SUM-FULL SETS ARE NOT ZERO-SUM-FREE

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ABSTRACT. Let A be a finite, nonempty subset of an abelian group. We show that if every element of A is a sum of two other elements, then A has a nonempty zerosum subset. That is, a (finite, nonempty) sum-full subset of an abelian group is not zero-sum-free.

1. INTRODUCTION: SUM-FULL VS ZERO-SUM-FREE

A subset A of an abelian group is called *sum-full* if every element of A is a sum of two other elements, possibly equal to each other; that is, in the standard notation, if $A \subseteq 2A$. The subset is *zero-sum* if the sum of its elements is equal to 0; it is *zero-sum-free* if it does not contain itself a non-empty zero-sum subset.

In these terms, a problem posted in 2010 by Gjergji Zaimi at the MathOverflow web site [4] reads:

Can a finite, nonempty, sum-full set of real numbers be zero-sum-free? Similar or related problems appeared in [3, 5, 6].

The problem has attracted some attention from the mathematical community, perhaps due to the fact that *a priori*, there is no obvious link between the properties of being sum-full and being zero-sum-free. Despite the interest, the only publication spanned by this problem we are aware of is [1], by Taras Banakh and Alex Ravsky.

It is immediately seen that a finite, nonempty, sum-full, zero-sum-free set cannot contain zero, and that it must contain at least two negative and at least two positive numbers (which can be made *three* with a minor effort). Furthermore, considering subsets of any torsion-free abelian group, not necessarily the additive group of real numbers, leads to an equivalent problem. Indeed, there is no reason to confine ourselves to torsion-free groups; the same question can be asked for any abelian group. Apart from several basic observations of this sort, the problem remained wide open: no

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counterexample in any group has ever been found, and virtually no progress towards a nonexistence result has been made.

In this note we present a complete solution to this problem.

Theorem 1. Let A be a finite, nonempty subset of an abelian group. If A is sum-full, then it is not zero-sum-free; that is, if every element of A is representable as a sum of two other elements, then A has a nonempty zero-sum subset.

The proof of Theorem 1 is of combinatorial nature; it is both surprisingly short and elementary, requiring nothing beyond very basic linear algebra.

Let \mathcal{M}_n denote the set of all integer square matrices of order n with all elements on the main diagonal greater than or equal to -1, all elements off the main diagonal non-negative, and all row sums equal to 1. The following lemma is at the heart of the proof of Theorem 1.

Lemma 1. For any matrix $M \in \mathcal{M}_n$ there exist nonzero vectors $u, v \in \{0, 1\}^n$ such that $M^t u = v$; that is, there exists a system of rows of M such that their sum is a nonzero, zero-one vector.

Deriving Theorem 1 from Lemma 1 is a matter of several lines. Namely, given a finite, nonempty, sum-full subset $A = \{a_1, \ldots, a_n\}$ of an abelian group, for each $k \in [n]$ we fix a representation $a_k = a_i + a_j$ with $i, j \in [n]$. Interpreting the resulting representations as *n*-dimensional vectors, consider the matrix M having these vectors as its rows. (Thus, for instance, if in the representation $a_k = a_i + a_j$ the indices i, j, k are pairwise distinct, then M contains three nonzero elements in row k: the element on the main diagonal is equal to -1, and the elements in columns i and j are equal to 1.) Clearly, we have $M \in \mathcal{M}_n$. By the lemma, there is a nonzero vector $v = (v_1, \ldots, v_n) \in \{0, 1\}^n$ representable as a linear combination of the rows of M. On the other hand, by the construction, every row of M is orthogonal to the vector $a := (a_1, \ldots, a_n)$; hence, v is orthogonal to a, too. Denoting by S the set of those indices $k \in [n]$ with $v_k = 1$, we then get $\sum_{k \in S} a_k = 0$, showing that A is not zero-sum-free.

We prove Lemma 1 in the next section. The proof is inductive; this is why the class \mathcal{M}_n is slightly wider than the class of matrices which can actually arise from sum-full sets.

Although Theorem 1 solves the original problem completely, in Section 3 we present yet another solution for the special case of the elementary abelian 3-groups. The reason for presenting a dedicated solution in this particular case is that the solution is based on a totally different approach relating sum-full sets and Sidon sets. Moreover, the central lemma used in this case is in fact valid for any abelian group. Postponing the details to Section 3, here we confine ourselves to the remark that for the elementary abelian 2-groups, the assertion of Theorem 1 is almost immediate.

Finally, we note that Theorem 1 stays true in the more general settings of *sequences* (multisets) instead of sets. This follows by observing that repeated elements can be removed from a set without acquiring or violating the sumfullness property.

2. Proof of Lemma 1

Our goal in this section is to prove Lemma 1, and hence Theorem 1.

We use induction on n. For $n \leq 2$ the assertion is easy to verify; assume thus that $n \geq 3$. Write $M = (m_{ij})_{1 \leq i,j \leq n}$ and for $i \in [n]$, denote by r_i the *i*th row of M.

If there exists $i \in [n]$ with $m_{ii} \ge 0$, then r_i is a nonzero zero-one vector; considering the row sum including r_i as its only summand, we get the assertion. We therefore assume that $m_{ii} = -1$ for all $i \in [n]$.

By the assumption just made, the sum of all off-diagonal elements of M is 2n. If the sum in every column is 2, then the sum of all the rows of M is the all-one vector, completing the proof in this case. Assume that there is a column with the sum of its off-diagonal elements 0 or 1. Changing the order of the columns if necessary, we further assume that the first column has this property.

If the sum of all off-diagonal elements of the first column is 0, then the first column contains only zero elements except that $m_{11} = -1$, and we apply the induction hypothesis to the matrix obtained from M by removing its first row and first column. This yields a system of rows of this new matrix with a nonzero zero-one sum. Prefixing 0 at the beginning of each of these rows, we get a system of rows of the *original* matrix M with a nonzero, zero-one sum, as wanted.

Hence, we can assume that the sum of the off-diagonal elements of the first column of M is 1; that is, one of the off-diagonal elements is 1, while the rest vanish. Without loss of generality, $m_{21} = 1$ while $m_{j1} = 0$ for $j \in [3, n]$. In addition to the element $m_{21} = 1$, the second row of M contains an element equal to 1 in a column other than the first or the second one, and we assume, for definiteness, that it is the third column: $m_{23} = 1$. With all the assumptions made so far, M can be visualized as follows:

$$M = \begin{pmatrix} -1 & * & * & * & \cdots & * \\ 1 & -1 & 1 & 0 & \cdots & 0 \\ \hline 0 & * & -1 & * & \cdots & * \\ 0 & * & * & -1 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & * & * & * & \cdots & -1 \end{pmatrix}$$

We now come to the crucial stage of the argument. Consider the square matrix M' of order n-2 obtained from M by adding its second column to the third one, and then removing the first two columns and first two rows of the resulting matrix. Numbering the rows and columns of M' from 3 to n, we write $M' = (m'_{ij})_{3 \le i,j \le n}$.

Recalling that by r_i we have denoted the *i*th row of M, let r'_i denote the *i*th row of M'. It is easily seen that $M' \in \mathcal{M}_{n-2}$; hence, by the induction hypothesis, there is a subset $I \subseteq [3, n]$ such that the vector $\sum_{i \in I} r'_i$ is nonzero and zero-one.

We remark that $m'_{ii} = m_{ii} = -1$ for all $i \in [4, n]$, but $m'_{33} \ge 0$ is possible (this happens if and only if $m_{32} > 0$). That is, all elements on the main diagonal of M', with the possible exception of m'_{33} , are equal to -1.

For $j \in \{2,3\}$ let $\sigma_j := \sum_{i \in I} m_{ij}$. If $3 \notin I$, then both σ_2 and σ_3 are sums of nonnegative terms; hence, are nonnegative themselves. Also, $\sigma_2 + \sigma_3 = \sum_{i \in I} m'_{i3} \in \{0,1\}$. This shows that $\sigma_2, \sigma_3 \in \{0,1\}$, and, as a result, $\sum_{i \in I} r_i$ is a (nonzero) zero-one vector, as wanted.

We thus assume that $3 \in I$. In this case we have $\sigma_2 + \sigma_3 \in \{0, 1\}$ and $\sigma_2 \ge 0$. If also $\sigma_3 \ge 0$, then $\sigma_2, \sigma_3 \in \{0, 1\}$ and we reach again the conclusion that $\sum_{i \in I} r_i$ is nonzero and zero-one.

We are left with the situation where $3 \in I$ and $\sigma_3 < 0$. In this case we have $\sigma_2 \in \{1, 2\}$ and $\sigma_3 = -1$, and it follows that $r_2 + \sum_{i \in I} r_i$ is a zero-one vector.

3. The elementary abelian 3-groups

In this section we prove a version of Theorem 1 restricted to the elementary abelian 3-groups.

Theorem 2. Let A be a finite, nonempty subset of an elementary abelian 3-group. If A is sum-full, then it is not zero-sum-free.

Here is our main lemma.

Lemma 2. Let A be a finite, generating, sum-full, zero-sum-free subset of an abelian group. Then for any proper subgroup H, there are $a_1, a_2, a_3, a_4 \in A \setminus H$ with $a_1 + a_2 = a_3 + a_4$ and $\{a_1, a_2\} \neq \{a_3, a_4\}$.

Proof. Since A is generating and H is proper, there is an element $a_1 \in A \setminus H$. Since A is sum-full, there are $a_2, b_1 \in A$ with $a_1 = a_2 + b_1$. In view of $a_1 \notin H$, at least one of a_2 and b_1 is not in H; switching a_2 and b_1 if needed, we assume that $a_2 \notin H$. Iterating,

we get an infinite chain of equalities

$$a_1 = a_2 + b_1,$$

$$a_2 = a_3 + b_2,$$

$$\vdots$$

with $a_1, a_2, \ldots \in A \setminus H$ and $b_1, b_2, \ldots \in A$. Since A is finite, there are indices $1 \leq i < j$ such that $a_i = a_j$, while $a_s \neq a_t$ whenever $i \leq s < t \leq j - 1$. Taking the sum of the corresponding equalities, after a cancellation we obtain

$$b_i + \dots + b_{j-1} = 0.$$

Since A is zero-sum-free, the summands in the left-hand side are not pairwise distinct; thus we have, say, $b_{\sigma} = b_{\tau}$ with $i \leq \sigma < \tau \leq j - 1$. This leads to

$$a_{\sigma} - a_{\sigma+1} = b_{\sigma} = b_{\tau} = a_{\tau} - a_{\tau+1}$$

and then

$$a_{\sigma} + a_{\tau+1} = a_{\sigma+1} + a_{\tau}$$

proving the assertion unless either $a_{\sigma} = a_{\tau}$, or $a_{\sigma} = a_{\sigma+1}$ hold. However, in the former case we get a contradiction with the assumption $a_s \neq a_t$ $(i \leq s < t \leq j - 1)$, while in the latter case we have $b_{\sigma} = a_{\sigma} - a_{\sigma+1} = 0$, contradicting the assumption that A is zero-sum-free.

A subset S of an abelian group is called a Sidon set if it does not contain nontrivial additive quadruples; that is, if an equality $s_1 + s_2 = s_3 + s_4$ with $s_1, s_2, s_3, s_4 \in S$ is possible only for $\{s_1, s_2\} = \{s_3, s_4\}$. Clearly, any independent set in a vector space, and in particular any basis of a vector space, is Sidon.

The following corollary of Lemma 2 shows that a sum-full, zero-sum-free set cannot be "too close" to a Sidon set.

Corollary 1. Suppose that A is a finite, generating, sum-full, zero-sum-free subset of an abelian group. If $B \subseteq A$ is Sidon (in particular, if B is minimal generating), then $A \setminus B$ is generating.

For the proof, just apply the lemma to the subgroup H generated by the set $A \setminus B$: having H proper would produce a direct contradiction with the assertion of the lemma in view of $A \setminus H \subseteq A \setminus (A \setminus B) = B$.

The notion of a zero-sum-free set extends literally onto *sequences* with repeated elements allowed. Namely, a sequence is zero-sum-free if it does not contain nonempty zero-sum subsequences. The following classical result of Olson [2] establishes an upper bound for the largest possible size of a zero-sum-free sequence in a finite vector space. **Lemma 3** (Olson). Let A be a finite sequence of vectors of the finite vector space V over the p-element field, with a prime p. If A is zero-sum-free, then $|A| \leq (p-1) \dim V$.

Indeed, Olson has shown that for the finite abelian *p*-group with the invariants $p^{\alpha_1}, \ldots, p^{\alpha_m}$, the largest size of a zero-sum-free sequence is $p^{\alpha_1} + \cdots + p^{\alpha_m} - m$; this easily implies Lemma 3.

Proof of Theorem 2. We consider the underlying group as a vector space over the field \mathbb{F}_3 and denote it V.

Suppose that the theorem is wrong, and let $A \subseteq V$ be a counterexample; that is, A does not contain a zero-sum subset, while every element of A is a sum of two other elements. Without loss of generality we assume that A generates V; therefore, V is finite.

Choose arbitrarily an element $a \in A$ and find a representation a = b+c with some (not necessarily distinct) $b, c \in A$. It is easily verified that the set $\{a, b, c\}$ is Sidon; hence, by Corollary 1, its complement $A \setminus \{a, b, c\}$ is generating. Fix a basis $B \subseteq A \setminus \{a, b, c\}$. Applying Corollary 1 a second time, the complement $A \setminus B$ is generating. Since some of the vectors of $A \setminus B$ (namely, a, b, and c) are linearly dependent, we have

$$\dim V < |A \setminus B| = |A| - \dim V$$

whence $|A| > 2 \dim V$, contradicting Lemma 3.

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