ZERO-SUM-FREE SEQUENCES WITH FEW SUBSEQUENCE SUMS

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Abstract. Extending the results of Savchev-Chen and Yuan, we show that a zero-sum-free sequence of length n over an abelian group spans at least $2n$ distinct subsequence sums, unless it has a simple, explicitly described structure.

1. Introduction

A sequence of elements of an abelian group is called *zero-sum-free* if it does not contain a finite, nonempty subsequence with the zero sum of its terms. The problem of estimating the smallest possible number of distinct subsequence sums of a finite zero-sum-free sequence was raised by Eggleton and Erdős [EE72], and since then has attracted much attention; we refer the reader to the survey by Gao and Geroldinger [GG06] for a comprehensive introduction to the subject area and historical overview.

In one of the early papers on this subject, Olson and White [OW77] have shown that a zero-sum-free sequence of length n determines at least $2n$ distinct subsequence sums provided that the subgroup generated by the elements of the sequence is not cyclic; here the empty subsequence with the zero sum of its terms is counted, too. For the infinite cyclic group (identified with the group of integers), a complete description of the length-n sequences with fewer than $2n$ subsequence sums was obtained in [L99, Proposition 1] under the assumption that all elements of the sequence are positive; the general case where the elements are allowed to be negative has never been considered, to the best of our knowledge.

As an immediate corollary of the result of Olson and White, one recovers [EE72, Theorem 3]: a zero-sum-free sequence generating a finite, noncyclic group of order m has length $n \le m/2$.

Given a sequence α in an abelian group, let $\Sigma(\alpha)$ denote the set of all subsequence sums of α (including the empty subsequence); thus, if $\alpha = (a_1, \dots, a_n)$, then

$$\Sigma(\alpha) = \left\{ \sum_{i \in I} a_i \colon I \subseteq [1, n] \right\}.$$

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In 2007, Savchev and Chen, and independently Yuan, proved a remarkable result characterizing zero-sum-free sequences of length $n > m/2$ in the finite *cyclic* group of order m.

Theorem 1 (Savchev-Chen [SC07], Yuan [Y07]). *Suppose that $\alpha = (a_1, \dots, a_n)$ is a zero-sum-free sequence of elements of the cyclic group of order m. If $n > m/2$, then, having the elements of α suitably renumbered, there are a group element a of order m and integers $1 = x_1 \le \dots \le x_n$ satisfying $x_{k+1} \le 1 + x_1 + \dots + x_k,\ k \in [1, n-1]$, such that $a_k = x_k a$ for all $k \in [1, n]$. Moreover, $\Sigma(\alpha) = \{0, a, 2a, \dots, (x_1 + \dots + x_n)a\}$ and $x_1 + \dots + x_n < m$.*

By $\mathrm{ord}(a)$ we denote the order of a group element a.

In this note we prove the following refinement of the result of Savchev-Chen-Yuan in the spirit of Olson-White.

Theorem 2. *Suppose that $\alpha = (a_1, \dots, a_n)$ is a zero-sum-free sequence of elements of an abelian group. If $|\Sigma(\alpha)| < 2n$ then, having the elements of α suitably renumbered, there are a group element a and integers $1 = x_1 \le \dots \le x_n$ satisfying $x_{k+1} \le x_1 + \dots + x_k,\ k \in [1, n-1]$, such that $a_k = x_k a$ for all $k \in [1, n]$. Moreover, $\Sigma(\alpha) = \{0, a, 2a, \dots, (x_1 + \dots + x_n)a\}$, and if a is of finite order, then $x_1 + \dots + x_n < \mathrm{ord}(a)$.*

Theorem 2 is easily seen to imply Theorem 1; the major difference between the two results is that Theorem 2 yields the conclusion in the situation where the set of subsequence sums is small for whatever reason, not necessarily because "the whole group is exhausted". Loosely speaking, Theorem 2 shows that forcing α to be structured is not the fact that α is "long", as compared to the size of the group, but rather that α has few subsequence sums.

The bound $2n$ for the number of subsequence sums is best possible: if a_1 and a_2 are group elements of sufficiently large order with $a_2 \notin \{-na_1, \dots, -a_1, 0, a_1, \dots, na_1\}$, and $\alpha = (a_1, \dots, a_1, a_2)$ where the element a_1 is repeated $n-1$ times, then α is zero-sum-free and $|\Sigma(\alpha)| = 2n$, but α does not have the structure described in Theorem 2, as it follows by observing that $\Sigma(\alpha)$ is not an arithmetic progression.

A finite sequence α is *zero-sum* if the sum of its terms is 0; it is *minimal* zero-sum if, in addition, none of its nonempty subsequences is zero-sum. Theorem 1 is known to have the following equivalent restatement: if $\alpha = (a_1, \dots, a_n)$ is a minimal zero-sum sequence of elements of the cyclic group of order $m < 2(n-1)$, then there are a group element a and positive integers $x_1, \dots, x_n$ such that $x_1 + \dots + x_n = \mathrm{ord}(a)$ and $a_k = x_k a$ for all $k \in [1, n]$. As a corollary of Theorem 2, we obtain the same conclusion under weaker assumptions.

Corollary 1. *Suppose that* $\alpha = (a_1, \dots, a_n)$ *is a minimal zero-sum sequence of elements of a finite abelian group. If* $|\Sigma(\alpha)| < 2(n-1)$*, then there are a group element* a *and positive integers* $x_1, \dots, x_n$ *such that* $x_1 + \dots + x_n = \operatorname{ord}(a)$ *and* $a_k = x_k a$ *for all* $k \in [1, n]$.

Proof. Let α' be the sequence of length $n-1$ obtained from α by removing the term a_n. Clearly, α' is zero-sum-free, and $|\Sigma(\alpha')| \le |\Sigma(\alpha)| < 2(n-1)$. By Theorem 2, there are a group element a and positive integers $x_1, \dots, x_{n-1}$ such that $x_1 + \dots + x_{n-1} < \operatorname{ord}(a)$ and $a_k = x_k a$ for all $k \in [1, n-1]$. Let $x_n := \operatorname{ord}(a) - (x_1 + \dots + x_{n-1})$. Then x_n is a positive integer satisfying $x_1 + \dots + x_n = \operatorname{ord}(a)$, and we have

$$x_n a = (\operatorname{ord}(a) - (x_1 + \dots + x_{n-1}))a = -a_1 - \dots - a_{n-1} = a_n.$$

$\square$

The proof of Theorem 2 is presented in the next section; it is a modified version of the original proof of Savchev and Chen. In view of the above-mentioned result of Olson-White, it would suffice to prove the theorem in the case where the underlying group is cyclic. However, we give a complete, self-contained proof which works for any abelian group, whether it is finite, cyclic, or neither. An application is considered in the concluding Section 3.

2. Proof of Theorem 1

We start with some basic notions and facts about the zero-sum-free sequences.

For a finite sequence α, we denote by $\sigma(\alpha)$ the sum of all terms of α, and by α_k the truncated sequence consisting of the first k terms of α; here k is a positive integer not exceeding the length of α. Thus, writing $\alpha = (a_1, \dots, a_n)$,

$$\sigma(\alpha) = a_1 + \dots + a_n \text{ and } \alpha_k = (a_1, \dots, a_k), \quad 1 \le k \le n.$$

As defined in Section 1, given a sequence α, by $\Sigma(\alpha)$ we denote the set of all subsequence sums of α, including the empty subsequence.

We notice that if $\alpha = (a_1, \dots, a_n)$ is a zero-sum-free sequence, then $\Sigma(\alpha_{n-1})$ is a proper subset of $\Sigma(\alpha)$; indeed, $\sigma(\alpha)$ is an element of the later, but not of the former set. It follows that $|\Sigma(\alpha)| \ge n + 1$ for any zero-sum-free sequence α of length n.

We say that a nonempty, zero-sum-free sequence α is *sharp* if $\Sigma(\alpha) = \Sigma(\alpha_{n-1}) \cup \{\sigma(\alpha)\}$; in other words, α is sharp if it is nonempty, zero-sum-free, and $|\Sigma(\alpha)| = |\Sigma(\alpha_{n-1})| + 1$, where n is the length of α. (This terminology is different from that used in [SC07].)

The following claim contains some basic observations, of which the first two originate from [SF01], and the third one from [SC07] where the proofs of all three can be found.

For a group element g, we denote by $\langle g \rangle$ the cyclic subgroup generated by g; thus, if the order of g is finite, then the subgroup is finite, too, and $\operatorname{ord}(g) = |\langle g \rangle|$.

Claim 1 (Smith-Freeze, Savchev-Chen). *If $\alpha = (a_1, \dots, a_n)$ is a sharp, zero-sum-free sequence of elements of an abelian group, then*

(a) *$\Sigma(\alpha_{n-1})$ is a union of the arithmetic progression $\{0, a_n, 2a_n, \dots, sa_n\}$, where $1 \le s < \operatorname{ord}(a_n) - 1$, and a (possibly, zero) number of nonzero $\langle a_n \rangle$-cosets;*

(b) *$\sigma(\alpha_{n-1}) = sa_n$;*

(c) *a_n is the unique group element such that appending it to α_{n-1} as a last term yields a sharp, zero-sum-free sequence.*

The inequality $s < \operatorname{ord}(a_n) - 1$ in (a) should be interpreted as "either the order of a_n is infinite, or the order is finite and then the stated inequality holds". Other inequalities of this kind are used throughout without further explanations.

Our next claim is a version of [SC07, Lemma 6] extended onto arbitrary abelian groups (not necessarily cyclic or finite), and with the length assumption relaxed to the assumption that the subsequence sum set is small.

Claim 2. *Suppose that $\alpha = (a_1, \dots, a_n)$ is a zero-sum-free sequence with $|\Sigma(\alpha)| < 2n$. If $|\Sigma(\alpha_m)| \ge 2m$ for some $m \in [1, n-1]$, then the terms $a_{m+1}, \dots, a_n$ of α can be permuted (while keeping the order of the first m terms) so that the resulting sequence is sharp.*

Proof. We can assume that m is maximal subject to $|\Sigma(\alpha_m)| \ge 2m$; thus, in particular, the sequence obtained from α_m by appending any of the terms $a_{m+1}, \dots, a_n$ is sharp. By Claim 1(c), we have $a_{m+1} = \dots = a_n$, and by Claim 1(a), the set $\Sigma(\alpha_m)$ is a union of nonzero cosets of the subgroup $\langle a_n \rangle$ and an arithmetic progression $\{0, a_n, 2a_n, \dots, sa_n\}$ where $1 \le s < \operatorname{ord}(a_n) - 1$. Therefore each of $\Sigma(\alpha_{m+1}), \dots, \Sigma(\alpha_n)$ has the very same structure, the only difference between these sets being that each time we pass from α_k to α_{k+1}, the value of s increases by 1. It follows that for each $k \in [m+1, n]$, and in particular for $k = n$, the sequence α_k is sharp. □

Claim 3. *Suppose that the zero-sum-free sequences $\alpha = (a_1, \dots, a_n)$ and $\alpha' = (a'_1, \dots, a'_n)$ are rearrangements of each other. If both α and α' are sharp, then $a_n = a'_n$.*

Proof. By the definition of a sharp sequence, we have $\Sigma(\alpha) = \Sigma(\alpha_{n-1}) \cup \{\sigma(\alpha)\}$ and $\Sigma(\alpha') = \Sigma(\alpha'_{n-1}) \cup \{\sigma(\alpha')\}$ where the unions are disjoint. Since $\Sigma(\alpha') = \Sigma(\alpha)$ and $\sigma(\alpha') = \sigma(\alpha)$, we conclude that $\Sigma(\alpha'_{n-1}) = \Sigma(\alpha_{n-1})$.

Consider the sequence $\alpha'' := (a_1, \dots, a_{n-1}, a'_n)$. This sequence is zero-sum-free in view of $\Sigma(\alpha_{n-1}) = \Sigma(\alpha'_{n-1})$, and it is sharp as

$$\left|\left(\Sigma(\alpha''_{n-1}) + a'_n\right) \setminus \Sigma(\alpha''_{n-1})\right| = \left|\left(\Sigma(\alpha_{n-1}) + a'_n\right) \setminus \Sigma(\alpha_{n-1})\right| = \left|\left(\Sigma(\alpha'_{n-1}) + a'_n\right) \setminus \Sigma(\alpha'_{n-1})\right| = 1.$$

The assertion follows now from Claim 1 (c). □

We say that a sequence $\alpha = (a_1, \dots, a_n)$ of elements of an abelian group is *connected* if each term of α is an algebraic sum of some of the preceeding terms; that is, if $a_k \in \Sigma(\alpha_{k-1}) - \Sigma(\alpha_{k-1})$, for all $k \in [2, n]$. If, indeed, $a_k \in \Sigma(\alpha_{k-1})$ holds, then we say that α is *strongly* connected.

If $\xi = (x_1, \dots, x_n)$ is a strongly connected sequence of nonnegative integers, then $x_k \le \sigma(\xi_{k-1})$. If, in addition, $x_1 = 1$, then from $\Sigma(\xi_k) = \Sigma(\xi_{k-1}) \cup (\Sigma(\xi_{k-1}) + x_k)$ it follows that $\Sigma(\xi_k) = [0, \sigma(\xi_k)]$, for all $k \in [1, n]$. (In the terminology of [SC07], the equality $\Sigma(\xi_k) = [0, \sigma(\xi_k)]$ means that ξ is *behaving*.)

Claim 4. *Every zero-sum-free sequence α with $|\Sigma(\alpha)| < 2n$, where n is the length of α, can be rearranged so that the resulting sequence is either connected, or sharp.*

Proof. Suppose that $\alpha = (a_1, \dots, a_n)$ where the elements are numbered so that α starts with the longest connected sequence possible. If this longest connected sequence has length n, then α is connected; suppose thus that the length is $m-1$, where $m \le n$. Then a_m cannot be represented as a difference of two elements of $\Sigma(\alpha_{m-1})$, meaning that the sets $\Sigma(\alpha_{m-1})$ and $\Sigma(\alpha_{m-1}) + a_m$ are disjoint. Therefore $|\Sigma(\alpha_m)| \ge 2|\Sigma(\alpha_{m-1})| \ge 2m$. The result now follows from Claim 2. □

For an integer set S and an element a of an abelian group, we write $S \cdot a := \{sa \colon s \in S\}$.

It is easy to see that if $\alpha = (a_1, \dots, a_n)$ is connected and zero-sum-free, then for each $k \in [1, n]$ there is an integer x_k such that $a_k = x_k a_1$ and moreover, $x_1 = 1$, and if a_1 has finite order, then $0 \le x_k < \operatorname{ord}(a_1)$. We say that the sequence of integers $\xi = (x_1, \dots, x_n)$ is *associated* with α. Clearly, in this situation we have $\Sigma(\alpha) = \Sigma(\xi) \cdot a_1$.

Claim 5. *If $\alpha = (a_1, \dots, a_n)$ is a zero-sum-free, connected sequence of elements of an abelian group, then the associated integer sequence ξ has positive terms and is* strongly connected*. Moreover, $\sigma(\xi) < \operatorname{ord}(a_1)$.*

Proof. We use induction on n. The case $n = 1$ is trivial. Suppose that $n > 1$ and write $\xi = (x_1, \dots, x_n)$. By the induction hypothesis, $x_1, \dots, x_{n-1} \ge 1$, $\sigma(\xi_{n-1}) < \operatorname{ord}(a_1)$, and ξ_{n-1} is strongly connected, as a result of which $\Sigma(\xi_{n-1}) = [0, \sigma(\xi_{n-1})]$; consequently, $\Sigma(\alpha_{n-1}) = [0, \sigma(\xi_{n-1})] \cdot a_1$.

Since α is connected, we have

$$a_n \in \Sigma(\alpha_{n-1}) - \Sigma(\alpha_{n-1}) = [-\sigma(\xi_{n-1}), \sigma(\xi_{n-1})] \cdot a_1.$$

On the other hand, since α is zero-sum-free, $a_n \notin -\Sigma(\alpha_{n-1}) = [-\sigma(\xi_{n-1}), 0] \cdot a_1$. It follows that

$$x_n a_1 = a_n \in [1, \sigma(\xi_{n-1})] \cdot a_1.$$

If the order of a_1 is infinite, then we immediately conclude that $x_n \in [1, \sigma(\xi_{n-1})]$. If a_1 is of finite order, then we arrive at the same conclusion choosing $t \in [1, \sigma(\xi_{n-1})]$ to satisfy

$x_n a_1 = t a_1$ and observing that $1 \le x_n, t < \operatorname{ord}(a_1)$. Recalling that $\Sigma(\xi_{n-1}) = [0, \sigma(\xi_{n-1})]$, we derive that $x_n \in \Sigma(\xi_{n-1})$; hence ξ is strongly connected and, consequently, $\Sigma(\xi) = [0, \sigma(\xi)]$. This shows that every integer from the interval $[1, \sigma(\xi)]$ is the sum of the elements of a subsequence of ξ. Clearly, this subsequence is nonempty. Therefore, every element of the set $[1, \sigma(\xi)] \cdot a_1$ is the sum of the elements of a nonempty subsequence of α. Since α is zero-sum-free, we conclude that $0 \notin [1, \sigma(\xi)] \cdot a_1$, implying the last assertion. □

Claim 6. *If ξ is a strongly connected sequence of positive integers, then so is the nondecreasing rearrangement of ξ.*

Proof. Let ξ' be the nondecreasing rearrangement of ξ. We know that every term of ξ is the sum of some of the other terms; therefore, every term is in fact the sum of some strictly smaller terms. It follows that every term of ξ' is the sum of some of the preceding terms (since *all* terms smaller than the one under consideration precede it). □

As a direct consequence of Claims 5 and 6, we have

Claim 7. *If $\alpha = (a_1, \dots, a_n)$ is a zero-sum-free, connected sequence of elements of an abelian group, then the conclusion of Theorem 2 holds true: having the elements of α suitably renumbered, there are a group element a and integers $1 = x_1 \le \dots \le x_n$ with $x_1 + \dots + x_n < \operatorname{ord}(a)$ such that $a_k = x_k a$ for all $k \in [1, n]$, and $x_{k+1} \le x_1 + \dots + x_k$ for all $k \in [1, n-1]$. Moreover, $\Sigma(\alpha) = [0, x_1 + \dots + x_n] \cdot a$.*

Finally, we can prove our main result.

Proof of Theorem 2. If α admits a connected rearrangement, then the result follows from Claim 7; we thus assume that none of the rearrangements of α are connected and show that this assumption leads to a contradiction.

With Claim 4 in mind, we can assume that α is sharp. Let $c := a_n$, and let m be the multiplicity of c in α; that is, $m = |\{i \in [1, n] \colon a_i = c\}|$. Consider a rearrangement of α beginning with the term c repeated m times. Since this rearrangement is not connected, there exists $k \in [2, n]$ such that $\Sigma(\alpha_{k-1})$ is disjoint from $\Sigma(\alpha_{k-1}) + a_k$, resulting in $|\Sigma(\alpha_k)| = 2|\Sigma(\alpha_{k-1})| \ge 2k$. By Claim 2 we can further rearrange the terms of α, starting from the $(k+1)$-th term, so that the resulting rearrangement α' is sharp. By Claim 3, the last term of α' is then equal to c, contradicting the way α' is constructed. □

3. High-multiplicity elements in zero-sum-free sequences with few subset sums

The Savchev-Chen-Yuan theorem has numerous applications. Using Theorem 2 instead, one can improve or generalize some of the corresponding results. We consider just one example.

Suppose that α is a zero-sum-free sequence of length n over the cyclic group of order m. Bovey, Erdős, and Niven [BEN75] have shown that if $n > \frac{1}{2}m$, then α contains a term of multiplicity at least $2n - m + 1$; as remarked in [BEN75], this estimate is best possible whenever $\frac{2}{3}(m-1) \le n < m$. An improvement for the complementary range $\frac{1}{2}m < n \le \frac{2}{3}(m-1)$ was obtained by Gao and Geroldinger [GG98], and then Savchev and Chen [SC07] applied their result to prove an estimate which is sharp for all values of $n > \frac{1}{2}m$; namely, the maximum multiplicity is at least

$$\begin{cases} n - \left\lfloor \frac{m-1}{3} \right\rfloor, & \frac{1}{2}m < n \le \frac{2}{3}(m-1), \\ 2n - m + 1, & \frac{2}{3}(m-1) \le n < m. \end{cases}$$

Using Theorem 2 instead of Theorem 1, we prove

Theorem 3. *Suppose that α is a zero-sum-free sequence of length n over an abelian group. If α has at most $m < 2n$ distinct subsequence sums, then there is a group element that appears in α with the multiplicity at least*

$$\max_{r \ge 1} \frac{2}{r} \left(n - \frac{m-1}{r+1} \right).$$

Since a zero-sum-free sequence in the cyclic group of order m can have at most m distinct subsequence sums, the case $r = 1$ gives the Boven-Erdős-Niven estimate, and taking $r = 2$ we recover the Savchev-Chen bound. Substituting $r \ge 3$ does not lead to any further improvement, which is very expectable bearing in mind that the Savchev-Chen bound is tight. Notice, however, that Theorem 3 is applicable under substantially weaker assumptions.

Proof of Theorem 3. Applying Theorem 2, and having α appropriately ordered, we write $\alpha = (a_1, \dots, a_n)$ and $\xi = (x_1, \dots, x_n)$ where $1 = x_1 \le \dots \le x_n$ are the integers appearing in the conclusion of Theorem 2. For each $j \ge 1$, denote by ν_j be the number of indices $i \in [1, n]$ with $x_i = j$. We have $\nu_1 + \nu_2 + \nu_3 + \dots = n$ and $\nu_1 + 2\nu_2 + 3\nu_3 + \dots = \sigma(\xi) = |\Sigma(\alpha)| - 1 \le m - 1$. Therefore, denoting by μ the largest multiplicity of an element of α, and observing that $\mu = \max_{j \ge 1} \nu_j$, for any integer $r \ge 1$ we obtain

$$\begin{aligned} \frac{1}{2} r(r+1)\mu + m &\ge (r\nu_1 + (r-1)\nu_2 + \dots + \nu_r) \\ &\qquad + (1 + \nu_1 + 2\nu_2 + \dots + r\nu_r + (r+1)\nu_{r+1} + \dots) \\ &\ge (r+1)n + 1 \end{aligned}$$

which is equivalent to $\mu \ge \frac{2}{r}\left(n - \frac{m-1}{r+1}\right)$. $\square$

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