A few words about the Vector Field Problem

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For example suppose we have group structure on a set and a topology then one would want to know whether multiplication and inverse are continuous functions or not. When it happens, the set in consideration is called topological group. One of the obvious question is following: given a topology on a set, can we find a compatible group structure which would make it a topological group?
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For example suppose we have group structure on a set and a topology then one would want to know whether multiplication and inverse are continuous functions or not. When it happens, the set in consideration is called **topological group**. One of the obvious question is following: given a topology on a set, can we find a compatible group structure which would make it a topological group?

Similarly one wonders when can a differentiable manifold be given compatible **Lie group** structure?
A differentiable manifold $X$ is connected and locally looks like a ball in $\mathbb{R}^n$ with no corners. Here $n$ is fixed and it is known as the dimension of $X$. 
Differentiable manifold

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**Figure:** Self intersection

![Figure: Self intersection](image)

**Figure:** Homeomorphism doesn’t preserve differentiable structure

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Example- Circle, Torus(product of manifolds), Sphere, etc.
Suppose $X$ is a differentiable manifold.
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Suppose $\alpha : (-\varepsilon, \varepsilon) \longrightarrow X \subseteq \mathbb{R}^m$ is a differentiable map.

Then $\dot{\alpha}(0) = \frac{d}{dt} \alpha(0)$ is defined.
This is called a Tangent Vector of $X$ at $x$.

For example, one can see that every vector $v \in \mathbb{R}^l$ is vector to any point $x \in \mathbb{R}^l$. 
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\begin{tikzpicture}
  \draw (0,0) .. controls (1,1) and (2,-1) .. (3,0);
  \draw[->] (2,1) -- (2,2);
  \node at (2,2.5) {$x$};
\end{tikzpicture}
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Let's consider $S^2$. 
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Since any manifold $X$ locally looks like an open subset of sphere of dimension $m$ so arguing in similar manner we have $\tau_x X$ is vector space of dimension $m$. 
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This behaviour is called VARYING CONTINUOUSLY. Such a map is called vector field. A vector field $v$ is called nowhere vanishing vector field if $v(x) \neq 0$ for any $x \in M$. 
Existence

Lets try to see how a vector field on sphere would look.

We can see in this case that there are two points where vector field takes value zero. In fact— if we consider any continuous vector field on $S^2$, it takes value 0 at some point. Using degree of a map, one can check that if $n$ is even then there can not be a nowhere vanishing vector field on $S^n$.
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Lemma

Let $M$ be a connected, compact, differentiable manifold. There exist a nowhere vanishing vector field on $M$ if and only if Euler characteristic $\chi(M) = 0$. 

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**Lemma**

*Let M be a connected, compact, differentiable manifold. There exist a nowhere vanishing vector field on M if and only if Euler characteristic $\chi(M) = 0$.***
Similarly we can ask whether we can find $v_x, w_x$ for $x \in X$ such that $v_x$ and $w_x$ are linearly independent for every $x \in X$.  

Consider torus $T$ in $\mathbb{R}^2$. We know $T = S^1 \times S^1$. Let $v$ denote nonzero everywhere linearly independent vector field for first copy of $S^1$ and $w$ for second copy. Then $(v,0)$ and $(0,w)$ are nonzero everywhere linearly independent vector fields.

Note that if $M$ is a differentiable manifold of dimension $n$ then we can have at most $n$ everywhere linearly independent vector fields.
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![Diagram of a torus with vectors $v$ and $w$](image)

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One of the most efficient approaches to determine span of a differentiable manifold $M$ is by considering tangent bundle $\tau M$ as a bundle over $M$ (not just as a manifold in its own right) and noticing that vector field is a section of $\tau M$.
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\[
\begin{align*}
M & \xrightarrow{\nu} \tau M \\
\xrightarrow{id} & \\
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\]
Definition (Parallelizability, stable parallelizability)

- $M$ is differentiable manifold.
- $M$ is parallelizable $\iff \tau_M \sim \mathbb{R}^{\dim(M)} \oplus \epsilon \mathbb{R}$
- $M$ is stably parallelizable $\iff \tau_M \oplus \epsilon \mathbb{R} \sim \mathbb{R}^{\dim(M) + 1} \oplus \theta$ for some vector bundle $\theta$.

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- $\text{span}(M) := \max \{ r | \tau_M \sim \mathbb{R}^r \oplus \eta \}$ for some vector bundle $\eta$.
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Problem is that we can't differentiate bundle from stable bundle so we wish to understand relation between $\text{span}$ and stable $\text{span}$ of a manifold as much as possible.

We have a few results (due to Koschorke) relating $\text{span}$ and stable $\text{span}$ which depend on vanishing of Stiefel-Whitney class. Let's recall some of them here.

(i) If $n \equiv 0 \pmod{2}$, then $\text{span}(M) = 0$ or stable $\text{span}(M)$.

(ii) If $n \equiv 1 \pmod{4}$ and $w_2(M) = 0$, then $\text{span}(M) = 1$ or stable $\text{span}(M)$.

(iii) If $n \equiv 3 \pmod{8}$ and $w_1(M) = w_2(M) = 0$, then $\text{span}(M) = 3$ or stable $\text{span}(M)$. 


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We are considering following manifold (which is a quotient of complex Stiefel manifold by a finite central subgroup):

\[ W_{n,k} = \text{space of all unitary } k \text{-frames in } \mathbb{C}^n. \]

That is, \((v_1, \ldots, v_k) \in W_{n,k} \) if and only if \(v_i \in \mathbb{C}^n, \langle v_i, v_i \rangle = 1 \) and \(\langle v_i, v_j \rangle = 0 \) if \(i \neq j\) where \(\langle , \rangle\) denotes the standard Hermitian metric on \(\mathbb{C}^n\).

Equivalently we have \( W_{n,k} = U(n) / U(n-k) \) where \(U(n-k)\) is imbedded in \(U(n)\) as the subgroup that fixes the first \(k\) standard basis vectors \(e_1, \ldots, e_k \in \mathbb{C}^n\).

Let \(m \in \mathbb{N}\). \(\Gamma_m \subset S^1\), \(\Gamma_m\) subgroup generated by a primitive \(m\)-th root of unity.

\(\Gamma_m\) acts on \(W_{n,k}\) by \(z \cdot (v_1, \ldots, v_k) = (zv_1, \ldots, zv_k)\) where \(z \in \Gamma_m\).

We denote the orbit space by \(W_{n,k}; m\) and it will be referred to as the \(m\)-projective Stiefel manifolds.

Notice that when \(k = 1\), \(W_{n,1}; m = L_n(m)\) lens space.
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That is, \((v_1, \ldots, v_k) \in W_{n,k}\) if and only if \(v_i \in \mathbb{C}^n, \langle v_i, v_i \rangle = 1\) and \(\langle v_i, v_j \rangle = 0\) if \(i \neq j\) where \(\langle , \rangle\) denotes the standard Hermitian metric on \(\mathbb{C}^n\).
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\[ W_{n,k} = U(n)/U(n-k) \] where \(U(n-k)\) is imbedded in \(U(n)\) as the subgroup that fixes the first \(k\) standard basis vectors \(e_1, \ldots, e_k \in \mathbb{C}^n\).
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One can see that \(W_{n,k;m} = U(n)/(\mathbb{Z}_m \times U(n-k))\).
Proposition

Let $2 \leq k < n$ and let $m \geq 2$. One has

$$\text{span}(W_{n,k;m}) = \text{stable span}(W_{n,k;m})$$

in each of the following cases: (i) $k$ is even, (ii) $n$ is odd, and (iii) $n \equiv 2 \pmod{4}$. 

Theorem

If there exists an $r \geq 1$ such that $(nk)^r$ is not divisible by $m^2$, then $W_{n,k;m}$ is not stably parallelizable. In particular, if $W_{n,k;m}$ is stably parallelizable, then $m$ divides $nk$. 

We are trying to refine these results and wish to solve vector field problem for $m$-projective space completely.
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Theorem

If there exists an \(r \geq 1\) such that \(\binom{nk}{r}\) is not divisible by \(m^{2r}\), then \(W_{n,k;m}\) is not stably parallelizable. In particular, if \(W_{n,k;m}\) is stably parallelizable, then \(m\) divides \(nk\).
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Thank you!