# THE POPULARITY GAP 

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#### Abstract

Suppose that $A$ is a finite, nonempty subset of a cyclic group of either infinite or prime order. We show that if the difference set $A-A$ is "not too large", then there is a nonzero group element with at least as many as $(2+o(1))|A|^{2} /|A-A|$ representations as a difference of two elements of $A$; that is, the second largest number of representations is, essentially, twice the average. Here the coefficient 2 is best possible.

We also prove continuous and multidimensional versions of this result, and obtain similar results for sufficiently dense subsets of an arbitrary abelian group.


## 1. Background and summary of results

A large body of problems, conjectures, and results in additive combinatorics assert, in different ways, that, normally, the number-of-representations function exhibits an irregular behaviour. The general framework for this line of research is as follows.

For finite subsets $A$ and $B$ and an element $g$ of an abelian group, let $r_{A, B}(g):=$ $|A \cap(g-B)|$; that is, $r_{A, B}(g)$ is the number of representations of $g$ as a sum of an element of $A$ and an element of $B$. How close to a constant function can the function $r_{A, B}$ be?

In this paper we consider representations by differences; that is, we are concerned with the special case where $B=-A$. Notation-wise, we abbreviate $r_{A,-A}$ as $r_{A}$.

Clearly, the largest value attained by the function $r_{A}$ is $|A|$. Our principal results show that for the cyclic groups of either infinite or prime order, the second largest value attained by $r_{A}$ is, essentially, at least twice the average value.

For a prime $p$, by $\mathbb{C}_{p}$ we denote the cyclic group of order $p$.
Theorem 1. Suppose that $p$ is a prime, $A \subseteq \mathbb{C}_{p}$ is a nonempty subset, and $\delta \in(0,1 / 3)$. If

$$
|A-A| \leq \min \{K|A|, 2(p+1) / 3\}, \quad K<|A|^{\delta},
$$

then there is a nonzero element $d \in \mathbb{C}_{p}$ such that $r_{A}(d)>2 K^{-1}|A|(1-2 \delta \ln (2 / \delta))$.
The following integer analog of Theorem 1 is, in fact, its immediate consequence.
Theorem 2. Suppose that $A$ is a finite, nonempty set of integers, and that $\delta \in(0,1 / 3)$. If

$$
|A-A| \leq K|A|, \quad K<|A|^{\delta},
$$

then there is an integer $d \neq 0$ such that $r_{A}(d)>2 K^{-1}|A|(1-2 \delta \ln (2 / \delta))$.

Comparing Theorems 1 and 2, we see that the bounds obtained are not affected by the "modulo- $p$ overlapping". Loosely speaking, there is no difference between the group $\mathbb{C}_{p}$ and the group of integers $\mathbb{Z}$ as far as the second largest value of the function $r_{A}$ is concerned.

The following theorem establishes a similar estimate in terms of the length of the set $A$ instead of the size of its difference set.

Theorem 3. Suppose that $L>1$, and that $A \subseteq[1, L]$ is a nonempty set of integers. If $|A-A|<|A|^{1+\delta}$ with $\delta \in(0,1 / 3)$, then there is an integer $d \neq 0$ such that $r_{A}(d)>$ $\frac{|A|^{2}}{L}(1-2 \delta \ln (2 / \delta))$.

We note that the assumption $|A-A|<|A|^{1+\delta}$ of Theorem 3 holds trivially if $A$ is sufficiently dense in $[1, L]$; say, $|A|>(2 L)^{1 /(1+\delta)}$ suffices.

Theorem 3 follows readily from Theorem 2 by letting $K:=|A-A| /|A|$ and observing that then $2 K^{-1}|A|=2|A|^{2} /|A-A|>|A|^{2} / L$.

Apart from the factor $1-2 \delta \ln (2 / \delta)$, the lower bounds obtained in Theorems 1-3 are, essentially, best possible; they are attained, for example, when $A$ is an interval or an appropriately sampled random set.

Interestingly, Theorems 1-3 cannot be obtained by a straightforward averaging as the number of elements $d \in A-A$ with $r_{A}(d)$ large can have "zero measure".

Claim 1. For any $\varepsilon \in(0,1)$ there is a finite, nonempty set $A \subset \mathbb{Z}$ such that

$$
\left|\left\{d: r_{A}(d) \geq(1-\varepsilon) \frac{2|A|^{2}}{|A-A|}\right\}\right| \leq 2 \varepsilon|A-A|
$$

Since the function $r_{A}: d \mapsto|A \cap(A+d)|$ is generalized by the functions $\left(d_{1}, \ldots, d_{k-1}\right) \mapsto$ $\left|A \cap\left(A+d_{1}\right) \cap \cdots \cap\left(A+d_{k-1}\right)\right|$ (see the next section), it is natural to extend Theorem 1 onto the intersection of $k \geq 3$ translates of the set $A$. In this direction, we prove the following multidimensional generalization.

Theorem 4. Suppose that $p$ is a prime, $A \subseteq \mathbb{C}_{p}$ is a nonempty subset, and $\delta \in(0,1 /(3 k))$ with an integer $k \geq 3$. If

$$
|A-A| \leq \min \{K|A|, 2(p+1) / 3\}, \quad K<|A|^{\delta},
$$

then there are pairwise distinct, nonzero elements $d_{1}, \ldots, d_{k-1} \in \mathbb{C}_{p}$ such that

$$
\left|A \cap\left(A+d_{1}\right) \cap \cdots \cap\left(A+d_{k-1}\right)\right|>2^{k-1} K^{-(k-1)}|A|\left(1-3 \delta k^{2} \ln (1 / k \delta)\right)
$$

From Theorem 4 one can easily derive multidimensional analogs of Theorems 2 and 3; we omit the details.

Our next result exhibits irregularities in the behaviour of the function $r_{A}$ for large subsets $A$ of an arbitrary finite abelian group.

Theorem 5. Suppose that $A$ is a subset of a finite abelian group $G$ such that $|A| \geq 2^{10}$ and $|A-A|=(1-\varepsilon)|G|$ with $0<\varepsilon \leq 2^{-5}$. If $|A-A| \leq|A|^{1+\sqrt{\varepsilon} / 2}$, then there is a nonzero element $d \in G$ such that $r_{A}(d) \geq \frac{|A|^{2}}{|A-A|}\left(1+\frac{1}{8} \sqrt{\varepsilon}\right)$.

Finally, we prove a continuous analog of Theorem 3. For a function $f \in L^{1}(\mathbb{R})$, let

$$
(f \circ f)(x):=\int_{\mathbb{R}} f(t) f(x+t) d t
$$

Theorem 6. Suppose that $f$ is a real, nonnegative function with $\operatorname{supp}(f) \subseteq[0,1]$. If $f$ is not constant on its support then, letting $\rho:=\|f\|_{2} /\|f\|_{1}$, for any $0<\delta \leq \min \left\{\frac{1}{2 \rho^{12}}, \frac{1}{2^{8}}\right\}$ we have

$$
\sup _{x \in[-1,1] \backslash[-\delta, \delta]} \frac{(f \circ f)(x)}{\|f\|_{1}^{2}} \geq 1-8 \max \left\{\sqrt[8]{2 \delta}, \frac{\ln \log _{\rho}(1 / 2 \delta)}{\log _{\rho}(1 / 2 \delta)}\right\}
$$

We note that the matching upper bound $\|f \circ f\|_{\infty} \leq\|f\|_{1}^{2}$ is trivial.
The quantity $\log _{\rho}(1 / 2 \delta)$ characterizes the length of the forbidden interval $[-\delta, \delta]$ as measured against the "scatter" of $f$. If $\log _{\rho}(1 / 2 \delta)$ is small enough, then we can have $\operatorname{supp}(f) \subseteq[0, \delta]$; in this case $\operatorname{supp}(f \circ f) \subseteq[-\delta, \delta]$ and there do not exist $x \notin[-\delta, \delta]$ with $(f \circ f)(x)>0$. Theorem 6 assumes $\log _{\rho}(1 / 2 \delta) \geq 12$, and the remainder term is $o(1)$ in the regime where $\log _{\rho}(1 / 2 \delta) \rightarrow \infty$ and $\delta \rightarrow 0$.

Our argument involves two major components: the higher energies technique [SS13] and a result in the spirit of [L01] establishing an extremal property of the interval subsets of prime-order groups. The central role is played by the following quantity.

For finite subsets $A$ and $D$ of an abelian group and integer $k \geq 1$, let $T_{D}^{(k)}(A)$ be the number of $k$-tuples $\left(a_{1}, \ldots, a_{k}\right) \in A^{k}$ such that $a_{i}-a_{j} \in D$ for any $1 \leq i, j \leq k$. In particular, $T_{D}^{(k)}(D)$ is the number of $k$-tuples $\left(d_{1}, \ldots, d_{k}\right)$ such that

$$
\left\{0, d_{1}, \ldots, d_{k}\right\}-\left\{0, d_{1}, \ldots, d_{k}\right\} \subseteq D
$$

The key result we prove about this quantity shows that in a group of prime order, if $0 \in D=(-D)$, then $T_{D}^{(k)}(A)$ can only get larger if $A$ and $D$ are replaced with the intervals $\bar{A}, \bar{D}$ of sizes $|\bar{A}|=|A|$ and $|\bar{D}|=|D|$, respectively, centered at 0 ; that is, either $\bar{A}=[-m, m]$, or $\bar{A}=[-(m-1), m]$ with $m=\lfloor|A| / 2\rfloor$ in both cases, and similarly for $\bar{D}$.

Proposition 1. For any prime $p \geq 5$, integer $k \geq 1$, and nonempty subsets $A, D \subseteq \mathbb{C}_{p}$ with $0 \in D=(-D)$, we have

$$
T_{D}^{(k)}(A) \leq T_{\bar{D}}^{(k)}(\bar{A})
$$

where $\bar{A}, \bar{D} \subseteq \mathbb{C}_{p}$ are intervals with $|\bar{A}|=|A|$ and $|\bar{D}|=|D|$, centered at 0 .
As a consequence of Proposition 1 we prove the following, slightly technical, corollary.

Corollary 1. For any prime $p \geq 5$, integer $k \geq 3$, and subset $D \subseteq \mathbb{C}_{p}$ with $0 \in D=$ $(-D)$ and $k \leq|D| \leq \frac{2}{3}(p+1)$, we have

$$
T_{D}^{(k)}(D) \leq 3 k 2^{-k-1}|D|^{k}
$$

We now turn to the proofs of the results discussed above. In the next section we introduce the basic notation, collect auxiliary tools needed for the proofs, and prove Claim 1. In Section 3 we prove the key Proposition 1. Theorem 1 is derived from Proposition 1 in Section 4 where also Corollary 1 is proved; as we have noticed, Theorems 2 and 3 follow from Theorem 1 and therefore do not require any additional consideration. Theorems 5 and 4 are proved in Sections 5 and 6, respectively. A proof of Theorem 6, along with a somewhat broader context for the problems studied in this paper, is outlined in the concluding Section 7.

## 2. Notation and preliminaries

In this section $A$ and $D$ are finite, nonempty subsets of an abelian group $G$.
Following [SS13] (but using a different notation), for an integer $k \geq 2$ we let

$$
\begin{equation*}
R_{A}^{(k)}\left(x_{1}, \ldots, x_{k-1}\right):=\left|A \cap\left(A+x_{1}\right) \cap \cdots \cap\left(A+x_{k-1}\right)\right|, \quad x_{1}, \ldots, x_{k-1} \in G \tag{1}
\end{equation*}
$$

This quantity generalizes the function $r_{A}$ which is obtained as a particular case $k=2$. Alternatively, $R_{A}^{(k)}\left(x_{1}, \ldots, x_{k-1}\right)$ is the number of representations of $\left(x_{1}, \ldots, x_{k-1}\right)$ as a difference of an element of the "diagonal" $\{(a, \ldots, a): a \in A\}$ and an element of the Cartesian power $A^{k-1}$. Clearly,

$$
\begin{equation*}
\sum_{x_{1}, \ldots, x_{k-1} \in G} R_{A}^{(k)}\left(x_{1}, \ldots, x_{k-1}\right)=|A|^{k} \tag{2}
\end{equation*}
$$

Less obvious is the identity

$$
\begin{equation*}
\sum_{x_{1}, \ldots, x_{k-1} \in G}\left(R_{A}^{(k)}\left(x_{1}, \ldots, x_{k-1}\right)\right)^{l}=\sum_{y_{1}, \ldots, y_{l-1} \in G}\left(R_{A}^{(l)}\left(y_{1}, \ldots, y_{l-1}\right)\right)^{k} \tag{3}
\end{equation*}
$$

valid for any $k, l \geq 2$ and any finite subset $A \subseteq G$, see [SV12, Lemma 2.8].
The common value of the two sums in (3) is denoted $\mathrm{E}_{k, l}(A)$; thus, $\mathrm{E}_{k, l}(A)=\mathrm{E}_{l, k}(A)$. We write

$$
\mathrm{E}_{k}(A):=\mathrm{E}_{k, 2}(A)=\sum_{x \in G}|A \cap(A+x)|^{k}
$$

Let $\mu(A):=\max \left\{r_{A}(d): d \in A-A, d \neq 0\right\}$; that is, $\mu(A)$ is the second largest value attained by the function $r_{A}$. We have

$$
\begin{equation*}
\mathrm{E}_{k}(A)=\sum_{x \in G}|A \cap(A+x)|^{k} \leq|A|^{k}+(\mu(A))^{k-1}|A|^{2} \tag{4}
\end{equation*}
$$

Recall that in Section 1 we have defined

$$
T_{D}^{(k)}(A):=\left|\left\{\left(a_{1}, \ldots, a_{k}\right) \in A^{k}: a_{i}-a_{j} \in D, 1 \leq i, j \leq k\right\}\right|, \quad k \geq 1
$$

Thus, for instance, if $0 \in D$, then $T_{D}^{(1)}(A)=|A|$, and if $A-A \subseteq D$, then $T_{D}^{(k)}(A)=|A|^{k}$. Furthermore, $T_{D}^{(k)}(A+g)=T_{D}^{(k)}(A)$ for any group element $g$, and $T_{D}^{(2)}(A)$ is the total number of representations of the elements $d \in D$ as $d=a_{1}-a_{2}$ with $a_{1}, a_{2} \in A$. The relevance of this quantity in our context is explained by the observation that if $D=A-A$ and $\left(a_{1}, \ldots, a_{k-1}\right) \in \operatorname{supp}\left(R_{A}^{(k)}\right)$, then $a_{i} \in D$ and $a_{i}-a_{j} \in D$ for all $1 \leq i, j \leq k-1$; as a result,

$$
\begin{equation*}
\left|\operatorname{supp}\left(R_{A}^{(k)}\right)\right| \leq T_{D}^{(k-1)}(D) \tag{5}
\end{equation*}
$$

Combining identities (2) and (3) and estimates (5) and (4), and using the Cauchy-Schwarz inequality, we get

$$
\begin{align*}
|A|^{2 k+2} & =\left(\sum_{x_{1}, \ldots, x_{k} \in G} R_{A}^{(k+1)}\left(x_{1}, \ldots, x_{k}\right)\right)^{2} \\
& \leq\left|\operatorname{supp}\left(R_{A}^{(k+1)}\right)\right| \cdot \sum_{x_{1}, \ldots, x_{k} \in G}\left(R_{A}^{(k+1)}\left(x_{1}, \ldots, x_{k}\right)\right)^{2} \\
& =\left|\operatorname{supp}\left(R_{A}^{(k+1)}\right)\right| \cdot \sum_{x \in D}\left(R_{A}^{(2)}(x)\right)^{k+1} \\
& =\left|\operatorname{supp}\left(R_{A}^{(k+1)}\right)\right| \cdot \mathrm{E}_{k+1}(A) \\
& \leq T_{D}^{(k)}(D) \cdot\left(|A|^{k+1}+(\mu(A))^{k}|A|^{2}\right) \tag{6}
\end{align*}
$$

The resulting inequality is, in fact, our main path in this paper to estimating $\mu(A)$.
We close this section with the proof of Claim 1 from Section 1.
Proof of Claim 1. Fix an integer $n \geq \frac{1}{\varepsilon}+2$, write $P:=[1, n]$, and let $\Lambda \subseteq \mathbb{Z}$ be a finite Sidon set (see, for instance, [O04]) satisfying $(2 P-2 P) \cap(2 \Lambda-2 \Lambda)=\{0\}$ and $|\Lambda|>\sqrt{2 / \varepsilon}+1$. Consider the sumset $A:=P+\Lambda$. We have $|A|=|\Lambda| n$ and $|A-A|=$ $(2 n-1)\left(|\Lambda|^{2}-|\Lambda|+1\right)$. Any element $d \in A-A$ can be represented as $d=\lambda_{1}-\lambda_{2}+p_{1}-p_{2}$ with $\lambda_{1}, \lambda_{2} \in \Lambda$ and $p_{1}, p_{2} \in P$; moreover, if $d \notin P-P$, then $\lambda_{1}$ and $\lambda_{2}$ are uniquely determined by $d$, and $r_{A}(d)=n-\left|p_{1}-p_{2}\right|$. Therefore if $r_{A}(d)>(1-\varepsilon) 2|A|^{2} /|A-A|$, then

$$
n-\left|p_{1}-p_{2}\right|>(1-\varepsilon) \frac{2 n^{2}|\Lambda|^{2}}{(2 n-1)\left(|\Lambda|^{2}-|\Lambda|+1\right)}>(1-\varepsilon) n
$$

implying $\left|p_{1}-p_{2}\right|<\varepsilon n$. This shows that there are at most as many as $\left(|\Lambda|^{2}-|\Lambda|\right)(2 \varepsilon n+1)$ elements $d \notin P-P$ with $r_{A}(d)>(1-\varepsilon) 2|A|^{2} /|A-A|$. Since $|P-P|=2 n-1$, we get at most

$$
\begin{aligned}
\left(|\Lambda|^{2}\right. & -|\Lambda|)(2 \varepsilon n+1)+2 n-1 \\
& =\varepsilon(2 n-1)\left(|\Lambda|^{2}-|\Lambda|\right)+(\varepsilon+1)\left(|\Lambda|^{2}-|\Lambda|\right)+2 n-1 \\
& =\varepsilon(2 n-1)\left(|\Lambda|^{2}-|\Lambda|+1\right)+(\varepsilon+1)\left(|\Lambda|^{2}-|\Lambda|+1\right)+(2 n(1-\varepsilon)-2)
\end{aligned}
$$

elements totally. Finally, we notice that the first summand in the right-hand side is $\varepsilon|A-A|$ while, in view of the assumptions $n \geq \frac{1}{\varepsilon}+2$ and $|\Lambda|>\sqrt{2 / \varepsilon}+1$, the last two
summands can be estimated as

$$
(\varepsilon+1)\left(|\Lambda|^{2}-|\Lambda|+1\right)<\frac{1}{2} \varepsilon(2 n-1)\left(|\Lambda|^{2}-|\Lambda|+1\right)=\frac{1}{2} \varepsilon|A-A|
$$

and

$$
2 n(1-\varepsilon)-2<2 n-1<\frac{1}{2} \varepsilon(2 n-1)\left(|\Lambda|^{2}-|\Lambda|+1\right)=\frac{1}{2} \varepsilon|A-A| .
$$

## 3. Proof of Proposition 1

We start with a lemma establishing two simple identities that the quantities $T_{D}^{(k)}(A)$ satisfy.

Lemma 1. For any integer $k \geq 1$ and any finite subsets $A, D$ of an abelian group with $0 \in D=(-D)$ we have

$$
\begin{equation*}
T_{D}^{(k+1)}(A)=\sum_{\substack{d_{1}, \ldots, d_{k} \in D \\ d_{i}-d_{j} \in D}} R_{A}^{(k+1)}\left(d_{1}, \ldots, d_{k}\right) \tag{7}
\end{equation*}
$$

(with the indices $i$ and $j$ running independently over all values in $\{1, \ldots, k\}$ ), and

$$
\begin{equation*}
T_{D}^{(k+1)}(A)=\sum_{a \in A} T_{D}^{(k)}(A \cap(D+a)) \tag{8}
\end{equation*}
$$

Proof. From the definition,

$$
T_{D}^{(k+1)}(A)=\left|\left\{\left(a_{1}, \ldots, a_{k+1}\right) \in A^{k+1}: a_{i}-a_{j} \in D, 1 \leq i, j \leq k+1\right\}\right| .
$$

Letting $d_{i}:=a_{k+1}-a_{i}(i=1, \ldots, k)$, we rewrite this equality as

$$
\begin{aligned}
T_{D}^{(k+1)}(A) & =\sum_{\substack{d_{1}, \ldots, d_{k} \in D \\
d_{i}-d_{j} \in D}}\left|\left\{a_{k+1} \in A: a_{k+1}-d_{1}, \ldots, a_{k+1}-d_{k} \in A\right\}\right| \\
& =\sum_{\substack{d_{1}, \ldots, d_{k} \in D \\
d_{i}-d_{j} \in D}}\left|A \cap\left(A+d_{1}\right) \cap \cdots \cap\left(A+d_{k}\right)\right|,
\end{aligned}
$$

proving (7). In a similar way, re-denoting $a_{k+1}$ by $a$, we get

$$
\begin{aligned}
T_{D}^{(k+1)}(A) & =\sum_{a \in A}\left|\left\{\left(a_{1}, \ldots, a_{k}\right) \in A^{k}: a_{1}-a, \ldots, a_{k}-a \in D, a_{i}-a_{j} \in D\right\}\right| \\
& =\sum_{a \in A}\left|\left\{\left(a_{1}, \ldots, a_{k}\right) \in(A \cap(D+a))^{k}: a_{i}-a_{j} \in D\right\}\right| \\
& =\sum_{a \in A} T_{D}^{(k)}(A \cap(D+a))
\end{aligned}
$$

which establishes (8).
For $1 \leq n \leq p$ we let $J_{n}:=[1, n] \subseteq \mathbb{C}_{p}$.

Lemma 2. For any $k \geq 2$ and any fixed $d_{1}, \ldots, d_{k-1} \in \mathbb{C}_{p}$, the sequence

$$
\left\{R_{J_{n}}^{(k)}\left(d_{1}, \ldots, d_{k-1}\right): 1 \leq n \leq p-1\right\}
$$

is convex.
Proof. Let

$$
B_{n}:=J_{n} \cap\left(J_{n}+d_{1}\right) \cap \cdots \cap\left(J_{n}+d_{k-1}\right) ;
$$

thus, $\left|B_{n}\right|=R_{J_{n}}^{(k)}\left(d_{1}, \ldots, d_{k-1}\right)$. We have $B_{1} \subseteq B_{2} \subseteq \cdots$ and $B_{n}+1 \subseteq B_{n+1}$. Furthermore, we observe that if $x, x+1 \in B_{n}$ for some $x \in \mathbb{C}_{p}$ and $2 \leq n \leq p-1$, then indeed $x \in B_{n-1}$; we leave the verification to the reader.

To prove that the sequence $\left|B_{n}\right|$ is convex, we show that

$$
\left(B_{n} \backslash B_{n-1}\right)+1 \subseteq B_{n+1} \backslash B_{n}, \quad 2 \leq n \leq p-2
$$

(which implies $\left|B_{n}\right|-\left|B_{n-1}\right| \leq\left|B_{n+1}\right|-\left|B_{n}\right|$ ). Indeed, if $x \in B_{n} \backslash B_{n-1}$, then $x+1 \in B_{n+1}$, while from $x \in B_{n}$ and $x \notin B_{n-1}$ we derive that $x+1 \notin B_{n}$ by the observation just made.

For a finite subset $S$ of a cyclic group of either infinite or prime order, we define $\bar{S}:=[-(m-1), m]$ if $|S|=2 m$, and $\bar{S}:=[-m, m]$ if $|S|=2 m+1$ (with an integer $m \geq 0$ ); that is, $\bar{S}$ is the interval of size $|\bar{S}|=|S|$, contained in the same underlying group, and centered around 0 , possibly with a small "right-end overweight".

Proof of Proposition 1. We use induction by $k$. For $k=1$ we have the equalities

$$
T_{D}^{(1)}(A)=|A|=|\bar{A}|=T_{\bar{D}}^{(1)}(\bar{A}) ;
$$

assume now that $k \geq 2$.
For integer $1 \leq n \leq p-1$ let $t(n):=T_{\bar{D}}^{(k)}\left(\overline{J_{n}}\right)$; equivalently, $t(n)=T_{\bar{D}}^{(k)}\left(J_{n}\right)$. Since the sum of convex sequences is convex, by (7) and Lemma 2, the sequence $t(n)$ is convex in the range $1 \leq n \leq p-1$.

From (8) and the induction hypotheses,

$$
\begin{aligned}
T_{D}^{(k+1)}(A) & =\sum_{a \in A} T_{D}^{(k)}(A \cap(D+a)) \\
& \leq \sum_{a \in A} T_{\bar{D}}^{(k)}(\overline{A \cap(D+a)}) \\
& =\sum_{a \in A} t(|A \cap(D+a)|) .
\end{aligned}
$$

By [L01, Theorem 1], the sequence $\{|A \cap(D+a)|: a \in A\}$ is majorized by the sequence $\{|\bar{A} \cap(\bar{D}+a)|: a \in \bar{A}\}$, in the sense that for any $1 \leq h \leq|A|$, the sum of the $h$ largest terms of the former sequence does not exceed the sum of the $h$ largest terms of the latter
one. On the other hand, as observed above, $t(n)$ is a convex sequence. Consequently, by Karamata's inequality, we have

$$
\sum_{a \in A} T_{\bar{D}}^{(k)}(\overline{A \cap(D+a)}) \leq \sum_{a \in \bar{A}} T_{\bar{D}}^{(k)}(\bar{A} \cap(\bar{D}+a)) .
$$

Recalling (8), we conclude that the sum in the right-hand side is $T_{\bar{D}}^{(k+1)}(\bar{A})$, which proves the assertion.

## 4. Proof of Theorem 1

Lemma 3. Suppose that $p \geq 5$ is a prime, and that $D=[-m, m] \subseteq \mathbb{C}_{p}$ with $m=$ $(|D|-1) / 2$. If $m<\frac{1}{3} p$, then $T_{D}^{(k)}(D)=(m+1)^{k+1}-m^{k+1}$.

We give two proofs of this lemma, the second being of more geometric nature.
First proof of Lemma 3. Let $J:=[0, m]^{k}$, and for an integer $s \in[0, m]$, denote by $\vec{s}$ the $k$-tuple $(s, \ldots, s)$. From the assumption $m<\frac{1}{3} p$ it is not difficult to derive that the set of all those $k$-tuples $\left(x_{1}, \ldots, x_{k}\right) \in D^{k}$ with $x_{i}-x_{j} \in D$ for all $i, j \in[1, k]$ (which are the $k$-tuples counted in $\left.T_{D}^{(k)}(D)\right)$ is the union of the translates $J-\vec{s}$ over all $s \in[0, m]$. Therefore, using the inclusion-exclusion formula,

$$
T_{D}^{(k)}(D)=\left|\bigcup_{s=0}^{m}(J-\vec{s})\right|=\sum_{\varnothing \neq S \subseteq[0, m]}(-1)^{|S|-1}\left|\bigcap_{s \in S}(J-\vec{s})\right|
$$

With $s_{\min }, s_{\max } \in[0, m]$ denoting the smallest and the largest elements of $S$, respectively, the intersection in the right-hand side is the cube $\left[-s_{\min }, m-s_{\max }\right]^{k}$. Consequently,

$$
T_{D}^{(k)}(D)=\sum_{l=0}^{m}(m+1-l)^{k} \sum_{S}(-1)^{|S|-1},
$$

where the inner sum extends onto all subsets $\varnothing \neq S \subseteq[0, m]$ with $s_{\max }-s_{\min }=l$. It remains to notice that this sum is equal to $m+1$ if $l=0$ and to $-m$ if $l=1$, and that it vanishes for any $l \geq 2$ (indeed, for any fixed $0 \leq x<y \leq m$ with $y \geq x+2$ there are equally many even-sized and odd-sized subsets $S \subseteq[x, y]$ with $x, y \in S)$.

Second proof of Lemma 3. As in the first proof, let $J:=[0, m]^{k}$, and for $s \in[0, m]$, denote by $\vec{s}$ the $k$-tuple $(s, \ldots, s)$; we want to show that the translates $J-\vec{s}(s \in[0, m])$ jointly contain as many as $(m+1)^{k+1}-m^{k+1}$ elements.

The cube $J$ has $2 k$ faces, any face is obtained by setting one of the coordinates to 0 or $m$. Of these $2 k$ faces, $k$ are "visible" from the origin, with one parallel invisible face corresponding to any visible one. Any such pair of faces contributes to $T_{D}^{(k)}(D)$ a parallelepiped of volume $(m+1)^{k-1}(m+1)=(m+1)^{k}$ (because the distance between the two hyperplanes containing any pair of faces is $m$ ). Each of these parallelepipeds contains the set $\{-\vec{s}: s \in[0, m]\}$. One can check by induction that the intersection of any $j$ parallelepipeds is a cube of codimension $j-1$, and with the edge length $m+1$.

Hence, its volume is $(m+1)^{k+1-j}$. Also, the cube $[1, m]^{k} \subseteq \bigcup_{s \in[0, m]}(J-\vec{s})$ has size $m^{k}$ and does not belong to any of these parallelepipeds. Thus by the inclusion-exclusion formula we have the total contribution of

$$
\begin{aligned}
\sum_{j=1}^{k}\binom{k}{j}(-1)^{j+1}(m+1)^{k+1-j} & +m^{k} \\
& =-(m+1)\left(m^{k}-(m+1)^{k}\right)+m^{k}=(m+1)^{k+1}-m^{k+1}
\end{aligned}
$$

Using elementary calculus, one can show that if $|D|=2 m+1 \geq k \geq 3$, then the expression $(m+1)^{k+1}-m^{k+1}$ in the statement of Lemma 3 does not exceed $3 k 2^{-k-1}|D|^{k}$. This establishes Corollary 1 as an immediate consequence of the lemma and Proposition 1.

Proof of Theorem 1. We write $D:=A-A$ and let $k:=\left\lfloor\delta^{-1}\right\rfloor$; thus $k \geq 2$.
The case $|A|=1$ is ruled out by the assumption $K<|A|^{\delta}$, and we assume that $|A| \geq 2$. By the Cauchy-Davenport theorem,

$$
\begin{equation*}
|A|^{\delta}>K \geq \frac{|D|}{|A|} \geq 2-\frac{1}{|A|} \geq \frac{3}{2} \tag{9}
\end{equation*}
$$

whence $|D| \geq|A| \geq(3 / 2)^{1 / \delta}>1 / \delta \geq k$. Therefore, $T_{D}^{(k)}(D) \leq 3 k 2^{-k-1}|D|^{k}$ by Corollary 1. Combining this estimate with (6), we get

$$
|A|^{2 k+2} \leq\left(|A|^{k+1}+(\mu(A))^{k}|A|^{2}\right) \cdot 3 k 2^{-k-1}|D|^{k} ;
$$

thus, at least one of

$$
\frac{5}{8}|A|^{2 k+2} \leq|A|^{k+1} \cdot 3 k 2^{-k-1}|D|^{k}
$$

and

$$
\frac{3}{8}|A|^{2 k+2} \leq(\mu(A))^{k}|A|^{2} \cdot 3 k 2^{-k-1}|D|^{k}
$$

holds true. This means that we have either

$$
|A|^{k+1} \leq \frac{12}{5} k 2^{-k}|D|^{k}
$$

or

$$
|A|^{2 k} \leq(\mu(A))^{k} \cdot 4 k 2^{-k}|D|^{k}
$$

In the first case, in view of $K^{k} \leq K^{1 / \delta}<|A|$, we obtain

$$
|A| \leq \frac{12}{5} k 2^{-k} K^{k}<\frac{12}{5} k 2^{-k}|A|
$$

which is impossible in view of $2^{k-2}>\frac{3}{5} k$.
In the second case, in view of $k>1 /(2 \delta)$, we have

$$
\mu(A) \geq \frac{2|A|^{2}}{|D|} e^{-\ln (4 k) / k} \geq 2 K^{-1}|A| e^{-2 \delta \ln (2 / \delta)} \geq 2 K^{-1}|A|(1-2 \delta \ln (2 / \delta))
$$

## 5. Proof of Theorem 5

We start with an estimate for the quantity $T_{D}^{(k)}(D)$ in the situation where $D$ is a large subset of a finite abelian group.

Lemma 4. Suppose that $k \geq 1$ is an integer, $G$ is a finite abelian group, and $D \subseteq G$ is a subset with $0 \in D=(-D)$. If $|G \backslash D|=\tau|D|$ with $0<\tau \leq \min \left\{1 / 2,2 /\left(k^{2}-k+2\right)\right\}$, then

$$
\begin{equation*}
T_{D}^{(k)}(D) \leq\left(1-\frac{1}{4} k(k-1) \tau\right)|D|^{k} \tag{10}
\end{equation*}
$$

Proof. Let $C:=G \backslash D$. Throughout the proof, it will be convenient to identify the sets $C$ and $D$ with their indicator functions; thus, for instance, $D(x)+C(x)=1$ for any $x \in G$.

Let

$$
\Omega_{k}:=\left\{\left(x_{1}, \ldots, x_{k}\right) \in D^{k}: x_{i}-x_{j} \in D, 1 \leq i, j \leq k\right\} .
$$

We use induction on $k$.
The case $k=1$ agrees with (10) in view of $T_{D}^{(1)}(D)=|D|$. Furthermore,

$$
\begin{aligned}
T_{D}^{(2)}(D) & =\left|\Omega_{2}\right| \\
& =\sum_{x \in D}|D \cap(D+x)| \\
& =\sum_{x \in D}(|D|-|C+x|+|C \cap(C+x)|) \\
& \leq(|D|-|C|)|D|+\sum_{x \in G}|C \cap(C+x)| \\
& =(|D|-|C|)|D|+|C|^{2} \\
& =\left(1-\tau+\tau^{2}\right)|D|^{2}
\end{aligned}
$$

which, in view of the assumption $\tau \leq \frac{1}{2}$, settles the case $k=2$.
We now estimate the quantity $T_{D}^{(k+1)}(D)$ assuming that $k+1 \geq 3$ and $\tau \leq 2 /\left(k^{2}+k+2\right)$.
For any $y_{1}, \ldots, y_{k} \in G$ we have

$$
\begin{align*}
D\left(y_{1}\right) \cdots D\left(y_{k}\right)= & D\left(y_{2}\right) \cdots D\left(y_{k}\right)-C\left(y_{1}\right) D\left(y_{2}\right) \cdots D\left(y_{k}\right) \\
= & D\left(y_{3}\right) \cdots D\left(y_{k}\right)-C\left(y_{1}\right) D\left(y_{2}\right) \cdots D\left(y_{k}\right)-C\left(y_{2}\right) D\left(y_{3}\right) \cdots D\left(y_{k}\right) \\
& \vdots  \tag{11}\\
= & 1-\sum_{i=1}^{k} C\left(y_{i}\right) D\left(y_{i+1}\right) \cdots D\left(y_{k}\right)
\end{align*}
$$

and, similarly,

$$
D\left(y_{i+1}\right) \cdots D\left(y_{k}\right)=1-\sum_{j=i+1}^{k} C\left(y_{j}\right) D\left(y_{j+1}\right) \cdots D\left(y_{k}\right) .
$$

Multiplying this identity by $C\left(y_{i}\right)$ and substituting the result into (11) we get

$$
\begin{aligned}
D\left(y_{1}\right) \cdots D\left(y_{k}\right) & =1-\sum_{i=1}^{k}\left(C\left(y_{i}\right)-\sum_{j=i+1}^{k} C\left(y_{i}\right) C\left(y_{j}\right) D\left(y_{j+1}\right) \cdots D\left(y_{k}\right)\right) \\
& =1-\sum_{i=1}^{k} C\left(y_{i}\right)+\sum_{1 \leq i<j \leq k} C\left(y_{i}\right) C\left(y_{j}\right) D\left(y_{j+1}\right) \cdots D\left(y_{k}\right) .
\end{aligned}
$$

We let $y_{i}=x-x_{i}$ and take sums to obtain

$$
\begin{aligned}
T_{D}^{(k+1)}(D)= & \sum_{\substack{\left(x_{1}, \ldots, x_{k}\right) \in \Omega_{k} \\
x \in D}} D\left(x-x_{1}\right) \cdots D\left(x-x_{k}\right) \\
= & \sum_{\substack{\left(x_{1}, \ldots, x_{k}\right) \in \Omega_{k} \\
x \in D}} 1-\sum_{\substack{\left(x_{1}, \ldots, x_{k}\right) \in \Omega_{k} \\
x \in D}} \sum_{i=1}^{k} C\left(x-x_{i}\right) \\
& \quad+\sum_{\substack{\left(x_{1}, \ldots, x_{k}\right) \in \Omega_{k} \\
x \in D}} \sum_{1 \leq i<j \leq k} C\left(x-x_{i}\right) C\left(x-x_{j}\right) D\left(x-x_{j+1}\right) \cdots D\left(x-x_{k}\right) \\
= & \sigma_{1}-\sigma_{2}+\sigma_{3},
\end{aligned}
$$

where each of $\sigma_{1}, \sigma_{2}, \sigma_{3}$ denotes the corresponding sum in the right-hand side. Clearly, we have $\sigma_{1}=T_{D}^{(k)}(D)|D|$. Furthermore,

$$
\begin{aligned}
\sigma_{2} & =k \sum_{\substack{\left(x_{1}, \ldots, x_{k}\right) \in \Omega_{k} \\
x \in D}} C\left(x-x_{k}\right) \\
& =k \sum_{\substack{\left(x_{1}, \ldots, x_{k}\right) \in \Omega_{k} \\
x \in G}}(1-C(x)) C\left(x-x_{k}\right) \\
& =\tau|D| \cdot k T_{D}^{(k)}(D)-k \sum_{\substack{\left(x_{1}, \ldots, x_{k}\right) \in \Omega_{k} \\
x \in G}} C(x) C\left(x-x_{k}\right) \\
& \geq \tau|D| \cdot k T_{D}^{(k)}(D)-k \tau^{2}|D|^{2} T_{D}^{(k-1)}(D)
\end{aligned}
$$

(the inequality follows by allowing $x_{k}$ to take all values from $G$ ).
Finally, for any fixed $1 \leq i<j \leq k$,

$$
\sum_{x_{i}, x \in D} C\left(x-x_{i}\right) C\left(x-x_{j}\right) \leq \sum_{x_{i} \in D}\left|\left(C+x_{i}\right) \cap\left(C+x_{j}\right)\right| \leq|C|^{2}=\tau^{2}|D|^{2}
$$

Consequently, $\sigma_{3} \leq\binom{ k}{2} T_{D}^{(k-1)}(D) \tau^{2}|D|^{2}$. Therefore

$$
\begin{aligned}
T_{D}^{(k+1)}(D) \leq & T_{D}^{(k)}(D)|D|-\tau k T_{D}^{(k)}(D)|D|+k \tau^{2} T_{D}^{(k-1)}(D)|D|^{2} \\
& +\binom{k}{2} \tau^{2} T_{D}^{(k-1)}(D)|D|^{2} \\
= & (1-\tau k) T_{D}^{(k)}(D)|D|+\frac{1}{2} k(k+1) \tau^{2} T_{D}^{(k-1)}(D)|D|^{2}
\end{aligned}
$$

In view of the induction hypothesis (10), it suffices to prove that

$$
\begin{aligned}
(1-\tau k)\left(1-\frac{1}{4} k(k-1) \tau\right)+\frac{1}{2} k(k+1) \tau^{2}\left(1-\frac{1}{4}(k-1)(k-2) \tau\right) & \\
& \leq 1-\frac{1}{4} k(k+1) \tau
\end{aligned}
$$

Expanding and simplifying, this inequality is equivalent to

$$
\left(k^{2}-1\right)(k-2) \tau^{2}-2\left(k^{2}+k+2\right) \tau+4 \geq 0
$$

which clearly holds true under the assumptions $k \geq 2$ and $\tau \leq 2 /\left(k^{2}+k+2\right)$ made at the beginning of the proof.

Proof of Theorem 5. Let $D:=A-A, \tau:=|G \backslash D| /|D|$, and $k:=\lceil 1 / \sqrt{\tau}\rceil+1$. Notice that the assumption $|D|=(1-\varepsilon)|G|$ implies $\tau=\varepsilon /(1-\varepsilon)$ and $\varepsilon=\tau /(\tau+1)$; along with $\varepsilon \leq 2^{-5}$, this leads to

$$
\begin{equation*}
\frac{1}{k}>\frac{\sqrt{\tau}}{2 \sqrt{\tau}+1}=\frac{\sqrt{\varepsilon}}{2 \sqrt{\varepsilon}+\sqrt{1-\varepsilon}}>0.747 \sqrt{\varepsilon} \tag{12}
\end{equation*}
$$

Furthermore, from $|D|<|A|^{1+\sqrt{\varepsilon} / 2}$ and $|A| \geq 2^{10}$ we get

$$
\begin{aligned}
& |A|^{k-1}|D|^{k} /|A|^{2 k} \leq|A|^{-(k+1)+k(1+\sqrt{\varepsilon} / 2)}=|A|^{k \sqrt{\varepsilon} / 2-1} \\
& \quad<|A|^{\sqrt{\varepsilon / \tau} / 2+\sqrt{\varepsilon}-1}=|A|^{\sqrt{1-\varepsilon} / 2+\sqrt{\varepsilon}-1}<|A|^{-0.331}<0.101 .
\end{aligned}
$$

As a result, applying the estimate (6) and Lemma 4,

$$
\begin{aligned}
|A|^{2 k} & \leq T_{D}^{(k)}(D) \cdot\left(|A|^{k-1}+(\mu(A))^{k}\right) \\
& \leq\left(1-\frac{1}{4} k(k-1) \tau\right) \cdot\left(|A|^{k-1}+(\mu(A))^{k}\right)|D|^{k} \\
& <\frac{3}{4} \cdot\left(0.101|A|^{2 k}+(\mu(A))^{k}|D|^{k}\right)
\end{aligned}
$$

whence

$$
(\mu(A))^{k}|D|^{k}>1.232|A|^{2 k}
$$

and then

$$
\mu(A)>\frac{|A|^{2}}{|D|}(1.232)^{1 / k}
$$

Finally, recalling (12),

$$
1.232^{1 / k}>1+\frac{1}{k} \ln (1.232)>1+\frac{1}{8} \sqrt{\varepsilon}
$$

## 6. Proof of Theorem 4

Proof of Theorem 4. Let $D:=A-A$, and let $\mu^{(k)}(A):=\max R_{A}^{(k)}\left(d_{1}, \ldots, d_{k-1}\right)$ where the maximum extends over all $(k-1)$-tuples $\left(d_{1}, \ldots, d_{k-1}\right) \in D^{k-1}$ with pairwise distinct, nonzero components $d_{i}$.

Let $l:=\lfloor 1 /(2 \delta(k-1))\rfloor$; we notice that, in view of

$$
\frac{1}{2 \delta(k-1)}-\frac{1}{3 \delta k}>\frac{1}{2 \delta(k-1)}-\frac{1}{3 \delta(k-1)}=\frac{1}{6 \delta(k-1)}>\frac{1}{6 \delta k}>1
$$

we have

$$
\begin{equation*}
\frac{1}{3 \delta k}<l \leq \frac{1}{2 \delta(k-1)} \tag{13}
\end{equation*}
$$

Similarly to (6), from (2) and (3), using Hölder's inequality we get

$$
\begin{align*}
|A|^{k(l+1)} & =\left(\sum_{d_{1}, \ldots, d_{l} \in D}\left(R_{A}^{(l+1)}\left(d_{1}, \ldots, d_{l}\right)\right)\right)^{k}  \tag{14}\\
& \leq\left(T_{D}^{(l)}(D)\right)^{k-1} \cdot \sum_{d_{1}, \ldots, d_{l} \in D}\left(R_{A}^{(l+1)}\left(d_{1}, \ldots, d_{l}\right)\right)^{k} \\
& =\left(T_{D}^{(l)}(D)\right)^{k-1} \cdot \sum_{d_{1}, \ldots, d_{k-1} \in D}\left(R_{A}^{(k)}\left(d_{1}, \ldots, d_{k-1}\right)\right)^{l+1} .
\end{align*}
$$

Let $\sigma_{0}$ denote the part of the sum in the right-hand side extending over the $(k-1)$-tuples $\left(d_{1}, \ldots, d_{k-1}\right)$ with either $d_{i}=0$, or $d_{i}=d_{j}$ with some $i, j \in[1, n], i \neq j$, and let $\sigma_{1}$ be the sum over the $k$-tuples $\left(d_{1}, \ldots, d_{k-1}\right)$ with all components $d_{i}$ distinct from 0 and from each other.

Using (13), we obtain

$$
\begin{aligned}
\left(T_{D}^{(l)}(D)\right)^{k-1} \sigma_{0} & \leq|D|^{l(k-1)} \cdot k^{2}|A|^{l} \sum_{d_{1}, \ldots, d_{k-2} \in D} R_{A}^{(k-1)}\left(d_{1}, \ldots, d_{k-2}\right) \\
& \leq|A|^{(1+\delta) l(k-1)} k^{2}|A|^{k+l-1} \\
& =k^{2}|A|^{k(l+1)-1+\delta l(k-1)} \\
& \leq k^{2}|A|^{k(l+1)-1 / 2} \\
& \leq \frac{1}{2}|A|^{k(l+1)} .
\end{aligned}
$$

Comparing this estimate with (14), we conclude that $\left(T_{D}^{(l)}(D)\right)^{k-1} \sigma_{1} \geq \frac{1}{2}|A|^{k(l+1)}$. On the other hand, we have $\sigma_{1} \leq\left(\mu^{(k)}(A)\right)^{l}|A|^{k}$. Therefore,

$$
\left(\mu^{(k)}(A)\right)^{l}|A|^{k}\left(T_{D}^{(l)}(D)\right)^{k-1} \geq \frac{1}{2}|A|^{k(l+1)}
$$

whence, by Corollary 1

$$
\begin{aligned}
\mu^{(k)}(A) & >\left(\left(3 l 2^{-l-1}|D|^{l}\right)\right)^{-(k-1) / l} \cdot 2^{-1 / l}|A|^{k} \\
& =\frac{2^{k-1}|A|}{|K|^{k-1}}((3 / 2) l)^{-(k-1) / l} 2^{-1 / l} \\
& >\frac{2^{k-1}|A|}{|K|^{k-1}}(3 l)^{-(k-1) / l} .
\end{aligned}
$$

Finally, using (13) again,

$$
(3 l)^{-(k-1) / l}>1-3(k-1) \frac{\ln (3 l)}{3 l}>1-3 k \frac{\ln (1 / k \delta)}{1 / k \delta}>1-3 \delta k^{2} \ln (1 / k \delta) .
$$

## 7. Proof of Theorem 6

So far, we have dealt with finite sets, which can be identified with their characteristic functions. More generally, instead of the sets, one can consider any finitely (or compactly) supported nonnegative function $f$, and study the quantities

$$
R_{f}^{(k)}\left(x_{1}, \ldots, x_{k-1}\right):=\sum_{x \in \operatorname{supp} f} f(x) f\left(x+x_{1}\right) \cdots f\left(x+x_{k-1}\right), \quad x_{1}, \ldots, x_{k-1} \in G
$$

(cf. (1)) aiming to show that some of these quantities are "atypically large".
In keeping with this interpretation, instead of (2) and (3) we use the identities

$$
\sum_{x_{1}, \ldots, x_{k-1} \in G} R_{f}^{(k)}\left(x_{1}, \ldots, x_{k-1}\right)=\|f\|_{1}^{k}
$$

and

$$
\sum_{x_{1}, \ldots, x_{k-1} \in G}\left(R_{f}^{(k)}\left(x_{1}, \ldots, x_{k-1}\right)\right)^{l}=\sum_{y_{1}, \ldots, y_{l-1} \in G}\left(R_{f}^{(l)}\left(y_{1}, \ldots, y_{l-1}\right)\right)^{k}
$$

implying

$$
\begin{align*}
\|f\|_{1}^{2 k+2} & \leq \sum_{x_{1}, \ldots, x_{k}}\left(R_{f}^{(k+1)}\left(x_{1}, \ldots, x_{k}\right)\right)^{2} \cdot\left|\operatorname{supp}\left(R_{f}^{(k+1)}\right)\right| \\
& \leq T_{D}^{(k)}(D) \sum_{x \in G}\left(R_{f}^{(2)}(x)\right)^{k+1} \tag{15}
\end{align*}
$$

where $D=\operatorname{supp}(f)$, cf. (6).
To illustrate this approach, consider a finite, nonempty subset $A$ of an abelian group $G$, and let $D:=A-A$ and $f=r_{A}$; thus, $\operatorname{supp}(f)=D$, and $\|f\|_{1}=|A|^{2}$. Suppose that
$K:=|D-D| /|D|<|D|^{\delta}$ for a sufficiently small $\delta>0$. If $G$ is cyclic and either infinite, or of prime order $p \geq \frac{3}{2}|D|-1$, then following the proofs of Theorems 1 and 2 one can show that there exists a nonzero group element $s$ such that

$$
(f \circ f)(s)=\left|\left\{(x, y, z, w) \in A^{4}: x+y-z-w=s\right\}\right| \geq\left(2+o_{\delta}(1)\right) \frac{|A|^{4}}{K|D|}
$$

As another illustration, we prove Theorem 6.
Proof of Theorem 6. Our argument uses continuous analogs of the quantities and results developed above in the discrete settings. We sketch the proof leaving it to the reader to work out the missing details.

Let $D:=[-1,1]$ and $\omega:=\|f\|_{1}^{-2} \sup _{x \in \mathbb{R} \backslash[-\delta, \delta]}(f \circ f)(x)$. We have

$$
\int_{\mathbb{R}}\left(f(x)-\|f\|_{1}\right)^{2} d x=\|f\|_{2}^{2}-\|f\|_{1}^{2}=\left(\rho^{2}-1\right)\|f\|_{1}^{2}
$$

and by the Cauchy-Schwarz inequality, for any $s \in \mathbb{R}$,

$$
\begin{aligned}
\omega\|f\|_{1}^{2} & \geq(f \circ f)(s) \\
& =\|f\|_{1}^{2}+\int_{\mathbb{R}} f(x)\left(f(x+s)-\|f\|_{1}\right) d x \\
& \geq\|f\|_{1}^{2}-\|f\|_{2} \sqrt{\rho^{2}-1}\|f\|_{1} .
\end{aligned}
$$

Consequently,

$$
\omega \geq 1-\rho \sqrt{\rho^{2}-1}
$$

Suppose first that $\rho \leq e^{2 \sqrt[4]{2 \delta}}$. Then $\rho<1+4 \sqrt[4]{2 \delta}$ and $\rho \sqrt{\rho+1}<\rho \sqrt{2 \rho} \leq \sqrt{2} e^{3 \sqrt[4]{2 \delta}}$, and it follows that

$$
\rho \sqrt{\rho^{2}-1}<2 \sqrt[8]{2 \delta} \cdot \sqrt{2} e^{3 \sqrt[4]{2 \delta}}<8 \sqrt[8]{2 \delta}
$$

implying the assertion.
Suppose now that $\rho \geq e^{2 \sqrt[4]{2 \delta}}$.
As a continuous version of (15), we have

$$
\begin{aligned}
\|f\|_{1}^{2 k+2} & \leq \int_{-1}^{1}(f \circ f)^{k+1}(x) d x \cdot T_{D}^{(k)}(D) \\
& \leq\left(2 \delta\|f\|_{2}^{2 k+2}+\omega^{k}\|f\|_{1}^{2 k+2}\right) \cdot T_{D}^{(k)}(D) \\
& =\left(2 \delta \rho^{2 k+2}+\omega^{k}\right)\|f\|_{1}^{2 k+2} \cdot T_{D}^{(k)}(D) .
\end{aligned}
$$

On the other hand, using an argument similar to that in the proof of Lemma 3, it is easy to see that $T_{D}^{(k)}(D)=k+1$. Therefore,

$$
\begin{equation*}
\omega^{k} \geq \frac{1}{k+1}-2 \delta \rho^{2 k+2} \tag{16}
\end{equation*}
$$

To optimize, we let $L_{1}=\log _{\rho}(1 / 2 \delta), L:=L_{1}-\log _{\rho}\left(L_{1}\right)$, and $k:=\lfloor L / 2\rfloor-1$; thus, $L_{1} \geq 12$ by the assumptions. Furthermore, from the inequality $\sqrt{z} \ln (1 / z) \leq 2 \sqrt[4]{z}$ (valid
for any positive $z \leq 1$ ), substituting $z=2 \delta$ we get $\sqrt{2 \delta} \ln (1 / 2 \delta) \leq \ln (\rho)$ whence $L_{1} \leq \frac{1}{\sqrt{2 \delta}}$. It follows that $\log _{\rho}\left(L_{1}\right) \leq \frac{1}{2} \log _{\rho}(1 / 2 \delta)=\frac{1}{2} L_{1}$ and then $L=L_{1}-\log _{\rho}\left(L_{1}\right) \geq \frac{1}{2} L_{1} \geq 6$, implying $k \geq L / 4$. Also,

$$
(2 k+2) \rho^{2 k+2} \leq L \rho^{L} \leq L_{1} \rho^{L_{1}-\log _{\rho}\left(L_{1}\right)}=\rho^{L_{1}}=\frac{1}{2 \delta} .
$$

Therefore, from (16),

$$
\omega^{k} \geq \frac{1}{2(k+1)}
$$

It follows that

$$
\omega \geq e^{-\frac{1}{k} \ln (2(k+1))} \geq 1-\frac{1}{k} \ln (2(k+1)) \geq 1-\frac{4}{L} \ln (L)
$$

and to complete the proof we recall that $\frac{1}{2} L_{1} \leq L \leq L_{1}$.

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