atkinson, T. Stitt: Restricted permutations and the wreath product, Discrete Math. 259 (2002), 19–36

[5] D. E. Knuth: *Fundamental Algorithms, The Art of Computer Programming* Vol. 1 (Second Edition), Addison-Wesley, Reading, Mass. (1973).

[6] *Special issue on permutation patterns*, Electron. J. Combin. 9(2) (2003), editors M. D. Atkinson, D. A. Holton.