INTERNAL CATEGORICAL STRUCTURES IN HOMOTOPICAL ALGEBRA

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Abstract. This is a survey on the use of some internal higher categorical structures in algebraic topology and homotopy theory. After providing a general view of the area and its applications, we concentrate on the algebraic modelling of connected $(n+1)$-types through cat^n-groups.

1. Introduction. In this paper we present a survey of an area of mathematics which arose historically within algebraic topology in an attempt to describe higher order versions of the fundamental group in homotopy theory.

We have tried to highlight the interplay between the categorical input and the topological and homotopical one. With this in mind, we provide an overview of the area in Section 2. Here we indicate the main categorical structures used, the main areas of applications and the form in which these categorical structures arise in the applications. In the subsequent sections we concentrate on one of the main achievements in the area, which is the construction of an algebraic model of connected n-types. Section 3 contains a description of the structures used in the modelling of connected $(n+1)$-types, the cat^n-groups while Section 4 deals with some more technical aspects of cat^n-groups as an algebraic category. Section 5 provides a concise exposition of the proof that cat^n-groups model connected $(n+1)$-types; we have followed the approach to this proof given by Bullejos, Cegarra and Duskin [12]. We conclude in Section 5 with a broad outline of recent developments.

2. Overview. This section contains a brief survey of the area of mathematics covered by the title of the paper. The topic is located between algebraic topology and category theory and we could refer to it as "strict higher-dimensional algebra."

As a branch of algebraic topology, it arose from the investigation of higher-order versions of the fundamental group in homotopy theory, as envisaged in Grothendieck's "pursuing stacks" program [30]. The algebraic modelling of homotopy n-types was thus a central achievement in the area.

The categorical input comes essentially from two types of strict higher categorical structures: strict n-categories and n-fold categories. In concrete algebraic-topological situations, these appear internalized in a fixed based category such as groups and algebras. The resulting internalized structures admit equivalent descriptions: strict n-categories (resp. strict ω-categories)

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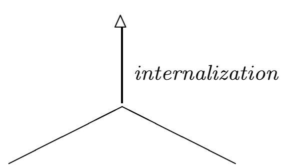
give rise to crossed n-complexes (resp. crossed complexes) [8] ; n-fold categories give rise to cat^n-groups (and cat^n-algebras), or equivalently crossed n-cubes (of groups, algebras, etc.) [24]. Another structure widely used in the area is that of n-hypercrossed complexes [16]. Hypercrossed complexes of groups are equivalent to simplicial groups via the Moore normalization functor, hence they provide a non-abelian version of the Dold-Kan theorem. n-Hypercrossed complexes of groups (resp. groupoids) are models of connected (resp. non-connected) $(n+1)$-types [16, 26]. A categorical description of hypercrossed complexes as algebras for a monad has been recently given in [5].

The reason for the internalization is best understood when considering cat^n-groups as homotopy models. The Kan loop-group functor G from reduced simplicial sets to simplicial groups has a right adjoint $\overline{W}$, and this adjoint pair induces an equivalence between the corresponding homotopy categories [40]. On the other hand, any path-connected space X is weakly homotopy equivalent to the geometric realization of its reduced singular complex $\mathcal{S}_0 X$. Hence, the functor $G\mathcal{S}_0$ from based topological spaces to simplicial groups translates the problem of modelling homotopy types into the realm of simplicial groups. In the non-path-connected case, one needs to use groupoids instead of groups, and the analogue of the Kan loop group functor is the construction in [23]. The following table summarizes the main structures in the area and their applications:

STRICT HIGHER-DIMENSIONAL ALGEBRA

1) Crossed n-complexes; d), e).
2) cat^n-groups, cat^n-algebras
   a), b), c), f).
3) n-Hypercrossed complexes; a), g).



ALGEBRAIC TOPOLOGY
a) Modelling homotopy n-types.
b) Higher order Van Kampen theorem.
c) Non-abelian cohomology.
d) Homological algebra.
e) Combinatorial group theory.
f) Topological quantum field theory.
g) Homotopy 3-types and double loop spaces.

CATEGORY THEORY
1) n-cat, ω-cat.
2) Cat^n

The applications range throughout various areas of algebraic topology. Cat^n-groups and crossed n-complexes satisfy a higher-order Van Kampen theorem [7], and this was used to perform computations of homotopy groups [9]. Non-abelian cohomology theories naturally require cat^1-groups (equivalently crossed modules) as coefficient objects [6].

In homological algebra, n-crossed complexes were used in the interpretation of group cohomology [32, 31] and, more generally, of cohomology in a certain class of algebraic categories [52]. More recently, (co)homology theories of cat^n-groups have been developed [15, 13], [44] and cat^n-algebras have been described operadically in [34]. Combinatorial group theory uses crossed modules as a way to encode information about group presentations [10]. An application of algebraic models of n-types is found in topological quantum field theory [50]. Homotopy 3-types modelled by 2-crossed modules have been directly linked to double loop spaces in [3], whereas the analogous problem for $n > 3$ seems, so far, entirely open.

3. n-Fold categories in groups. The category of n-fold categories internalized in the category **Gp** of groups can be described in various equivalent ways. These arose in applications, as well as in independent proofs that they model homotopy $(n+1)$-types in the path-connected case. This section is devoted to an exposition of some of these characterizations, starting with the case $n = 1$.

Given a category $\mathcal{D}$ with pullbacks, an *internal category in* $\mathcal{D}$ consists of a diagram

$$\phi : D_1 \times_{D_0} D_1 \xrightarrow{\ c\ } D_1 \ \overset{d_0}{\underset{d_1}{\rightrightarrows}} \ D_0 \quad (i : D_0 \to D_1)$$

where d_0, d_1 are "source" and "target" maps, i is the "identity" map, c is the "composition" map. We call D_1 the "arrow object," D_0 the "object object." These data satisfy the axioms of a category; that is, the following identities hold, where $\pi_0, \pi_1 : D_1 \times_{D_0} D_1 \to D_1$ are the two projections:

$$d_0 i = 1_{D_0} = d_1 i, \qquad d_1\pi_1 = d_1 c, \qquad d_0 \pi_0 = d_0 c$$

$$c\begin{pmatrix} 1_{D_1} \\ i d_0 \end{pmatrix} = 1_{D_1} = \begin{pmatrix} i d_1 \\ 1_{D_1} \end{pmatrix}, \qquad c(1_{D_1} \times_{D_0} c) = c(c \times_{D_0} 1_{D_1}).$$

Given internal categories ϕ and ϕ', an *internal functor* $F : \phi \to \phi'$ consists of a pair of morphisms $F_0 : D_0 \to D_0'$, $F_1 : D_1 \to D_1'$ satisfying the axioms:

$$d_0 F_1 = F_0 d_0, \qquad d_1 F_1 = F_0 d_1$$

$$F_1 i = i F_0, \qquad F_1 c = c(F_1 \times_{F_0} F_1).$$

We denote by $\mathbf{Cat}\,\mathcal{D}$ the category of internal categories and internal functors. If in addition $\mathcal{D}$ is semiabelian, then $\mathbf{Cat}\,\mathcal{D}$ is equivalent to the category $\mathbf{CM}(\mathcal{D})$ of crossed modules in $\mathcal{D}$. At this level of generality, this fact was established fairly recently, in [33]. It was known earlier for a particular class of semiabelian categories, which includes most algebraic examples; these are the "categories of groups with operations" [43, 49]. We shall restrict to this context, as it is sufficient to handle the case of cat^n-groups.

Categories of groups with operations are a particular case of the general notion of variety of universal algebra. This includes familiar examples such as groups, rings, Lie algebras and many others. Recall [39, p. 124] that this is given by specifying a set S, a set of operators Ω acting on S and a set of identities E, in the following sense. There is a function $\Omega \to \mathbb{N}$ which assigns to each $\omega \in \Omega$ a natural number, called the arity of ω. An action of Ω on S ia a function assigning to each operator ω an n-ary operation $\omega_A : S^n = S \times \overset{n}{\cdots} \times S \to S$ (S^0 is understood to be the singleton). For example, in the case of the variety of groups we have three operators, product, inverse, identity, of arities 2,1,0 respectively: the corresponding operations take $(x, y) \to xy$, $x \to x^{-1}$, $* \to 1$.

Continuing with the general case, from the elements of Ω we can form the set of derived operators: given $\omega \in \Omega$ of arity n and n derived operators $\lambda_1 \cdots \lambda_n$ of arities $m_1, \dots, m_n$, we can form a composite $\omega(\lambda_1 \cdots \lambda_n)$ of arity $m_1 + \cdots + m_n$; also given an operator λ of arity n and a function $f : \{1, \dots, n\} \to \{1, \dots, m\}$, we can use substitution to form an operator ϑ of arity m where $\vartheta(x_1 \dots x_m) = \lambda(x_{f_1} \dots x_{f_n})$. For instance, in the case of the variety of groups, taking $\omega = \lambda_1 = \lambda_2$ to be the product we obtain an operator $\omega(\lambda_1, \lambda_2)$ of order 4 and corresponding operations $G^4 \to G$, $(x_1, x_2, x_3, x_4) \to (x_1x_2)(x_3x_4)$; taking for λ the product and $f : \{1, 2\} \to \{1\}$ we obtain by substitution an operator of order 1, $\vartheta(x) = x^2$.

The set E of identities is a set of ordered pairs $< \lambda, \mu >$ of derived operators, where λ and μ have the same arity. An action A of Ω on S satisfies the identity $< \lambda, \mu >$ if $\lambda_A = \mu_A : S^n \to S$. For example, in the case of the variety of groups, E consists of the axiom for identity $1x = x = x1$, inverse $xx^{-1} = 1 = x^{-1}x$ and associative law $x(yz) = (xy)z$.

Summarizing, a *variety of universal algebras*, or $< \Omega, E >$-algebra, is a set S with an action A of Ω on S which satisfies all the identities of E. A morphism of $< \Omega, E >$-algebras is a function on the underlying sets which preserves all operators of Ω.

A *category $\mathcal{D}$ of groups with operations* is a variety of universal algebras which is, first of all, a variety of groups. Also, group identity is the only operator of arity 0. All other operators different from group multiplication, inverse and identity have order 1 and 2. We denote the set of these operators by Ω_1' and Ω_2' respectively. Finally, the following axioms hold: if $* \in \Omega_2'$, $*^0$ defined by $x *^0 y = y * x$ is in Ω_2'. Also, $a * (bc) = (a * b)(a * c)$ for all $* \in \Omega_2'$; $\omega(ab) = \omega(a)\omega(b)$, $\omega(a)*b = \omega(a*b)$ for all $\omega \in \Omega_1'$, $* \in \Omega_2'$.

In a category of groups with operations, there is a notion of action and of semidirect product. Given $A, B \in \mathbf{Ob}\mathcal{D}$, a B-structure A is a split extension of B by A in $\mathcal{D}$, that is the diagram in $\mathcal{D}$

$$A \overset{i}{\rightarrowtail} E \underset{s}{\overset{p}{\rightleftarrows}} B \qquad\qquad ps = \mathrm{id}_B. \tag{1}$$

If A is a B-structure, the semidirect product $A \widetilde{\times} B$ is the object of $\mathcal{D}$ which is $A \times B$ as a set, with operations

$$\begin{aligned} \omega(a,b) &= (\omega(a), \omega(b)) \\ (a',b')(a,b) &= (a's(b')\ a\ s(b')^{-1}, b'b) \\ (a',b') * (a,b) &= (a' * a)(a' * s(b))(s(b') * a) \end{aligned}$$

for all $a, a' \in A$, $b, b' \in B$, $\omega \in \Omega_1'$, $* \in \Omega_2'$. Given the split extension (1) there is an isomorphism $E \cong A \widetilde{\times} B$. For each $b \in B$, $a \in A$ we denote

$$\begin{aligned} {}^{b}a &= s(b)\ a\ s(b)^{-1} \\ b * a &= s(b) * a \\ a * b &= a * s(b). \end{aligned}$$

This notion of action allows us to formulate the notion of crossed module in $\mathcal{D}$.

Definition 3.1. *Let $\mathcal{D}$ be a category of groups with operations. A crossed module in $\mathcal{D}$ consists of a triple (A, B, ϕ) where A is a B-structure, $\phi \in Mor(\mathcal{D})$ such that for all $a, a_1, a_2 \in A$, $b \in B$, $* \in \Omega_2'$,*

CM 1 $\phi({}^{b}a) = b\phi(a)b^{-1}$,

CM 2 ${}^{\phi(a_1)}a_2 = a_1 a_2 a_1^{-1}$,

CM 3 $\phi(a_1) * a_2 = a_1 * a_2 = a_1 * \phi(a_2)$,

CM 4 $\begin{cases} \phi(b * a) = b * \phi(a) \\ \phi(a * b) = \phi(a) * b. \end{cases}$

A morphism from a crossed module (A, B, ϕ) to a crossed module (A', B', ϕ') consists of a pair of morphisms in $\mathcal{D}$, $f : A \to A'$ and $g : B \to B'$ such that $g\ \phi = f\ \phi'$ and, for all $a \in A$, $b \in B$,

$$f({}^{b}a) = {}^{g(b)}f(a), \qquad f(b * a) = g(b) * f(a).$$

Given a crossed module (A, B, Φ) in $\mathcal{D}$, consider the diagram

$$(A \widetilde{\times} B) \times_B (A \widetilde{\times} B) \underset{m}{\to} A \widetilde{\times} B \ \overset{d_0}{\underset{s}{\overset{\to}{\underset{d_1}{\overset{\to}{\leftarrow}}}}}\ B \tag{2}$$

where $s(b) = (1, b)$, $d_0(a, b) = b$, $d_1(a, b) = \phi(a)b$, $m((a, b), (c, \phi(a)b)) = (ca, b)$. Clearly d_0 and s are morphisms in $\mathcal{D}$. Axioms CM 1 and CM 4 imply that d_1 is a morphism in $\mathcal{D}$, while CM 2 and CM 3 imply that m is a morphism in $\mathcal{D}$. It is then straightforward to check that (2) is an internal category in $\mathcal{D}$.

Conversely, an internal category

$$C\times_B C \overset{m}{\rightarrow} C \underset{\underset{s}{\leftarrow}}{\overset{\overset{d_0}{\rightarrow}}{\underset{d_1}{\rightarrow}}} B \tag{3}$$

in $\mathcal{D}$, determines a split extension of B by $\ker d_0$, so that $C \cong \ker d_0 \widetilde{\times} B$. After some straightforward identification, (3) has the same form as (2), hence $(\ker d_0, B, d_{1|\ker d_0})$ is a crossed module in $\mathcal{D}$. Clearly the two processes give an equivalence of categories. Notice also that, given the internal category (3) and $(a,b) \in \ker d_0 \widetilde{\times} B \cong C$, we have $m((a,b)(a^{-1},\phi(a)b)) = (1,b) = s(b)$, $m((a^{-1},\phi(a)b)(a,b)) = (1,\phi(a)b) = s(\phi(a)b)$. In other words, any arrow-object $c \in C$ is invertible in the sense that there is another arrow-object c' such that $m(c,c')$ and $m(c',c)$ are identities. Internal categories having this property are called *internal groupoids*. Summarizing:

THEOREM 3.1. [49] *Let $\mathcal{D}$ be a category of groups with operations. Then the categories $\mathbf{CM}(\mathcal{D})$ and $\mathbf{Cat}\,\mathcal{D}$ are equivalent. Also, every object of $\mathbf{Cat}\mathcal{D}$ is an internal groupoid.*

Particularizing Definition 3.1 to the case where $\mathcal{D}$ is the category of groups, we obtain that a crossed module in groups consists of a triple (T,G,μ) where $\mu : T \rightarrow G$ is a group homomorphism, G acts on T and, for all $t,t' \in G,\ g \in G$,

$$\mu({}^{g}t) = g\mu(t)g^{-1}, \qquad {}^{\mu(t)}t' = tt't^{-1}.$$

A morphism of crossed modules $(f_T,f_G) : (T,G,\mu) \rightarrow (T',G',\mu')$ consists of a pair of group homomorphisms $f_T : T \rightarrow T',\ f_G : G \rightarrow G'$ such that $\mu' f_T = f_G\mu,\ f_T({}^{g}t) = {}^{f_G(g)}f_T(t)$ for all $g \in G,\ \ t \in T$.

We list some common examples of crossed modules in groups;

(i) (Whitehead) [53] $(\pi_2(X,A,*),\pi_1(A,*),\partial)$,
for $(X,A,*)$ a based pair of spaces, ∂ boundary map.

(ii) (N,G,i) $N \lhd G$, i inclusion, action of G on N by conjugation.

(iii) $(G, Aut\,G, \mu)$, $\mu(g)(h) = ghg^{-1}$, ${}^{\alpha}g = \alpha(g)$, $\alpha \in Aut\,G$, $g \in G$.

(iv) $(M,G,0)$, M a G-module.

From Theorem 3.1, a crossed module in groups (T,G,μ) corresponds to an internal category in $\mathbf{Gp}$

$$(T \rtimes G)\times_G (T \rtimes G) \overset{m}{\rightarrow} T \rtimes G \underset{\underset{s}{\leftarrow}}{\overset{\overset{d_0}{\rightarrow}}{\underset{d_1}{\rightarrow}}} G \tag{4}$$

where $d_0(t,g) = g,\ d_1(t,g) = \mu(t)g,\ s(g) = (1,g),\ m((t,g),(t',\mu(t)g)) = (t't,g)$. An easy calculation shows that m is a group homomorphism if and only if the commutator $[\ker d_0, ker d_1]$ is trivial. Thus the internal category (4) is equivalent to the reflexive graph

$$T \rtimes G \underset{s}{\overset{d_0}{\underset{d_1}{\rightrightarrows}}\!\!\longleftarrow} G, \qquad d_0 s = \mathrm{id}_G, \qquad d_1 s = \mathrm{id}_G$$

with the extra condition that $[\ker d_0, ker d_1] = 1$. In turn, this corresponds to the triple $(T \rtimes G, d_0', d_1')$, where $d_0' = d_0 s$, $d_1' = d_1 s$ such that $d_0' d_1' = d_1'$, $d_1' d_0' = d_0'$, $[\ker d_0', \ker d_1'] = 0$. In the literature, this structure is called a cat^1-group. More precisely:

DEFINITION 3.2. *A cat^1-group consists of a triple (G, d, t) where $d, t : G \to G$ are group homomorphisms and*

i) $dt = t, \;\; td = d$

ii) $[\ker d, \ker t] = 1$.

A *morphism of cat^1-groups* $(G, d, t) \to (G', d', t')$ consists of a morphism $f : G \to G'$ such that $fd = d'f$, $ft = t'f$. Denoting by $\boldsymbol{\mathcal{C}^1\mathcal{G}}$ the category of cat^1-groups, from the previous discussion we obtain

PROPOSITION 3.1. *The categories* $\mathbf{Cat(Gp)}$, $\mathbf{CM(Gp)}$, $\boldsymbol{\mathcal{C}^1\mathcal{G}}$ *are equivalent.*

Other characterizations of internal categories in groups have been given; a simplicial one will be mentioned in next section, and one in terms of double groupoids with connections [11] will not be discussed here, but is very relevant in computations.

The notion of cat^1-group admits a natural generalization to higher dimensions.

DEFINITION 3.3. *The category $\boldsymbol{\mathcal{C}^n\mathcal{G}}$ of* cat^n-groups *is defined as follows. A* cat^n-group *consists of a group G together with $2n$ endomorphisms $t_i, d_i : G \to G$, $\; 1 \le i \le n$ such that for all $1 \le i, j \le n$*

(i) $d_i t_i = t_i, \quad t_i d_i = d_i,$

(ii) $d_i t_j = t_j d_i, \quad d_i d_j = d_j d_i, \quad t_i t_j = t_j t_i, \qquad i \ne j,$

(iii) $[\ker d_i, \ker t_i] = 1.$

A morphism of cat^n-groups $(G, d_i, t_i) \to (G', d_i', t_i')$ is a group homomorphism $f : G \to G'$ such that $fd_i = d_i' f$, $ft_i = t_i' f \;\; 1 \le i \le n$.

Notice that the identity (iii) in Definition 3.3 is equivalent to

(iii)' $d_i(x) x^{-1} t_i(x) x^{-1} = t_i(x) x^{-1} d_i(x) x^{-1}, \;\; x \in G.$

Using this fact we can view $\boldsymbol{\mathcal{C}^n\mathcal{G}}$ as a category of groups with operations as follows. Let $\Omega_0 = \{0\}$, $\Omega_1 = \{-\} \cup \{t_i, d_i\} \;\; 1 \le i \le n$, $\Omega_2 = \{+\}$; the set of identities comprises the group laws and the identities (i), (ii), (iii)' above. Hence we can speak of crossed modules in $\boldsymbol{\mathcal{C}^n\mathcal{G}}$ or equivalently, by Theorem 3.1, of internal categories in $\boldsymbol{\mathcal{C}^n\mathcal{G}}$. In the next theorem this notion is used to identify $\boldsymbol{\mathcal{C}^n\mathcal{G}}$ with n-fold categories internal to $\mathbf{Gp}$.

DEFINITION 3.4. *The category* $\mathbf{Cat^n(Gp)}$ *of n-fold categories in* $\mathbf{Gp}$ *is defined inductively by* $\mathbf{Cat^1(Gp)} = \mathbf{Cat(Gp)}$, $\mathbf{Cat^n(Gp)} = \mathbf{Cat(Cat^{n-1}(Gp)}$.

THEOREM 3.2. *For each* $n \geq 1$ *the categories* $\boldsymbol{\mathcal{C}^n\mathcal{G}}$, $\mathbf{CM}(\boldsymbol{\mathcal{C}^{n-1}\mathcal{G}})$, $\mathbf{Cat^n}(\mathbf{Gp})$ *are equivalent.* **Proof.** We first show that $\boldsymbol{\mathcal{C}^n\mathcal{G}}$ and $\mathbf{CM}(\boldsymbol{\mathcal{C}^{n-1}\mathcal{G}})$ are equivalent. Let $(\mathcal{T},\mathcal{G},\mu)$ be an object of $\mathbf{CM}(\boldsymbol{\mathcal{C}^{n-1}\mathcal{G}})$ with $\mathcal{T} = (T, s_i, v_i)$, $\mathcal{G} = (G, d_i, t_i)$, $i = 1, \ldots, (n-1)$. Thus $\mathcal{T}$ is a $\mathcal{G}$-structure and, for all $g \in G$, $t, t' \in T$

$$\mu({}^{g}t) = g\mu(t)g^{-1}, \qquad {}^{\mu(t)}t' = tt't^{-1}.$$

The object of $\boldsymbol{\mathcal{C}^n\mathcal{G}}$ corresponding to $(\mathcal{T},\mathcal{G},\mu)$ is $(T \rtimes G, r'_i, w'_i)$ where $r'_i = r_i$, $w'_i = w_i$ for $i = 1, \ldots, (n-1)$, $r'_n(t,g) = (1,g)$, $w'_n(t,g) = (1, \mu(t)g)$.

Conversely, given a cat^n-group $\mathcal{H} = (H, s_i, t_i)$, let $\mathcal{T} = (\ker s_n, s_{i_|}, t_{i_|})$, $\mathcal{G} = (\operatorname{Im} s_n, s_{i_|}, t_{i_|})$ and $\mu = t_{n_{|ker s_n}}$, then $\mathcal{T}$ is a $\mathcal{G}$-structure and $(\mathcal{T},\mathcal{G},\mu)$ is an object of $\mathbf{CM}\,(\boldsymbol{\mathcal{C}^{n-1}\mathcal{G}})$ corresponding to $\mathcal{H}$. Hence

$$\boldsymbol{\mathcal{C}^n\mathcal{G}} \simeq \mathbf{CM}(\boldsymbol{\mathcal{C}^{n-1}\mathcal{G}}). \tag{5}$$

We prove by induction that $\boldsymbol{\mathcal{C}^n\mathcal{G}} \simeq \mathbf{Cat^n}(\mathbf{Gp})$. By Proposition 3.1 we know this holds for $n = 1$. Suppose, inductively, that $\boldsymbol{\mathcal{C}^{n-1}\mathcal{G}} \simeq \mathbf{Cat^{n-1}}(\mathbf{Gp})$. Then Theorem 3.1 and (5) imply $\boldsymbol{\mathcal{C}^n\mathcal{G}} \simeq \mathbf{CM}(\boldsymbol{\mathcal{C}^{n-1}\mathcal{G}}) \simeq \mathbf{Cat}(\mathcal{C}^{\mathbf{n-1}}\mathcal{G}) \simeq \mathbf{Cat}(\mathbf{Cat^{n-1}}(\mathbf{Gp})) = \mathbf{Cat^n}(\mathbf{Gp})$, completing the inductive step. □

There is also another characterization of n-fold categories in groups in terms of crossed n-cubes [24]. In the case $n = 2$ they are called "crossed squares" and were originally introduced in [36]. A crossed square is a "crossed module of crossed modules" and a crossed n-cube generalizes this notion to higher dimensions. The precise definition of crossed n-cube is quite lengthy and we refer the reader to [24] for further details.

4. Cat^n-groups as an algebraic category. In this section we discuss another aspect of cat^n-groups, which arises from the fact that they are internal structures in the algebraic category of groups. The material in this section is more technical than in the rest of this paper, but not necessary for the main line of development; it can therefore be omitted at first reading.

Recall that a category $\mathcal{C}$ is algebraic when there is a functor $U : \mathcal{C} \to \mathbf{Set}$ which is monadic; this means that it has left adjoint $F : \mathbf{Set} \to \mathcal{C}$ and $\mathcal{C}$ is equivalent to the category $\mathbf{Set}^{\mathbb{T}}$ of $\mathbb{T}$-algebras for the monad $\mathbb{T} = (UF, \eta, F\varepsilon U)$ arising from the adjunction $F \dashv U$.

If $\mathcal{D}$ is a category of groups with operations, it follows from a general fact [39, Ch. VI §8, Theorem 1] that $\mathcal{D}$ is algebraic. From the above description of $\boldsymbol{\mathcal{C}^n\mathcal{G}}$ as a category of groups with operations it follows that $\boldsymbol{\mathcal{C}^n\mathcal{G}}$ is an algebraic category. This general theory, however, only provides us with existence of a left adjoint $\mathcal{F}_n : \mathbf{Set} \to \boldsymbol{\mathcal{C}^n\mathcal{G}}$ to the underlying set functor. Recent applications have shown that explicit knowledge of $\mathcal{F}_n$ can be very useful; see for instance [15, 13, 44, 47, 46, 45].

For $n = 1$ the functor $\mathcal{F}_1 : \mathbf{Set} \to \mathbf{CM}(\mathbf{Gp})$ was first described explicitly in [15]. In [13, Lemma 9] there is a description of objects of $\boldsymbol{\mathcal{C}^n\mathcal{G}}$

which are projective with respect to the class of regular epimorphisms, for each $n \geq 1$. From general theory [4, vol.II - Prop. 3.86], $\mathcal{F}_n(X)$ is projective with respect to regular epis, hence the result of [13] yields information about $\mathcal{F}_n(X)$. Here we give an explicit description of $\mathcal{F}_n$ for each $n \geq 0$ following [45]. A similar description was independently given in [14] using the language of crossed n-cubes.

Let $R_n : \boldsymbol{\mathcal{C}^n\mathcal{G}} \to \mathbf{Gp}$ be the functor $R_n(G, d_i, t_i) = G$. The category $\boldsymbol{\mathcal{C}^n\mathcal{G}}$ is (co)complete. In particular, given $\mathcal{T}, \mathcal{G} \in \boldsymbol{\mathcal{C}^n\mathcal{G}}$, let $u_1 : \mathcal{T} \to \mathcal{T} \amalg \mathcal{G}$, $u_2 : \mathcal{G} \to \mathcal{T} \amalg \mathcal{G}$ be the two coproduct injections. Using the universality of coproducts, it is easily seen that every element of $R_n(\mathcal{T} \amalg \mathcal{G})$ has the form

$$u_1(t_1)u_2(g_1)u_1(t_2)u_2(g_2)\ldots \quad \text{or} \quad u_2(g_1)u_1(t_1)u_2(g_2)u_1(t_2)\ldots \tag{6}$$

where $t_i \in R_n\mathcal{T}$, $g_i \in R_n\mathcal{G}$ for all i.

Consider the functor $\mathcal{V}_n : \mathbf{CM}(\boldsymbol{\mathcal{C}^{n-1}\mathcal{G}}) \to \boldsymbol{\mathcal{C}^{n-1}\mathcal{G}}$ defined by

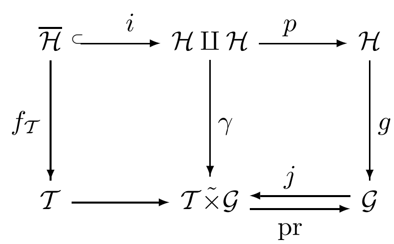
$$\mathcal{V}_n(\mathcal{T}, \mathcal{G}, \mu) = \mathcal{T} \times \mathcal{G}.$$

For any $\mathcal{H} \in \boldsymbol{\mathcal{C}^{n-1}\mathcal{G}}$, let $p : \mathcal{H} \amalg \mathcal{H} \to \mathcal{H}$ be defined by $pu_1 = 0$, $pu_2 = \mathrm{id}$; let $\overline{\mathcal{H}} = \ker p$. Consider the functor $\mathcal{L}_n : \boldsymbol{\mathcal{C}^{n-1}\mathcal{G}} \to \mathbf{CM}(\boldsymbol{\mathcal{C}^{n-1}\mathcal{G}})$ defined by

$$\mathcal{L}_n(\mathcal{H}) = (\overline{\mathcal{H}}, \mathcal{H} \amalg \mathcal{H}, i)$$

where i is the inclusion. We are going to show that $\mathcal{L}_n$ is left adjoint to $\mathcal{V}_n$. Let $(\mathcal{T}, \mathcal{G}, \mu) \in \mathbf{CM}(\boldsymbol{\mathcal{C}^{n-1}\mathcal{G}})$ and let $h : \mathcal{H} \to \mathcal{T}$, $g : \mathcal{H} \to \mathcal{G}$ be morphisms in $\boldsymbol{\mathcal{C}^{n-1}\mathcal{G}}$ inducing a morphism $(h, g) : \mathcal{H} \to \mathcal{T} \times \mathcal{G} = \mathcal{V}_n(\mathcal{T}, \mathcal{G}, \mu)$. We are going to produce a unique morphism $(f_\mathcal{T}, f_\mathcal{G}) : \mathcal{L}_n(\mathcal{H}) \to (\mathcal{T}, \mathcal{G}, \mu)$ in $\mathbf{CM}(\boldsymbol{\mathcal{C}^{n-1}\mathcal{G}})$ such that $(h, g) = (f_\mathcal{T} \times f_\mathcal{G})(u_1, u_2)$.

Consider the commutative diagram in $\boldsymbol{\mathcal{C}^{n-1}\mathcal{G}}$:



Here $\tilde{\times}$ is the semidirect product in the category of groups with operations $\boldsymbol{\mathcal{C}^{n-1}\mathcal{G}}$; γ is uniquely determined by $\gamma u_1 = ih$, $\gamma u_2 = jg$; $f_\mathcal{T}$ is the restriction of γ. Let $f_\mathcal{G} : \mathcal{H} \amalg \mathcal{H} \to \mathcal{G}$ be determined by $f_\mathcal{G} u_1 = \mu h$, $f_\mathcal{G} u_2 = g$.

For any $x \in R_{n-1}\mathcal{H}$, $w \in R_{n-1}(\mathcal{H} \amalg \mathcal{H})$, $w' \in R_{n-1}\overline{\mathcal{H}}$, we have

$$f_\mathcal{T}(u_1(x)w'u_1(x)^{-1}) = h(x)f_\mathcal{T}(w')h(x)^{-1} = {}^{\mu h(x)}f_\mathcal{T}(w') = {}^{f_\mathcal{G} u_1(x)}f_\mathcal{T}(w')$$

$$\begin{aligned} f_\mathcal{T}(u_2(x)w'u_2(x)^{-1}) &= \gamma(u_2(x)w'u_2(x)^{-1}) = jg(x)f_\mathcal{T}(w')jg(x)^{-1} \\ &= {}^{g(x)}f_\mathcal{T}(w') = {}^{f_\mathcal{G} u_2(x)}f_\mathcal{T}(w'). \end{aligned}$$

Since every element of $R_{n-1}(\mathcal{H} \amalg \mathcal{H})$ has the form (6), we conclude that, for all $w \in R_{n-1}(\mathcal{H} \amalg \mathcal{H})$, $w' \in R_{n-1}\overline{\mathcal{H}}$,

$$f_{\mathcal{T}}({}^{w}w') = f_{\mathcal{T}}(ww'w^{-1}) = {}^{f_{\mathcal{G}}(w)}f_{\mathcal{T}}(w') \tag{7}$$

and therefore also

$$\begin{aligned} \mu f_{\mathcal{T}}(wu_1(x)w^{-1}) &= \mu({}^{f_{\mathcal{G}}(w)}h(x)) = f_{\mathcal{G}}(w)\mu h(x) f_{\mathcal{G}}(w^{-1}) \\ &= f_{\mathcal{G}}(w) f_{\mathcal{G}} u_1(x) f_{\mathcal{G}}(w)^{-1} = f_{\mathcal{G}}(wu_1(x)w^{-1}). \end{aligned} \tag{8}$$

On the other hand, it is not difficult to prove that $R_{n-1}\overline{\mathcal{H}}$ is the subgroup of $R_n(\mathcal{H} \amalg \mathcal{H})$ generated by elements of the form $wu_1(x)w^{-1}$, $x \in R_{n-1}\mathcal{H}$, $w \in R_{n-1}(\mathcal{H} \amalg \mathcal{H})$. Hence from (8) we deduce that

$$\mu f_{\mathcal{T}} = f_{\mathcal{G}} i. \tag{9}$$

Thus (7) and (9) prove that $(f_{\mathcal{T}}, f_{\mathcal{G}})$ is a morphism in $\mathbf{CM}(\mathcal{C}^n\mathcal{G})$. It is clear that $(f_{\mathcal{T}}, f_{\mathcal{G}})(u_1, u_2) = (h, g)$ and that $(f_{\mathcal{T}}, f_{\mathcal{G}})$ is unique. This concludes the proof that $\mathcal{L}_n \dashv \mathcal{V}_n$.

Let $U : \mathbf{Gp} \to \mathbf{Set}$ be the forgetful functor and F its left adjoint. Let $\alpha_n : \mathbf{CM}(\mathcal{C}^{n-1}\mathcal{G}) \to \mathcal{C}^n\mathcal{G}$ be the functor realizing the equivalence of categories as in the proof of Theorem 3.2, and let β_n be its pseudo-inverse.We can always choose $\alpha_n \dashv \beta_n$. Let $\mathcal{U}_1 : \mathcal{C}^1\mathcal{G} \to \mathbf{Set}$, $\mathcal{U}_1 = U\mathcal{V}_1\beta_1$ and, inductively, $\mathcal{U}_n = \mathcal{U}_{n-1}\mathcal{V}_n\beta_n$.Then $\mathcal{U}_1$ has left adjoint $\mathcal{F}_1 = \alpha_1\mathcal{L}_1 F$; inductively, if $\mathcal{U}_{n-1}$ has left adjoint $\mathcal{F}_{n-1}$, then $\mathcal{U}_n$ has left adjoint $\mathcal{F}_n = \alpha_n\mathcal{L}_n\mathcal{F}_{n-1}$. Using Linton's criteria for monadicity [38], one can show that $\mathcal{U}_n$ is monadic. Summarizing:

THEOREM 4.1. *Let $U : \mathbf{Gp} \to \mathbf{Set}$ be the forgetful functor and F its left adjoint. There is a monadic functor $\mathcal{U}_n : \mathcal{C}^n\mathcal{G} \to \mathbf{Set}$ whose left adjoint $\mathcal{F}_n$ is given by $\mathcal{F}_0 = F$ and, for each $n \geq 1$ and any set X,*

$$\mathcal{F}_n X = \alpha_n(\overline{\mathcal{F}_{n-1}(X)}, \mathcal{F}_{n-1}(X) \amalg \mathcal{F}_{n-1}(X), i)$$

where i is the inclusion, $\amalg$ is the coproduct in $\mathcal{C}^{n-1}\mathcal{G}$, $\overline{\mathcal{F}_{n-1}(X)} = ker\ p$, $p : \mathcal{F}_{n-1}(X) \amalg \mathcal{F}_{n-1}(X) \to \mathcal{F}_{n-1}(X)$, $pu_1 = 0$, $pu_2 = id$ and α_n is the equivalence of categories $\alpha_n : \mathbf{CM}(\mathcal{C}^{n-1}\mathcal{G}) \to \mathcal{C}^n\mathcal{G}$.

5. Modelling homotopy types. The category of n-fold categories internal to groups constitutes an algebraic model of connected $(n+1)$-types. In the case $n = 1$ this result was first proved by Mac Lane and Whitehead [42] using crossed modules. For $n > 1$, the first proof of this result, due to Loday [36], was formulated in the category $\mathcal{C}^n\mathcal{G}$. Two other proofs appeared later; one due to Bullejois, Cegarra and Duskin [12] was formulated in the category $\mathbf{Cat^n}\,(\mathbf{Gp})$ of n-fold categories internal to groups; one due to Porter [48] using the category of crossed n-cubes.

In this section we provide a concise exposition of the proof given in [12]. The algebraic modelling of homotopy types is realized there as follows.

There is a classifying space functor

$$B : \mathbf{Cat^n}\,(\mathbf{Gp}) \rightarrow \text{ connected } (n+1)\text{-types}$$

with a left adjoint, the fundamental cat^n-group functor

$$\Pi_n : \text{connected } (n+1)\text{-types } \rightarrow \mathbf{Cat^n}\,(\mathbf{Gp}),$$

inducing an equivalence between the localization of $\mathbf{Cat^n}\,(\mathbf{Gp})$ with respect to some suitably defined weak equivalences and the homotopy category of connected $(n+1)$-types:

$$\mathbf{Cat^n}\,(\mathbf{Gp})/\!\sim \;\simeq\; \mathcal{H}o\,(\text{connected } (n+1)\text{-types}). \tag{10}$$

We start by describing the functor B. In analogy with the nerve of a small category, one defines the nerve of an internal category in $\mathcal{C}$, which is a simplicial object in $\mathcal{C}$. Hence, given an object of $\mathbf{Cat^n}\,(\mathbf{Gp}) \simeq \mathbf{Cat}(\mathbf{Cat^{n-1}}\,(\mathbf{Gp}))$, by applying the nerve we obtain a simplicial object in $\mathbf{Cat^{n-1}}\,(\mathbf{Gp})$ or, equivalently, an object of $\mathbf{Cat^{n-1}}\,(\mathbf{Simpl\ Gp}) \simeq \mathbf{Cat}\,(\mathbf{Cat^{n-2}}(\mathbf{Simpl\ Gp}))$. Iterating we obtain a multinerve functor

$$\mathcal{N} : \mathbf{Cat^n}\,(\mathbf{Gp}) \hookrightarrow \mathbf{Simpl^n}(\mathbf{Gp})$$

where $\mathbf{Simpl^n}(\mathbf{Gp})$ is the category of n-simplicial groups.

Let $\mathbf{Simpl\ Set}_0$ be the category of reduced simplicial sets. Recall that there is a functor $\overline{W} : \mathbf{Simpl\ Gp} \rightarrow \mathbf{Simpl\ Set}_0$ with a left adjoint $G : \mathbf{Simpl\ Set}_0 \rightarrow \mathbf{Simpl\ Gp}$, called Kan loop group functor, such that the canonical maps $G\overline{W}H_* \rightarrow H_*$ and $X_* \rightarrow \overline{W}GX_*$ are weak equivalences for all simplicial groups H_* and reduced simplicial sets X_*. A detailed description of the functors $\overline{W}$ and G can be found, for instance, in [27] and [40]. The classifying space of an n-simplicial group can be defined by the composition

$$B : \mathbf{Simpl}^{\,n}\mathbf{Gp} \overset{diag}{\rightarrow} \mathbf{Simpl\ Gp} \overset{\overline{W}}{\rightarrow} \mathbf{Simpl\ Set}_0 \overset{|\cdot|}{\rightarrow} \mathrm{Top}_*$$

where *diag* is the diagonal functor and $|\cdot|$ is the geometric realization.

The *classifying space* of an object of $\mathbf{Cat^n}\,(\mathbf{Gp})$ is by definition the classifying space of its multinerve. It can be proved that this is an $(n+1)$-type.

Proposition 5.1. [12] *Let $\mathcal{G}$ be an object of* $\mathbf{Cat^n}\,(\mathbf{Gp})$. *Then $\pi_i B\mathcal{G} = 0$ for all $i > (n+1)$.*

Remark 5.1. The classifying space of a cat^n-group can be expressed, up to homotopy, in an alternative way, as follows. Recall that there is a functor $W_{AM} : \mathbf{Simpl}^{\,2}\mathbf{Set} \rightarrow \mathbf{Simpl\,Set}$ called Artin-Mazur codiagonal (see[1]) and, further, that for any bisimplicial set X_{**}, $|diag\,X_{**}|$ and $|W_{AM}X_{**}|$ are weakly homotopy equivalent (a detailed proof of this fact can be found in [17]). Any group can be considered as a category with just one object, so there is a nerve functor $Ner : \mathbf{Gp} \to \mathbf{Simpl\,Set}$, which induces a functor $Ner^* : \mathbf{Simpl\,Gp} \to \mathbf{Simpl}^{\,2}\mathbf{Set}$. It can be seen [17] that $W_{AM}Ner^* = \overline{W}$. It follows from above that, for any simplicial group H_*, there is a weak homotopy equivalence $|\overline{W}H_*| \simeq |diag\,Ner^*\,H_*|$. In particular, given an n-simplicial group Y, this implies

$$BY \simeq |diag\,Ner^*\,diag\,Y|.$$

On the other hand, we clearly have a commutative diagram

$$\begin{array}{ccccccc}
\mathbf{Simpl}^{\,n}\mathbf{Gp} & \xrightarrow{diag} & \mathbf{Simpl\,Gp} & \xrightarrow{Ner^*} & \mathbf{Simpl}^{\,2}\mathbf{Set} & \xrightarrow{diag} & \mathbf{Simpl\,Set} \\
 & \searrow^{Ner^*} & & & & \nearrow_{diag} & \\
 & & & \mathbf{Simpl}^{\,n+1}\mathbf{Set} & & &
\end{array}$$

so that $BY \simeq |diag\,Ner^*\,Y|$. In particular, given a cat^n-group $\mathcal{G}$, we obtain a weak homotopy equivalence

$$B\mathcal{G} \simeq |diag\,Ner^*\,\mathcal{N}\mathcal{G}|. \tag{11}$$

Thus (11) says that we can calculate the classifying space of a cat^n-group, up to homotopy, by first taking the multinerve, then taking the nerve of the group in each dimension and then taking the geometric realization of the diagonal of the resulting $(n+1)$-simplicial set. This is often useful when working with cat^n-groups.

A morphism $f : \mathcal{G} \to \mathcal{G}'$ in $\mathbf{Cat^n}\,(\mathbf{Gp})$ is a *weak equivalence* if the induced morphism $Bf : B\mathcal{G} \to B\mathcal{G}'$ is a weak homotopy equivalence of spaces.

Next we describe the fundamental cat^n-group functor Π_n. Since the classifying space functor B factors through the categories of n-simplicial groups as well as the category of simplicial groups, we expect the same to hold for its left adjoint Π_n. There are two main steps in identifying the functor Π_n: first find a left adjoint $\mathcal{P}_n$ to the multinerve $\mathcal{N}$, then find a functor $\mathcal{T}_n$ from simplicial groups to n-simplicial groups

$$\mathbf{Cat^n}\,(\mathbf{Gp}) \underset{\mathcal{N}}{\overset{\mathcal{P}_n}{\leftrightarrows}} \mathbf{Simpl}^{\,n}\mathbf{Gp} \underset{\mathcal{T}_n}{\leftarrow} \mathbf{Simpl\,Gp} \underset{|\overline{W}|}{\overset{G\mathcal{S}_0}{\leftrightarrows}} \mathrm{Top}_*$$

so that (10) holds with $\Pi_n = \mathcal{P}_n\mathcal{T}_n G\mathcal{S}_0$.

The left adjoint to $\mathcal{N}$ is easily identified by induction as follows. Let $\mathcal{C}$ be a category of groups with operations. Then the nerve functor

$$Ner : \mathbf{Cat}\,\mathcal{C} \hookrightarrow \mathbf{Simpl}\,\mathcal{C}$$

has a left adjoint $\mathcal{P}$, called the fundamental groupoid functor. For any object G_* of $\mathbf{Simpl}\,\mathcal{C}$, $\mathcal{P}(G_*)$ has arrows object $G_1/d_2(N_2G)$, where N_* denotes the Moore complex, objects object G_0, source, target and identity maps induced by face and degeneracy operators respectively. The simplicial identities easily imply that this is an object of $\mathbf{Cat}\,\mathcal{C}$. The unit of the adjunction $G_* \to \mathcal{N}er\mathcal{P}(G_*)$ induces an isomorphism of π_1 and π_2, as is easily checked.

Notice that, when restricting to the subcategory $\mathbf{Simpl}\,\mathcal{C}$ of simplicial objects in $\mathcal{C}$ of Moore complex of length 1 (namely the Moore complex is trivial in dimension greater than 1), then the adjunction $\mathcal{P} \dashv \mathcal{N}er$ becomes an equivalence of categories. Taking $\mathcal{C} = \mathbf{Gp}$ we deduce that crossed modules in groups are also equivalent to simplicial groups of Moore complex of length 1.

By induction, suppose we have constructed the left adjoint $\mathcal{P}_{n-1}$ to the multinerve

$$\mathcal{N} : \mathbf{Cat^n}\,(\mathbf{Gp}) \hookrightarrow \mathbf{Simpl^{n-1}}(\mathbf{Gp}).$$

This induces an adjunction $\mathcal{P}^*_{n-1} \dashv \mathcal{N}^*$ between the corresponding categories of simplicial objects; from above, taking $\mathcal{C} = \mathbf{Cat^{n-1}}(\mathbf{Gp})$, we obtain the commutative diagram

$$\begin{array}{ccc}
\mathbf{Simpl\ Cat^{n-1}}(\mathbf{Gp}) & \underset{\mathcal{N}^*}{\overset{\mathcal{P}^*_{n-1}}{\rightleftarrows}} & \mathbf{Simpl^n\ Gp} \\
\Big\downarrow \| & & \mathcal{P}_n \Big\downarrow \Big\uparrow \mathcal{N} \\
\mathbf{Simpl\,Cat^{n-1}}(\mathbf{Gp}) & \underset{\mathcal{N}er}{\overset{\mathcal{P}}{\rightleftarrows}} & \mathbf{Cat}(\mathbf{Cat^{n-1}}(\mathbf{Gp})) \cong \mathbf{Cat^n}(\mathbf{Gp})
\end{array}$$

where $\mathcal{P}_n = \mathcal{P}\mathcal{P}^*_{n-1}$. By construction, $\mathcal{P}_n$ is left adjoint to $\mathcal{N}$, and this concludes the inductive step.

To construct the fundamental cat^n-group functor in such a way that (10) holds we would need the unit of the adjunction $\mathcal{P} \dashv \mathcal{N}$ to induce a map of classifying spaces $B(G_*) \to B\mathcal{N}\mathcal{P}_n(G_*)$ which is an $(n+1)$-weak equivalence. Given any n-simplicial group G_*, this is in general false unless $n = 1$. The key observation of [12] is that under certain asphericity conditions on G_* the above is true; furthermore, there exists a functor $\mathcal{T}_n : \mathbf{Simpl\,Gp} \to \mathbf{Simpl}^{\,n}\mathbf{Gp}$ with the property that for any simplicial group H_* and $n \geq 2$, the n-simplicial group $\mathcal{T}_n(H_*)$ satisfies the required asphericity conditions and the natural transformation $B(H_*) \to B\mathcal{T}_n(H_*)$ is a weak homotopy equivalence.

The functor $\mathcal{T}_n$ is given by or^*, the functor induced by the ordinal sum $or : \Delta \times \cdots^n \times \Delta \to \Delta$.

In the case $n = 2$, the right adjoint to $\mathcal{T}_2$ is called in the literature the Artin-Mazur codiagonal functor [1]. Further, $\mathcal{T}_2(H_*)$ is given by "total

Dec;" this is obtained as cotriple resolution of H_* via the cotriple associated to the pair of adjoint functors

$$\mathbf{Simpl\,Gp} \underset{Dec^1}{\overset{+}{\leftrightarrows}} Aug\,\mathbf{Simpl\,Gp}.$$

Here the "shift functor" Dec^1 maps a simplicial group to an augmented simplicial group obtained by forgetting the last face operator; its left adjoint forgets the augmentation. Hence $\mathcal{T}_2(H_*)$ has the form

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots \\
\cdots & H_5 & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & H_4 & \substack{\longrightarrow\\ \longrightarrow} & H_3 \\
 & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow & & \downarrow\downarrow\downarrow \\
\cdots & H_4 & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & H_3 & \substack{\longrightarrow\\ \longrightarrow} & H_2 \\
 & \downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
\cdots & H_3 & \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} & H_2 & \substack{\longrightarrow\\ \longrightarrow} & H_1
\end{array}$$

Thus, if X is a path-connected space and $H_* = G\mathcal{S}_0(X)$, the fundamental cat^2-group of X has the form

$$\begin{array}{ccc}
\dfrac{H_2 \times_{d_0} H_2}{\sim} & \substack{\longrightarrow\\ \longrightarrow} & H_1 \times_{d_0} H_1 \\
\downarrow\downarrow & & \downarrow\downarrow \\
H_1 \times_{d_0} H_1 & \substack{\longrightarrow\\ \longrightarrow} & H_1
\end{array}$$

6. Recent developments. In this section we give a very broad outline of some recent developments that link strict higher dimensional algebra to higher category theory. As explained in Section 2, the area which we refer to as strict higher-dimensional algebra arose within algebraic topology and has as its main categorical input strict n-categories / ω-categories and n-fold categories internalized in a fixed algebraic category such as groups and algebras.

An important development has been taking place almost in parallel in category theory with the discovery of several notions of weak n-category. It is outside the scope of this paper to provide a survey of this rapidly expanding area: we refer the reader to existing surveys [18, 37] and references there.

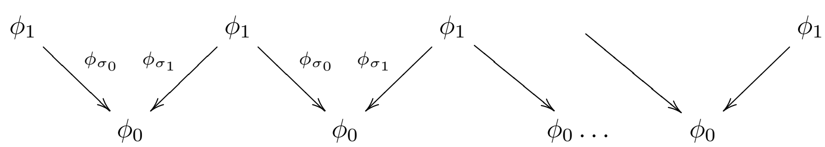
The modelling of homotopy types provides an important link between weak n-categories and homotopy theory. Namely, it is understood that a good definition of weak n-category should provide a model of n-types in the weak n-groupoid case. This "conjecture-test" has been verified for some of the current notions of weak n-category for instance, for Tamsamani's model [54] and Batanin's model [19, 2]; in the weak ω-category case, it holds automatically for Street's [51] and Verity's [56], [57] definitions.

On the other hand, since strict higher-dimensional algebra does provide models of all n-types, it is entirely natural to ask how do these models compare with weak n-groupoids. One aspect that makes this a very intriguing question is that internal n-fold categorical structures seem to have a very different feature from weak n-groupoids: namely, in the first there is no apparent "weakness" in the definition (this is why we think of them as part of strict higher-dimensional algebra). Thus it ought to be interesting to directly construct, from a cat^n-group, a weak $(n+1)$-groupoid representing the same homotopy type. This comparison program has been carried out using Tamsamani's model of weak n-groupoids, for $n = 3$ in [47] and in [46] for general n. The main outcome of this comparison is a semistrictification result for Tamsamani's weak n-groupoids. This relates to "coherence theorems" for higher categories which, in the general case, are known so far in dimension 2 [41] and 3 [28].

At present a comparison between $\mathbf{Cat^n(Gp)}$ and weak $(n+1)$-groupoids, other than Tamsamani's, is not known. In view of the above such comparison could be interesting for the theory of weak n-categories itself.

We shall now give a very broad sketch of some ideas involved in the comparison between cat^n-groups and Tamsamani's model.

Suppose $\mathcal{C}$ is a category with finite limits and $\phi \in \mathbf{Simpl}\,\mathcal{C}$ is a simplicial object in $\mathcal{C}$; put $\phi_n = \phi[n]$ and let $\phi_1 \times_{\phi_0} \cdots \times_{\phi_0} \phi_1$ be the limit of the diagram



where $\sigma_0, \sigma_1 : [0] \to [1]$, $\sigma_0(0) = 0$, $\sigma_1(0) = 1$. It is immediate to see that, for each $n \geq 2$, there is a unique map $\eta_n : \phi_n \to \phi_1 \times_{\phi_0} \cdots \times_{\phi_0} \phi_1$ such that $\mathrm{pr}_j \eta_j = \phi\nu_j$, where $\nu_j : [1] \to [n]$, $\nu_j(0) = j-1$, $\nu_j(1) = j$. The maps η_n are called *Segal maps* and they play a fundamental role in the cat^n-group and the Tamsamani model. In fact, a simplicial object in $\mathcal{C}$ is the nerve of an internal category in $\mathcal{C}$ precisely when all Segal maps are isomorphisms. Thus in view of Theorem 3.2 the definition of cat^n-groups could also be given, inductively, as follows:

i) $\mathbf{Cat^0(Gp)} = \mathbf{Gp}$.
ii) $\mathbf{Cat^n(Gp)}$ is the full subcategory of simplicial objects in $\mathbf{Cat^{n-1}(Gp)}$ such that all Segal maps are isomorphisms. (12)

One of the key ideas of Tamsamani's model is to encode the weakness of associativity and identity laws in a weak n-category in the requirement that the Segal maps are not isomorphisms but some suitably defined n-equivalences. In the case $n = 2$ one obtains a notion strictly related to bicategories, see [35]. Thus the definition of Tamsamani's weak n-groupoids is given inductively by requiring:

i) A weak 1-groupoid is just a groupoid. An equivalence of weak 1-groupoids is a functor which is fully faithful and essentially surjective on objects.
ii) A weak n-groupoid is a simplicial object $\mathcal{G}$ in weak $(n-1)$-groupoids such that $\mathcal{G}_0$ is discrete and the Segal maps are $(n-1)$-equivalences. (13)

Clearly, this definition is not complete until one specifies what is an n-equivalence, given inductively a notion of $(n-1)$-equivalence. The details of this important point can be found in [55] and [54]. Here we limit ourselves to comparing (12) and (13) on a qualitative basis. We can say (12) and (13) are similar in that they are both multi-simplicial structures: unravelling the inductive definitions (12) and (13) and using the nerve construction we see that $\mathbf{Cat^n(Gp)}$ can be viewed as a full subcategory of n-simplicial groups, Tamsamani's weak n-groupoids as a full subcategory of n-simplicial sets. Another similarity is that both definitions are given inductively on dimension.

There are however some important differences. In (13) we have a multi-simplicial structure which is discrete at level zero; this discreteness condition does not hold in (12). We say that (12) is a strict cubical structure, (13) is a weak globular one. This terminology refers, intuitively, to the shape of the 2-cells. A second difference is that (12) defines a structure internal to the category of groups, (13) defines a structure based on **Set**. Thus in the comparison problem there are two main issues: a) How to pass from a strict cubical structure to a weak globular one without losing homotopical information. b) How to pass from an internal structure in groups to a set-based structure while preserving the homotopy type.

Point b) is the easiest to deal with, through the use of the nerve functor $\mathbf{Gp} \to [\Delta^{op}, \mathbf{Set}]$. Point a) is much more subtle and involves a discretization process which is made possible by the fact that every cat^n-group can be represented, up to homotopy, by one in which some of the 'faces' are weakly equivalent to discrete ones. For $n = 2$, for instance, it can be shown that every cat^2-group is weakly equivalent to one in which the object of objects is a cat^1-group weakly equivalent to a discrete one: see [47] for details. This property of cat^n-groups can be proved using the functor $\mathcal{F}_n : \mathbf{Set} \to \mathbf{Cat^n(Gp)}$ which was described in Theorem 4.1.

We conclude by mentioning that in recent years there has been a surge of interest in category theory in n-fold categories themselves, especially for $n = 2$. The theory of double categories is being developed in great detail: see for instance [20–22, 29]. Also in [25] double categories and their lax versions have been studied in relation to applications to conformal field theory.

These recent developments seem to indicate that n-fold categorical structures, although deceptively simple in their definition, encode much of the complexity and richness of higher category theory and could provide stimulating new developments in the years to come.

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