DANIEL KRÁL', ORIOL SERRA, AND LLUÍS VENA: "A COMBINATORIAL PROOF OF THE REMOVAL LEMMA FOR GROUPS"

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The triangle removal lemma says, loosely speaking, that a graph of order n with $o(n^3)$ triangles can be made triangle-free by removing $o(n^2)$ edges. It seems that the most common rigorous statement of this lemma is as follows.

Lemma 1′ (The Triangle Removal Lemma, Standard Version)**.** *For any $\delta > 0$ there exists $c > 0$ such that if Γ is a graph of order n with at most cn^3 triangles, then there is a set of at most δn^2 edges of Γ, removing which destroys all the triangles.*

Green [G05] uses the following restatement, which can be shown equivalent; see Appendix.

Lemma 1″ (The Triangle Removal Lemma, Alternative Version)**.** *For any $c > 0$ there exists $\delta = \delta(c) > 0$ with $\lim_{c\to 0+} \delta(c) = 0$ such that if Γ is a graph of order n with at most cn^3 triangles, then there is a set of at most δn^2 edges of Γ, removing which destroys all the triangles.*

We refer the reader to [G05] for discussion, attribution, and connections with Szemerédi's regularity lemma, from which the triangle removal lemma easily follows.

One of the central results of [G05] is a kind of regularity lemma for abelian groups, as a corollary of which the following "removal lemma for abelian groups" is obtained.

Theorem 1 (Green [G05, Theorem 1.5])**.** *Let G be a finite abelian group of order $N := |G|$, and let $k \ge 3$ be an integer. If $A_1, \dots, A_k$ are subsets of G such that the equation $x_1 + \dots + x_k = 0$ has $o(N^{k-1})$ solutions in the variables $x_i \in A_i$ $(1 \le i \le k)$, then one can remove $o(N)$ elements from each set A_i so as to leave sets A_i' with the property that this equation has no solutions with $x_i \in A_i'$ $(1 \le i \le k)$.*

(We have presented the intuitive version of the theorem; it can be made precise following the same lines as in Lemmas 1′ and 1″.)

In [KSV09], Theorem 1 is given a completely different proof, relying on a graph-theoretic extension of the triangle removal lemma. Indeed, since the approach of [KSV09] is purely combinatorial (in contrast with Green's approach, based on Fourier analysis), it yields a more general result, extending onto non-abelian groups.

Theorem 2 (Král'-Serra-Vena [KSV09, Theorem 2]). *Let G be a finite group of order $N := |G|$, and let $k \geq 3$ be an integer. If $A_1, \dots, A_k$ are subsets of G such that the equation $x_1 \cdots x_k = 1$ has $o(N^{k-1})$ solutions in the variables $x_i \in A_i$ $(1 \leq i \leq k)$, then one can remove $o(N)$ elements from each set A_i so as to leave sets A_i' with the property that this equation has no solutions with $x_i \in A_i'$ $(1 \leq i \leq k)$.*

Notice, that the only difference between Theorems 1 and 2 is that in the latter theorem, G is not assumed to be abelian; accordingly, the multiplicative notation is used.

Corollary 1. *Let G be a finite group of odd order $N := |G|$. If the equation $xy = z^2$ has $o(N^2)$ solutions in the elements of a subset $A \subseteq G$, then $|A| = o(N)$.*

Though in [KSV09] some extensions onto certain systems of equations are also provided[1], here we confine ourselves to reproducing the proof of Theorem 2. The argument applies to the "distinct summands Cayley graph" (cf. [RS78, S04]) the following digraph removal lemma of Alon and Shapira.

Lemma 2 (Alon-Shapira [AS04, Lemma 4.1]). *For every $\delta, k > 0$ there exists $c > 0$ with the following property: if H is a digraph of order k, and Γ is a digraph of order n containing at most cn^k copies of H, then there is a set of at most δn^2 edges of Γ, removing which from Γ renders it H-free.*

Proof of Theorem 2. Consider the k-partite digraph Γ on k disjoint copies of the group G in which every arc joins an element from the ith copy with an element from the $(i+1)$th copy, for some $i \in [0, k-1]$, and the arc is present if and only if the ratio of the two elements belong to A_i. Formally, we re-index the subsets A_i with the elements of $\mathbb{Z}/k\mathbb{Z}$, and define Γ to be the digraph with the vertex set $G \times (\mathbb{Z}/k\mathbb{Z})$ and the arc set

$$\{((g,i),(ga_i,i+1)) \colon g \in G,\ i \in \mathbb{Z}/k\mathbb{Z},\ a_i \in A_i.\}. \qquad (*)$$

We assign the label $[i, a_i]$ to the arc in $(*)$. Thus, for each $i \in \mathbb{Z}/k\mathbb{Z}$ and $a_i \in A_i$, there are exactly N arcs in Γ, labeled $[i, a_i]$. It is instructive to think of these arcs as going from the ith partite set "in the direction a_i".

Notice, that the order of Γ is kN.

Let H be the directed cycle of length k. It is easily verified that every copy of H in Γ gives raise to a solution of the equation $x_0 \cdots x_{k-1} = 1$ in the variables $x_i \in A_i$ $(i \in \mathbb{Z}/k\mathbb{Z})$. Conversely, to every such solution $(a_0, \dots, a_{k-1})$ there correspond N vertex-disjoint copies of H in Γ: namely,

$$((g,0),(ga_0,1),\dots,(ga_0\dots a_{k-2},k-1),(ga_0\dots a_{k-1},0));\ g \in G. \qquad (**)$$

[1] see [KSV] for a systematic treatment of this topic.

Given $\delta > 0$, we find c as in Lemma 2. If the number of solutions of the equation in question is at most cN^{k-1}, then the number of copies of H in G is at most $cN^k < c(kN)^k$; hence, by Lemma 2, there is a set E of at most $\delta(kN)^2$ arcs of Γ such that every copy of H in Γ contains an arc from E.

For each $i \in \mathbb{Z}/k\mathbb{Z}$, let B_i be the set of all those $a_i \in A_i$ such that there are at least N/k edges in E labeled $[i, a_i]$. Clearly, we have $|B_i| \le \frac{|E|}{N/k} \le \delta k^3 N$, and to complete the proof it suffices to show that every copy of H in Γ contains an edge labeled $[i, b_i]$ with $b_i \in B_i$; that is, if $a_0 \cdots a_{k-1} = 1$, where $a_i \in A_i$ for $i \in \mathbb{Z}/k\mathbb{Z}$, then there exists $i \in \mathbb{Z}/k\mathbb{Z}$ such that $a_i \in B_i$. To this end we consider again the N disjoint cycles in $(**)$. Each of them contains an edge from E, and hence there exists $i \in \mathbb{Z}/k\mathbb{Z}$ such that at least N/k of these edges share the same label $[i, a_i]$. Thus, $a_i \in B_i$, as required. □

Appendix: Equivalence of Lemmas $1'$ and $1''$.

Lemma $1''$ implies Lemma $1'$ in an almost immediate way: given $\delta > 0$ and assuming Lemma $1''$, find $c > 0$ such that $\delta(c) \le \delta$; then whenever Γ is a graph of order n with at most cn^3 triangles, there is a set of at most $\delta(c)n^2 \le \delta n^2$ edges of Γ, removing which destroys all the triangles.

To derive Lemma $1''$ from Lemma $1'$, fix a sequence $\delta_1 > \delta_2 > \cdots$ with $\lim_{i\to\infty} \delta_i = 0$, and find $c_1, c_2, \ldots > 0$ such that, for every integer $i \ge 1$, if Γ is a graph of order n with at most $c_i n^3$ triangles, then at most $\delta_i n^2$ edges can be removed from Γ so that all the triangles are destroyed. Clearly, we can modify the sequence $c_1, c_2, \ldots$ (decreasing some of its terms) to ensure that it is monotonically decreasing and satisfies $\lim_{i\to\infty} c_i = 0$. Now let

$$\delta(c) := \begin{cases} 1 & \text{if } c > c_1, \\ \delta_i & \text{if } c_{i+1} < c \le c_i \end{cases};$$

thus, $\lim_{c\to 0+} \delta(c) = 0$. Now, if Γ is a graph of order n with at most cn^3 triangles, then, with i satisfying $c_{i+1} < c \le c_i$, in view of $cn^3 \le c_i n^3$ and by the choice of c_i, all these triangles can be destroyed by removing at most $\delta_i n^2 = \delta(c) n^2$ edges of Γ. Therefore, $\delta(c)$ satisfies the assertion of Lemma $1''$.

References

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