SUM-FREE SETS IN $\mathbb{Z}_5^n$

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ABSTRACT. It is well-known that for a prime $p \equiv 2 \pmod 3$ and integer $n \geq 1$, the maximum possible size of a sum-free subset of the elementary abelian group $\mathbb{Z}_p^n$ is $\frac{1}{3}(p+1)p^{n-1}$. We establish a matching stability result in the case $p=5$: if $A \subseteq \mathbb{Z}_5^n$ is a sum-free subset with $|A| > \frac{3}{2} \cdot 5^{n-1}$, then there are a subgroup $H < \mathbb{Z}_5^n$ of size $|H| = 5^{n-1}$ and an element $e \notin H$ such that $A \subseteq (e+H) \cup (-e+H)$.

1. Background and motivation.

A subset S of an abelian group is *sum-free* if the equation $x+y=z$ has no solutions in the elements of S; that is, if S is disjoint from $2S$ where we use the standard notation $2S := \{s_1+s_2 \colon s_1, s_2 \in S\}$. The idea of a sum-free set goes back to Schur [S16] who was motivated by the modular version of the Fermat equation $x^n + y^n = z^n$. Despite this initial motivation, sum-free sets are treated in [S16] as a combinatorial object of independent interest. Originating from [S16], the celebrated *Schur's theorem* ("the positive integers cannot be partitioned into finitely many sum-free subsets") is considered one of the origins of Ramsey theory.

In the 1960's sum-free sets were studied under the name "mutant sets"; see, for instance, [K69]. The subject gained popularity when it turned out to be related to a problem of Erdős. The reader is invited to check [GR05, TV17] for a historical account and further references.

How large can a sum-free subset of a given finite abelian group be? First considered in 1968 by Diananda and Yap [D68, DY69], this basic question did not receive a complete answer up until the year 2005 when it was eventually resolved by Green and Ruzsa [GR05].

Once the largest possible size is known, it is natural to investigate the corresponding stability problem: what is the structure of sum-free subsets of finite abelian groups of size close to the largest possible? In this respect, the cyclic groups of infinite order and prime order, and elementary abelian p-groups have received particular attention. Here we are concerned with the groups of the latter type.

The case $p = 2$ is of special interest due to its connections with the coding theory and the theory of finite geometries, see [CP92, KL03] for a detailed explanation. Motivated by the applications in these areas, Davydov and Tombak [DT89] established the structure of large sum-free subsets in the binary settings. To state their principal result, we briefly review the basic notions of periodicity and maximality.

The *period* of a subset A of an abelian group G is the subgroup $\pi(A) := \{g \in G\colon A + g = A\} \le G$; that is, $\pi(A)$ is the largest subgroup $H \le G$ such that A is a union of H-cosets. The set A is *periodic* if $\pi(A) \ne \{0\}$ and *aperiodic* otherwise. One also says that A is H*-periodic* if $H \le \pi(A)$; that is, if A is the inverse image of a subset of the quotient group G/H under the canonical homomorphism $G \to G/H$.

A sum-free set is *maximal* if it is not properly contained in another sum-free set.

By $\mathbb{Z}_p^n$ we denote the elementary abelian p-group of rank n.

Theorem 1 ([DT89, Theorem 1])**.** *Let $n \ge 4$ and suppose that $A \subseteq \mathbb{Z}_2^n$ is a maximal sum-free set. If $|A| > 2^{n-2} + 1$, then A is periodic.*

From Theorem 1 it is not difficult to derive a detailed structural characterization of large sum-free sets in $\mathbb{Z}_2^n$.

Theorem 1′ ([DT89])**.** *Let $n \ge 4$ and suppose that $A \subseteq \mathbb{Z}_2^n$ is sum-free. If $|A| \ge 2^{n-2} + 1$, then either A is contained in a nonzero coset of a proper subgroup, or there are an integer $k \in [4, n]$, a subgroup $H \le \mathbb{Z}_2^n$ of size $|H| = 2^{n-k}$, and a maximal sum-free subset $\mathcal{A} \subseteq \mathbb{Z}_2^n/H \simeq \mathbb{Z}_5^k$ of size $|\mathcal{A}| = 2^{k-2} + 1$ such that A is contained in the inverse image of $\mathcal{A}$ under the canonical homomorphism $\mathbb{Z}_2^n \to \mathbb{Z}_2^n/H$.*

As an easy consequence, we have the following corollary.

Corollary 1 ([DT89])**.** *Let $n \ge 4$ and suppose that $A \subseteq \mathbb{Z}_2^n$ is sum-free. If $|A| \ge 5 \cdot 2^{n-4} + 1$, then A is contained in a nonzero coset of a proper subgroup.*

Corollary 1 was independently obtained in [CP92].

In the ternary case, only an analog of Corollary 1 is known.

Theorem 2 ([L05])**.** *Let $n \ge 3$ and suppose that $A \subseteq \mathbb{Z}_3^n$ is sum-free. If $|A| \ge 5 \cdot 3^{n-3} + 1$, then A is contained in a nonzero coset of a proper subgroup.*

As shown in [L05], the bound $5 \cdot 3^{n-3} + 1$ is sharp.

In this note, we study the first open case $p = 5$ proving the following result.

Theorem 3. *Let $n \ge 1$ and suppose that $A \subseteq \mathbb{Z}_5^n$ is sum-free. If $|A| > \frac{3}{2} \cdot 5^{n-1}$, then there are a proper subgroup $H < \mathbb{Z}_5^n$ and an element $e \notin H$ such that $A \subseteq (e + H) \cup (-e + H)$.*

There are no reasons to believe that the assumption $|A| > \frac{3}{2}\cdot 5^{n-1}$ of Theorem 3 is sharp. On the other hand, it cannot be relaxed to $|A| > 5^{n-1}$.

Example 1. Suppose that $n \ge 3$ is an integer, and that $H < \mathbb{Z}_5^n$ is a subgroup of index 5. Fix arbitrarily an element $e \notin H$ and a subset $S \subseteq H$ with $S \cap (-S) = \varnothing$ and $S \cup (-S) = H \setminus \{0\}$, and let $A := (e+S) \cup \{2e, -2e\} \cup (-e-S)$. A straightforward verification shows that A is sum-free. Suppose now that A is contained in a union of two cosets of a subgroup $F < \mathbb{Z}_5^n$. Since A meets four H-cosets and just two F-cosets, we have $F \ne H$. Furthermore, one of these F-cosets contains at least half the elements of the set $e+S$. The intersection of this F-coset with the coset $e+H$ has therefore size at least $\frac{1}{2}|S| = (|H|-1)/4 > |H|/5$ while, on the other hand, the intersection of an H-coset with an F-coset is a coset of a proper subgroup of H, and as such, has size at most $|H|/5$, a contradiction showing that A is not contained in a union of two cosets of a proper subgroup.

We now turn to the proof of Theorem 3.

2. Proof of Theorem 3

Our argument is self-contained except that we need the following classical result of Kneser (but see [G13, Theorem 6.1] for our present formulation).

Theorem 4 (Kneser [K53, K55])**.** *If $A_1, \dots, A_k$ are finite, nonempty subsets of an abelian group, then letting $H := \pi(A_1 + \dots + A_k)$ we have*

$$|A_1 + \dots + A_k| \ge |A_1 + H| + \dots + |A_k + H| - (k-1)|H|.$$

Theorem 4 is referred to below as *Kneser's theorem.*

We start with a series of "general" claims. At this stage, it is not assumed that A is a sum-free set satisfying the assumptions of Theorem 3.

Lemma 1. *Let $n \ge 1$ be an integer and suppose that $A \subseteq \mathbb{Z}_5^n$ is sum-free. If $|A| > \frac{3}{2}\cdot 5^{n-1}$ and A is contained in a union of two cosets of a proper subgroup $H < \mathbb{Z}_5^n$, then there is an element $e \notin H$ such that $A \subseteq (e+H) \cup (-e+H)$.*

Proof. Since $2|H| \ge |A| > \frac{3}{2}\cdot 5^{n-1}$, we have $|H| = 5^{n-1}$. Suppose that $A = (e_1 + A_1) \cup (e_2 + A_2)$, where A_1, A_2 are contained in H, and $e_1, e_2 \in \mathbb{Z}_5^n$ lie in distinct H-cosets. From $|A| > \frac{3}{2}\cdot 5^{n-1}$ we get $|A_1| + |A_2| = |A| > \frac{3}{2}|H|$. Therefore $\min\{|A_1|, |A_2|\} > \frac{1}{2}|H|$, and by the pigeonhole principle, $2A_1 = 2A_2 = A_1 + A_2 = H$. It follows that $2A = (2e_1 + H) \cup (e_1 + e_2 + H) \cup (2e_2 + H)$. Since A is sum-free, each of the three cosets in the right-hand side is distinct from each of the cosets $e_1 + H$ and $e_2 + H$, which is possible only if $e_2 + H = -e_1 + H \ne H$. $\square$

By Lemma 1, to prove Theorem 3 it suffices to show that any sum-free set in $\mathbb{Z}_5^n$ of size larger than $\frac{3}{2}\cdot 5^{n-1}$ is contained in a union of two cosets of a proper subgroup.

Proposition 1. *Let $n\ge 1$ be an integer and suppose that $A\subseteq\mathbb{Z}_5^n$ is sum-free. If $|A|>\frac{3}{2}\cdot 5^{n-1}$, then A cannot have non-empty intersections with exactly three cosets of a maximal proper subgroup of $\mathbb{Z}_5^n$.*

Proof. The case $n=1$ is immediate. Assuming that $n\ge 2$, $A\subseteq\mathbb{Z}_5^n$ is sum-free, and $H<\mathbb{Z}_5^n$ is a maximal proper subgroup such that A intersects non-trivially exactly three H-cosets, we obtain a contradiction.

Fix an element $e\in\mathbb{Z}_5^n\setminus H$, and for each $i\in[0,4]$ let $A_i:=(A-ie)\cap H$; thus, $A=A_0\cup(e+A_1)\cup(2e+A_2)\cup(3e+A_3)\cup(4e+A_4)$ with exactly three of the sets A_i non-empty. Considering the actions of the automorphisms of $\mathbb{Z}_5$ on its two-element subsets (equivalently, passing from e to $2e,3e$, or $4e$, if necessary), we further assume that one of the following holds:

(i) $A_2=A_3=\varnothing$;
(ii) $A_0=A_4=\varnothing$;
(iii) $A_3=A_4=\varnothing$.

We consider these three cases separately.

Case (i): $A_2=A_3=\varnothing$. In this case, $A=A_0\cup(e+A_1)\cup(4e+A_4)$, and since A is sum-free, we have $(A_1+A_4)\cap A_0=\varnothing$. It follows that $|A_0|+|A_1+A_4|\le|H|$. Consequently, letting $F:=\pi(A_1+A_4)$, we have $|H|\ge|A_0|+|A_1|+|A_4|-|F|=|A|-|F|$ by Kneser's theorem. Observing that $|F|\le\frac{1}{5}|H|=5^{n-2}$, we conclude that

$$|A|\le|H|+|F|\le\frac{6}{5}|H|=6\cdot 5^{n-2}<\frac{3}{2}\cdot 5^{n-1},$$

a contradiction.

Case (ii): $A_0=A_4=\varnothing$. In this case $A=(e+A_1)\cup(2e+A_2)\cup(3e+A_3)$ with $(A_1+A_2)\cap A_3=\varnothing$, and the proof can be completed as in Case (i).

Case (iii): $A_3=A_4=\varnothing$. In this case from $(A_0+A_1)\cap A_1=\varnothing$, letting $F:=\pi(A_0+A_1)$, by Kneser's theorem we get

$$|H|\ge|A_0+A_1|+|A_1|\ge|A_0|+2|A_1|-|F|$$

whence, in view of $|F|\le\frac{1}{5}|H|$,

$$2|A_1|+|A_0|\le\frac{6}{5}\,|H|. \tag{1}$$

Similarly, from $(A_0 + A_2) \cap A_2 = \varnothing$ we get

$$2|A_2| + |A_0| \le \frac{6}{5}\,|H|. \tag{2}$$

Averaging (1) and (2) we obtain $|A| \le \frac{6}{5}|H| < \frac{3}{2} \cdot 5^{n-1}$, a contradiction. □

Proposition 2. *Let $n \ge 1$ be an integer and suppose that $A \subseteq \mathbb{Z}_5^n$ is sum-free, and that $H < \mathbb{Z}_5^n$ is a maximal proper subgroup. If there is an H-coset with more than half of its elements contained in A, then A has non-empty intersections with at most three H-cosets.*

Proof. Fix an element $e \in \mathbb{Z}_5^n \setminus H$, and for each $i \in [0,4]$ set $A_i := (A - ie) \cap H$; thus, $A = A_0 \cup (e + A_1) \cup \dots \cup (4e + A_4)$. Suppose that $|A_i| > 0.5|H|$ for some $i \in [0,4]$. Since $2A_i = H$ by the pigeonhole principle, we have $i > 0$ (as otherwise $2A_0 = H$ would not be disjoint from A_0). Normalizing, we can assume that $i = 1$. From $2A_1 \cap A_2 = \varnothing$ we now derive $A_2 = \varnothing$, and from $(A_1 - A_1) \cap A_0 = \varnothing$ we get $A_0 = \varnothing$. □

In view of Lemma 1 and Propositions 1 and 2, we can assume that the set $A \subseteq \mathbb{Z}_5^n$ of Theorem 3 contains fewer than $\frac{1}{2} \cdot 5^{n-1}$ elements in every coset of every maximal proper subgroup.

Lemma 2. *Let $n \ge 1$ be an integer, and suppose that $A, B, C \subseteq \mathbb{Z}_5^n$ satisfy $(A+B) \cap C = \varnothing$. If $\min\{|A|, |B|\} > 2 \cdot 5^{n-1}$ and $C \ne \varnothing$, then $|A| + |B| + 2|C| \le 6 \cdot 5^{n-1}$.*

Proof. Write $H := \pi(A + B - C)$, and define $k \in [0, n]$ by $|H| = 5^{n-k}$. We have

$$\min\{|A + H|, |B + H|\} > 2 \cdot 5^{n-1} = 2 \cdot 5^{k-1}|H| \tag{3}$$

while, by Kneser's theorem, and since $(A + B) \cap C = \varnothing$ implies $0 \notin A + B - C$ and, consequently, $(A + B - C) \cap H = \varnothing$,

$$5^n - |H| \ge |A + B - C| \ge |A + H| + |B + H| + |C + H| - 2|H|. \tag{4}$$

Combining (3) and (4), we obtain

$$5^n \ge 2(2 \cdot 5^{k-1} + 1)|H| + |C + H| - |H| \ge 4 \cdot 5^{k-1}|H| + |C + H| + |H|.$$

Consequently,

$$|C| \le |C + H| \le 5^{n-1} - |H|.$$

On the other hand, from (4),

$$|A| + |B| + |C| \le 5^n + |H|.$$

Taking the sum of the last two estimates gives the result. □

Proposition 3. *Let $n \ge 1$ be an integer and suppose that $A \subseteq \mathbb{Z}_5^n$ is a sum-free subset of size $|A| > \frac{3}{2} \cdot 5^{n-1}$. If $H < \mathbb{Z}_5^n$ is a maximal proper subgroup such that every H-coset contains fewer than $\frac{1}{2}|H|$ elements of A, then there is at most one H-coset containing more than $\frac{2}{5}|H|$ elements of A.*

Proof. Suppose for a contradiction that there are two (or more) H-cosets that are *rich* meaning that they contain more than $\frac{2}{5}|H|$ elements of A each. Fix an element $e \in \mathbb{Z}_5^n \setminus H$ and write $A_i = (A - ie) \cap H$, $i \in [0, 4]$. Without loss of generality, either A_0 and A_1, or A_1 and A_2, or A_1 and A_4 are rich.

If A_0 and A_1 are rich, then applying Lemma 2 with H as the underlying group, in view of $(A_0 + A_1) \cap A_1 = \varnothing$ we get $4 \cdot \frac{2}{5}|H| < |A_0| + |A_1| + 2|A_1| \le 6 \cdot 5^{n-2}$, which is wrong.

If A_1 and A_2 are rich then, observing that $(A_1 + A_1) \cap A_2 = \varnothing$, we recover the contradictory $4 \cdot \frac{2}{5}|H| < |A_1| + |A_1| + 2|A_2| \le 6 \cdot 5^{n-2}$.

Finally, if A_1 and A_4 are rich, then from

$$(A_1 + A_4) \cap A_0 = (A_1 + A_1) \cap A_2 = (A_4 + A_4) \cap A_3 = \varnothing$$

using Lemma 2 we obtain

$$\begin{aligned} |A_1| + |A_4| + 2|A_0| &\le 6 \cdot 5^{n-2}, \\ |A_1| + |A_1| + 2|A_2| &\le 6 \cdot 5^{n-2}, \\ |A_4| + |A_4| + 2|A_3| &\le 6 \cdot 5^{n-2}. \end{aligned}$$

Taking the sum,

$$3|A_1| + 3|A_4| + 2|A_0| + 2|A_2| + 2|A_3| \le 18 \cdot 5^{n-2};$$

that is, $2|A| + |A_1| + |A_4| \le 18 \cdot 5^{n-2}$. However, from $|A| > \frac{3}{2} \cdot 5^{n-1}$ and $\min\{|A_1|, |A_4|\} > \frac{2}{5} \cdot 5^{n-1}$ we derive $2|A| + |A_1| + |A_4| > 3 \cdot 5^{n-1} + \frac{4}{5} \cdot 5^{n-1} = 19 \cdot 5^{n-2}$, a contradiction. □

We use character sums to complete the argument and prove Theorem 3.

Proof of Theorem 3. Suppose that $n \ge 2$, and that $A \subseteq \mathbb{Z}_5^n$ is a sum-free set with $\alpha := |A|/5^n > \frac{3}{10}$; we want to show that A is contained in a union of two cosets of a proper subgroup.

Denoting by 1_A the indicator function of A, consider the Fourier coefficients

$$\hat{1}_A(\chi) := 5^{-n} \sum_{a \in A} \chi(a), \ \chi \in \widehat{\mathbb{Z}_5^n}.$$

Since A is sum-free, we have $A \cap (A-A) = \varnothing$, whence

$$\sum_{\chi} |\hat{1}_A(\chi)|^2 \cdot \hat{1}_A(\chi) = 0;$$

consequently,

$$\sum_{\chi \neq 1} |\hat{1}_A(\chi)|^2 \cdot \hat{1}_A(\chi) = -\alpha^3$$

and, as a result,

$$\sum_{\chi \neq 1} |\hat{1}_A(\chi)|^2 \cdot \Re(\hat{1}_A(\chi)) = -\alpha^3.$$

Comparing this to

$$\sum_{\chi \neq 1} |\hat{1}_A(\chi)|^2 = \alpha(1-\alpha)$$

(which is an immediate corollary of the Parseval's identity), we obtain

$$\sum_{\chi \neq 1} |\hat{1}_A(\chi)|^2 \big((1-\alpha)\,\Re(\hat{1}_A(\chi)) + \alpha^2\big) = 0.$$

We conclude that there exists a non-principal character $\chi \in \widehat{\mathbb{Z}_5^n}$ such that

$$\Re(\hat{1}_A(\chi)) \leq -\frac{\alpha^2}{1-\alpha}. \tag{5}$$

Let $F := \ker \chi$, fix $e \in \mathbb{Z}_5^n$ with $\chi(e) = \exp(2\pi i/5)$, and for each $i \in [0,4]$, let $\alpha_i := |(A-ie) \cap F|/|F|$. By Propositions 1 and 2, we can assume that $\max\{\alpha_i \colon i \in [0,4]\} < 0.5$, and then by Proposition 3 we can further assume that there is at most one index $i \in [0,4]$ with $\alpha_i > 0.4$; that is, of the five inequalities $\alpha_i \leq 0.4$ $(i \in [0,4])$, at most one fails, but holds true once the inequality is relaxed to $\alpha_i < 0.5$. We show that this set of assumptions is inconsistent with (5). To this end, we notice that

$$5\Re(\hat{1}_A(\chi)) = \alpha_0 + s_1 \cos(2\pi/5) + s_2 \cos(4\pi/5)$$

where $s_1 := \alpha_1 + \alpha_4$ and $s_2 := \alpha_2 + \alpha_3 \leq 0.9$. Comparing with (5), we get

$$\begin{aligned}
-\frac{5\alpha^2}{1-\alpha} &\geq \alpha_0 + s_1 \cos(2\pi/5) + s_2 \cos(4\pi/5) \\
&= \alpha_0 + s_1 \cos(2\pi/5) + (s_2 - 0.9)\cos(4\pi/5) + 0.9\cos(4\pi/5) \\
&\geq \alpha_0 + s_1 \cos(2\pi/5) + (s_2 - 0.9)\cos(2\pi/5) + 0.9\cos(4\pi/5) \\
&\geq \alpha_0 + (5\alpha - \alpha_0 - 0.9)\cos(2\pi/5) + 0.9\cos(4\pi/5) \\
&\geq (5\alpha - 0.9)\cos(2\pi/5) + 0.9\cos(4\pi/5),
\end{aligned}$$

while the resulting inequality

$$-\frac{5\alpha^2}{1-\alpha} \geq (5\alpha - 0.9)\cos(2\pi/5) + 0.9\cos(4\pi/5)$$

is easily seen to be wrong for all $\alpha \in [0.3, 1)$. This completes the proof of Theorem 3. $\square$

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