On online bin packing with LIB constraints

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Abstract

In many applications of packing, the location of small items below large items, inside the packed boxes, is forbidden. We consider a variant of the classic online one dimensional bin packing, in which items allocated to each bin are packed there in the order of arrival, satisfying the condition above. This variant is called online bin packing problem with LIB (Larger Item in the Bottom) constraints.

We give an improved analysis of FIRST FIT showing that its competitive ratio is at most $\frac{5}{2} = 2.5$, and design a lower bound of 2 on the competitive ratio of any online algorithm. In addition, we study the competitive ratio of FIRST FIT as a function of an upper bound $\frac{1}{d}$ (where d is a positive integer) on the item sizes. Our upper bound on the competitive ratio of FIRST FIT tends to 2 as d grows, while the lower bound of 2 holds for any value of d.

Finally, we consider several natural and well known algorithms, namely, BEST FIT, WORST FIT, ALMOST WORST FIT, and HARMONIC, and show that none of them has a finite competitive ratio for the problem.

1 Introduction

The bin packing problem has numerous applications and has been extensively studied in both the offline and the online environments [7, 13, 6, 2, 1].

The basic online problem is defined as follows. The goal is to pack a sequence of items, $a_1, a_2, \ldots$, where $a_i \in (0, 1]$, into a minimum number of unit-capacity blocks, called *bins*, such that the total size of the items in each bin does not exceed 1. The items must be packed one by one, in a way that a packing of an item cannot be influenced by future items, that is, future items are unknown at the time of packing. Repacking is not allowed. The goal is to minimize the number of bins containing at least one item, also called *used* bins. The operation of assigning an item to a new bin is called *opening* a new bin. The set of items which are assigned to one bin are packed there in the order of arrival.

In some applications, such as packing large cargo containers, the sizes of items are relatively small compared to the size of the bins. Therefore, it is usually interesting to study algorithms which act on inputs, where all item sizes are in an interval $(0, \frac{1}{d}]$, for an integer $d \geq 1$. This variant is called the *parametric case*.

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We use the *(asymptotic) competitive ratio* to analyze online algorithms for variants of bin packing. This measure allows to compare the cost of an online algorithm $\mathcal{A}$, denoted by $\mathcal{A}(I)$ for an input I (or simply $\mathcal{A}$), to the cost of an optimal offline solution OPT for the same input, denoted by $\text{OPT}(I)$ (or OPT). We compare the costs only for inputs for which the optimal cost is sufficiently large. The (asymptotic) competitive ratio of $\mathcal{A}$, $\mathcal{R}_{\mathcal{A}}$ is defined as follows.

$$\mathcal{R}_{\mathcal{A}} = \lim_{N\to\infty} \left(\sup_{I:\text{OPT}(I)\geq N} \frac{\text{ALG}(I)}{\text{OPT}(I)} \right).$$

In the online bin packing problem with LIB (Larger Item in the Bottom) constraints [9, 10, 4, 11], the sub-sequence of items which are packed into one bin, $a_{j_1}, a_{j_2}, \ldots, a_{j_k}$, such that $j_1 < j_2 < \cdots < j_k$ must satisfy $a_{j_1} \geq a_{j_2} \geq \cdots \geq a_{j_k}$, i.e., a new item can only be packed on top of an item whose size is no smaller than the size of the new item. A clear application comes from safety and stability requirements of packings. Boxes of fragile equipment may collapse if an item which is too large is packed on top of a smaller item.

Previous work. Algorithms, which were designed for this problem, are adaptations of well-known algorithms for standard online bin packing. In particular, it was shown by Manyem [9, 10] that a natural variant of NEXT FIT (NF) (see Johnson [6]), cannot achieve a finite competitive ratio. This variant uses a single active bin, and closes it to open a new active bin if the new item is either larger than the last item packed in the active bin, or if assigning it to the active bin would cause its contents to exceed a total size of 1. On the other hand, Manyem [10] showed that the competitive ratio of a natural variant of FIRST FIT (FF) (see definition in Section 2) is at most 3. A lower bound of 2 on the competitive ratio of FF was shown in [11]. Thus, it is known that the competitive ratio of FF lies in the interval $[2, 3]$. An additional algorithm, which was considered by Manyem, Salt and Visser [11], is a variation of the Harmonic (HARM) algorithms [8] (see Section 3 for the definition of this variant). They showed that the competitive ratio of this class of algorithms is at least 2. As for lower bounds, a lower bound of 1.78 on the competitive ratio of any algorithm was claimed in [4]. Unfortunately, there seems to be an error in this proof. Thus, no non-trivial lower bounds are known. Clearly, any lower bound for semi-online bin packing with non-increasing item sizes is also a lower bound for online bin packing with LIB constraints. However, this approach cannot result in high lower bounds, since the algorithm FFDECREASING (FFD), which executes FF on a sorted list of items (and automatically fulfills the LIB constraints), has a competitive ratio of $\frac{11}{9}$ [5, 3].

Our results. We give an improved analysis of FF for bin packing with LIB constraints and show that its competitive ratio is at most 2.5. We further study the parametric behavior of FF and show that it is at most $1 + \frac{d+1}{d}$, for any parameter $d \geq 2$.

In addition, we show that other related algorithms, such as BEST FIT (BF), WORST FIT (WF) and ALMOST WORST FIT (AWF), have an unbounded competitive ratio for any $d \geq 1$. Moreover, we refine the result of [11] for HARM, and show that the overall competitive ratio of algorithms in this class is unbounded, though each algorithm in the class has a finite competitive ratio.

Finally, we prove a general lower bound, showing that no algorithm for the problem can have a competitive ratio which is smaller than 2. This lower bound holds in fact for *any* parameter d, and therefore, our analysis of FF for large values of d is almost tight.

2 First Fit

A natural algorithm for the problem, which is in fact a variation of the well-known algorithm FF [7, 13] for classic bin packing, and is called FF as well, was studied in [9, 10, 11] and is defined as follows.

Upon arrival of the t-th item (of size a_t), let $B_1, B_2, \ldots, B_m$ be the current list of used bins, in the order in which they were opened. Let b_i be the size of the last item assigned to B_i, and β_i be the total size of items currently assigned to B_i (for $1 \leq i \leq m$). Let k' be the minimal index of a bin, k, for which both the following conditions are satisfied: $\beta_k \leq 1 - a_t$ and $b_k \geq a_t$. Then the new item is assigned to $B_{k'}$. If no such k exists, a new bin (of index $m+1$) is opened, and the new item is assigned there.

Informally, an item is assigned to the first bin ever opened in which it fits both according to the restriction on the total size of items in a bin, and to the LIB condition. If no such bin exists, it is packed into an empty bin.

It was shown in [10] that the competitive ratio of FF is at most 3. We refine the analysis and show that in fact the competitive ratio is at most 2.5. For the parametric case with $d \geq 2$, we show an upper bound of $2 + \frac{1}{d}$ on the competitive ratio of FF.

We first consider some properties of packings. In what follows, we consider sub-sequences of the input, where in such a sub-sequence, every item must be packed into a separate bin in every solution. This class of sequences includes strictly increasing sub-sequences, sub-sequences where all items are strictly larger than $\frac{1}{2}$, and concatenations of the two options.

A sub-sequence $a_{y_1}, a_{y_2}, \ldots, a_{y_s}$ of the input sequence is called a *sequence of representatives*, if $y_1 < y_2 < \cdots < y_s$ (i.e., the items arrived exactly in this order, possibly punctuated by other items), and each item of the sub-sequence is packed in a separate bin by FF. In addition, let B_{u_i} denote the bin in which item y_i is packed, then we require $u_1 < u_2 < \cdots < u_s$, that is, the bin of item y_i was opened by FF later than the bin of item y_{i-1}, for $2 \leq i \leq s$. A sequence of representatives, $a_{y_1}, a_{y_2}, \ldots, a_{y_s}$, is called *good*, if there exists an index $0 \leq g \leq s$ such that $a_{y_1} < a_{y_2} < \cdots < a_{y_g} \leq \frac{1}{2}$ and $a_{y_f} > \frac{1}{2}$ for all $f \geq g+1$.

Lemma 1 *Consider a good sequence of representatives $a_{y_1}, a_{y_2}, \ldots, a_{y_s}$, then* $\textsc{Opt} \geq s$.

Proof. Let $0 \leq g \leq s$ be an index such that $a_{y_1} < a_{y_2} < \cdots < a_{y_g} \leq \frac{1}{2}$ and $a_{y_f} > \frac{1}{2}$ for all $f \geq g+1$.

Consider first the sequence of representatives $a_{y_1}, a_{y_2}, \ldots, a_{y_g}$. We show by induction that these items require g bins in any valid packing. Clearly, if $g = 0$ no bins are needed, and if $g = 1$ then one bin is needed. Assume now that $g \geq 2$ and $a_{y_1}, a_{y_2}, \ldots, a_{y_{g-1}}$ require $g-1$

bins. Since $y_1 < y_2 < \cdots < y_g$, the item y_g arrives when all items $y_1, y_2, ..., y_{g-1}$ are packed. Since $a_{y_g} > a_{y_k}$ for any $k < g$, it cannot share a bin with any of the previous items, and a g-th is needed.

If $g < s$ then consider the items $y_{g+1}, y_{g+2}, \ldots, y_s$. Among these items, no two items can share a bin due to their size. Moreover, an item y_p cannot be combined with an item y_k for $k \le g < p$, since $y_k < y_p$ and $a_{y_k} \le \frac{1}{2} < a_{y_p}$. ■

Theorem 2 *The competitive ratio of* FF *for online bin packing with* LIB *constraints is at most* 2.5 *for* $d = 1$, *and at most* $2 + \frac{1}{d}$ *for* $d \ge 2$.

Proof. We start with the proof for the general case $d = 1$. Consider the output of FF. Let $\mathcal{B}_H$ denote the sub-sequence of bins which contain a total size of items of at least $\frac{2}{3}$, and let $\mathcal{B}_L$ denote the remaining bins. Let W denote the total size of all input items. We have $\text{OPT} \ge W$. If $\mathcal{B}_L = \emptyset$ then since $W \ge \frac{2}{3}|\mathcal{B}_H|$, we get $\text{FF} = |\mathcal{B}_H| \le 1.5\text{OPT}$. Therefore, in what follows, we assume $|\mathcal{B}_L| > 0$.

Let $v = |\mathcal{B}_L|$ and $\mu = x_1, x_2, \ldots, x_v$ be the indices of bins in $\mathcal{B}_L$.

Given an output of FF, consider a subsequence of bins, $\mathcal{B}_T = t_1, t_2, \ldots, t_u$, opened in this order, but not necessarily consecutively. We define a sub-sequence,

$$\sigma_T = z_{t_1}, z_{t_2}, \ldots, z_{t_u},$$

of the input items as follows. Let z_{t_u} denote the last item ever assigned to the last bin of $\mathcal{B}_T$, B_{t_u}. We use a recursive definition, where the item z_{t_j} $(j < u)$ is an item packed into the bin of index t_j, defined based on the item $z_{t_{j+1}}$, as follows. Informally, z_{t_j} is the last item ever assigned to B_{t_j} before the item $z_{t_{j+1}}$ arrives. That is, z_{t_j} is the maximum item in $1, 2, \ldots, z_{t_{j+1}} - 1$, which is packed into the bin B_{t_j}. Note that such an item must exist, since bin B_{t_j} was non-empty strictly before any item was assigned to bin $B_{x_{t+1}}$. We first note that any sub-sequence σ_T, which is constructed in this way, is a sequence of representatives; the items arrived in the order in which they appear in σ_T, and they are packed into different bins by FF, such that a later item is packed into a later bin. We denote the sequence σ_T built for $\mathcal{B}_L$ by $\sigma = z_{x_1}, z_{x_2}, \ldots, z_{x_v}$. Note that σ is not necessarily a good sequence of representatives. In what follows, we partition σ into sub-sequences and construct an additional sequence of representatives of a subset of bins of $\mathcal{B}_L$ in order to obtain lower bounds on the cost of an optimal solution.

Let p be a maximal index such that $a_{z_{x_i}} \le \frac{1}{2}$ for all $i \le p$, that is, the length of a maximal prefix of σ, which contains only items of size no larger than $\frac{1}{2}$. We let σ' denote this prefix of σ. Using the next claim, we can deduce that $\sigma' = z_{x_1}, z_{x_2}, \ldots, z_{x_p}$ is a good sequence of representatives.

Claim 3 *For a subset of bins,* $\mathcal{B}_T = t_1, t_2, \ldots, t_u$, *if* $a_{z_{t_i}} \le \frac{1}{2}$ *for all* $1 \le i \le u$, *and no bin of* $\mathcal{B}_T$ *has a total size of items of at least* $\frac{2}{3}$ *upon termination, then* σ_T *is a good sequence of representatives.*

Proof. If $u \le 1$, then the claim is trivial, so assume $u \ge 2$. Thus it is sufficient to show $a_{z_{t_i}} < a_{z_{t_{i+1}}}$ for $1 \le i \le u-1$, in order to show that this is a good sequence of representatives. Assume by contradiction that $a_{z_{t_i}} \ge a_{z_{t_{i+1}}}$ for some $1 \le i \le u-1$. Then the reason that $z_{t_{i+1}}$ was not assigned to bin B_{t_i} is due to space restrictions, that is, the resulting total size would have exceeded 1. The case $a_{z_{t_{i+1}}} \le \frac{1}{3}$ is impossible since the total size of items assigned to B_{t_i} upon termination is less than $\frac{2}{3}$, and therefore it is less than $\frac{2}{3}$ throughout the execution. Otherwise $a_{z_{t_i}} \ge a_{z_{t_{i+1}}} > \frac{1}{3}$. Note that z_{t_i} cannot be the first item ever assigned to bin B_{t_i}, since in this case then $a_{z_{t_i}} + a_{z_{t_{i+1}}} \le 1$ and at the time of arrival of item $z_{t_{i+1}}$, bin B_{t_i} contains only one item, and could receive item $z_{t_{i+1}}$. Otherwise, there is at least one item, h, such that $h < z_{t_i}$, assigned to bin B_{t_i}. Due to the LIB constraints, $a_h \ge a_{z_{t_i}} > \frac{1}{3}$. The total sum of items packed into bin B_{t_i} by FF exceeds $\frac{2}{3}$, which contradicts the fact that $B_{t_i} \in \mathcal{B}_T$. ∎

Thus, since any sub-sequence of σ is a sequence of representatives, and all bins of $\mathcal{B}_L$ have a total size of less than $\frac{2}{3}$ upon termination of FF, we can use Claim 3 with the set of bins containing items of σ' and get $\textsc{Opt} \ge p$. Therefore, if $p = v$, then we have $|\mathcal{B}_H| + |\mathcal{B}_L| \le 1.5\textsc{Opt} + \textsc{Opt} = 2.5\textsc{Opt}$. We next concentrate on the case $v \ge p+1$.

Let $0 \le \alpha \le v$ be a maximal index such that $a_{z_{x_i}} \le \frac{1}{3}$ for all $i \le \alpha$. Since $a_{z_{x_{p+1}}} > \frac{1}{2}$, we have $\alpha \le p < v$. We next show that the sequence σ contains a prefix of length α, in which all items do not exceed the size $\frac{1}{3}$, followed by a suffix of length $v - \alpha$, in which each item has a size in $(\frac{1}{3}, 1]$, which implies that these items have sizes in $(\frac{1}{3}, \frac{2}{3})$, since the bins of $\mathcal{B}_L$ cannot contain an item of a size in $[\frac{2}{3}, 1]$.

Claim 4 *Every item z_{x_i} of σ, where $i > \alpha$, satisfies $a_{z_{x_i}} > \frac{1}{3}$.*

Proof. Since $v \ge \alpha + 1$, by definition, $a_{z_{x_{\alpha+1}}} > \frac{1}{3}$. Assume by contradiction that there exists an item $i > \alpha + 1 \ge 1$, such that $a_{z_{x_i}} \le \frac{1}{3}$, and assume that z_{x_i} is the first such item. Since we consider the set $\mathcal{B}_L$, the bin $B_{z_{x_{i-1}}}$ never reaches a total size of items of $\frac{2}{3}$ or more, so the only reason not to pack item z_{x_i} into bin $B_{z_{x_{i-1}}}$ must be $a_{z_{x_{i-1}}} < a_{z_{x_i}} \le \frac{1}{3}$. If $i = \alpha + 2$, then this contradicts $a_{z_{x_{\alpha+1}}} > \frac{1}{3}$. Otherwise, $i - 1 \ge \alpha + 2$, and this contradicting the fact that z_{x_i} is the first item with $i \ge \alpha + 2$, which has a size no larger than $\frac{1}{3}$. ∎

Recall that we are left with the case $v > 0$ and $v > p$. We consider several sub-sequences of *bins* of μ (which was defined to be the sequence of the bin indices of $\mathcal{B}_L$), based on the sizes of the corresponding items of σ, as follows. The sequence μ_1 contains all bins $x_i \in \mathcal{B}_L$ for which $a_{z_{x_i}} \in (0, \frac{1}{2}]$. The sequence μ_2 contains all bins $x_i \in \mathcal{B}_L$ for which $a_{z_{x_i}} \in (0, \frac{1}{3}] \cup (\frac{1}{2}, 1]$. The sequence μ_3 contains all bins $x_i \in \mathcal{B}_L$ for which $a_{z_{x_i}} \in (0, \frac{1}{3}]$. Thus $\alpha = |\mu_3|$ and $|\mu_1| \ge p$. We let $\beta = |\mu_1| - \alpha$ and $\gamma = |\mu_2| - \alpha$.

Let ϕ_i denote the sub-sequence of items $z_{x_i} \in \sigma$, where $x_i \in \mu_i$, for $i = 1, 2, 3$. ϕ_2 is a good sequence of representatives, since the first α items are of size at most $\frac{1}{3}$, and they form a non-decreasing sequence, and the remaining items have size above $\frac{1}{2}$. Thus $\textsc{Opt} \ge \alpha + \gamma$.

Note that ϕ_1 is not necessarily a good sequence of representatives. For example, if the sequence contains four items of the sizes 0.41, 0.65, 0.24, 0.4, then three bins are created, all of which belong to $\mathcal{B}_L$, the sequence σ is 0.41, 0.65, 0.4, and $\phi_1 = 0.41, 0.4$. Thus, in order to prove $\textsc{Opt} \geq \alpha + \beta$, we need to analyze μ_1 further. For that, we define a different sub-sequence of items ψ_1, which consists of exactly one item of each bin of μ_1, and prove that ψ_1 is a good sequence of representatives.

Consider therefore the bins of μ_1 and denote them by $y_1, y_2, \ldots, y_w$. Letting $\mathcal{B}_T = \mu_1$, we define a new sub-sequence of items, $\psi_1 = \sigma_T = \{s_{y_1}, s_{y_2}, \ldots, s_{y_w}\}$, which is a sequence of representatives. In this construction, s_{y_w} is the last item assigned to the bin B_{y_w}, and for each $i < w$, s_{y_i} is the last item, which is already assigned to bin B_{y_i}, at the time of the arrival of item y_{i+1}. In order to prove that this is a good sequence of representatives, by Claim 3, it is sufficient to show that all these items are of size in $(0, \frac{1}{2}]$.

Let π_i denote the index of the unique item of bin y_i in σ, and recall that $a_{\pi_i} \leq \frac{1}{2}$. We prove that for each bin, $\pi_i \leq s_{y_i}$, i.e., that for every bin $y_i \in \mu_1$, the item of ψ_1 of this bin arrived no earlier than the item of σ of this bin. Therefore, since they are packed in the same bin by FF, this implies $a_{s_{y_i}} \leq a_{\pi_i} \leq \frac{1}{2}$.

To prove $\pi_i \leq s_{y_i}$, we can use induction and actually prove the following. Consider a sequence of consecutive bins of B_L, packed by FF, denoted by $A_1, \ldots, A_c$. A_1 and A_c are two consecutive bins of μ_1, whereas bins $A_2, A_3, \ldots, A_{c-1} \notin \mu_1$, are the sub-sequence of bins in $\mathcal{B}_L$, opened by FF after A_1 and before A_c (if such bins exist). Let i_c and i'_c be two (not necessarily distinct) items packed in A_c, such that $i_c \leq i'_c$, $i_c \in \sigma$ while $i'_c \in \psi_1$. Let i'_1 be an item of maximum index packed in A_1 such that $i'_1 < i'_c$. By definition of ψ_1, $i'_1 \in \psi_1$. Let i_j be an item of σ packed into A_j, for $1 \leq j \leq C-1$. We need to show $i_1 \leq i'_1$. If this is proved, then it is possible to show by induction, starting from the last bin of μ_1, that each item of ψ_1 arrived no earlier than the item of σ, which is packed into the same bin by FF. This holds for bin y_w, since the item of ψ_1 of this bin is the very last item of this bin. To prove it for bin y_j $(j < w)$ after it was proved for bin y_{j+1}, we use the argument for $A_1 = y_j$ and $A_c = y_{j+1}$.

By definition of σ, we have $i_1 < i_2 < \cdots < i_c \leq i'_c$. Thus $i_1 < i'_c$, and therefore i_1 is a valid option for becoming i'_1. Since i'_1 is chosen to be an item of maximum index of item packed into bin A_1, which satisfies the property of being no larger than i'_c, we get that indeed $i'_1 \geq i_1$.

Thus, due to the existence of a good sequence of representatives in bins $y_1, y_2, \ldots, y_w$, we have $\textsc{Opt} \geq w = \alpha + \beta$. Combining with $\textsc{Opt} \geq \alpha + \gamma$ we have $\textsc{Opt} \geq \alpha + \frac{\beta+\gamma}{2}$.

We can give a better lower bound on the total size as follows. Due to the existence of large enough items (based on the definition of the sub-sequences), we get $W \geq \beta \cdot \frac{1}{3} + \gamma \cdot \frac{1}{2} + |\mathcal{B}_H| \cdot \frac{2}{3}$, or $|\mathcal{B}_H| + \frac{1}{2}(\beta + \gamma) \leq |\mathcal{B}_H| + \frac{1}{2}\beta + \frac{3}{4}\gamma \leq 1.5\textsc{Opt}$.

Therefore, we have

$$\text{FF} = |\mathcal{B}_L| + |\mathcal{B}_H| = \alpha + \beta + \gamma + |\mathcal{B}_H| = (\alpha + \frac{\beta+\gamma}{2}) + (\frac{\beta+\gamma}{2} + |\mathcal{B}_H|) \leq 2.5\textsc{Opt} \ .$$

Consider next the case $d \geq 3$ (the result for $d = 2$ follows from the case $d = 1$). Given an output of FF, we remove all bins which contain a total size of items of at least $\frac{d}{d+1}$. We denote the number of such bins by H. The set of bins $\mathcal{B}_T$ is the set of remaining bins, and the sub-sequence σ_T is constructed as above. We show that in this case, the items of σ_T, which are a sequence of representatives, form a good sequence of representatives. We denote the bins of σ_T by $x_1, x_2, \ldots, x_v$. All items have sizes below $\frac{1}{2}$, thus we need to show $a_{z_{x_i}} < a_{z_{x_{i+1}}}$. Assume that the opposite holds. The item $z_{x_{i+1}}$ is assigned to bin x_{i+1} and not to bin x_i since bin x_i is occupied by more than $1 - a_{z_{x_{i+1}}}$ at that time. Since the bin is occupied by less than $\frac{d}{d+1}$, we have $a_{z_{x_{i+1}}} > \frac{1}{d+1}$. However, $a_{z_{x_{i+1}}} \le \frac{1}{d}$, so $1 - a_{z_{x_{i+1}}} > 1 - \frac{1}{d}$.

We consider all the items of bin x_i, which arrived before item $z_{x_{i+1}}$, and let κ denote the number of such items. Each such item has a size in $(\frac{1}{d+1}, \frac{1}{d}]$, since it is no smaller than the size of the item $z_{x_{i+1}}$ (using $a_{z_{x_i}} \ge a_{z_{x_{i+1}}}$ and due to LIB constraints). We have shown that the total size of such items is in the interval $(\frac{d-1}{d}, \frac{d}{d+1})$. If $\kappa \ge d$, then the total size of items in bin x_i is more than $\frac{d}{d+1}$. If $\kappa \le d-1$, then the total size of these items is at most $\frac{d-1}{d}$. This contradiction implies $a_{z_{x_i}} < a_{z_{x_{i+1}}}$ as required.

Thus $\textsc{Opt} \ge v$ and $\textsc{Opt} \ge \frac{d}{d+1}H$, and since $\mathrm{FF} = H + v$, we get a competitive ratio of at most $2 + \frac{1}{d}$. ■

3 Other algorithms

We consider several common bin packing algorithms and define their adaptations for the bin packing problem with LIB constraints. The first type of algorithms consists of algorithms which act as follows, upon an arrival of a new item. The algorithm finds the subset of non-empty bins, into which the item can be packed, without exceeding the total size of 1 in the bin, and without violating the LIB constraints. If no such bin exists, a new bin is opened for the item. If at least one such bin exists, all such bins are considered. Out of these bins, BF chooses the bin with smallest remaining space. WF chooses the bin with the largest remaining space. AWF chooses the bin which has the second largest remaining space, if there are at least two available bins in the subset of bins which is valid for the current item, and otherwise, it chooses the only available bin.

We show that none of these algorithms has a bounded competitive ratio. Note the difference with standard bin packing, where BF and AWF behave similarly to FF (having all a competitive ratio of 1.7), and WF has a competitive ratio of 2 [6, 7].

Theorem 5 BF *has an unbounded competitive ratio for bin packing with* LIB *constraints, for any value of d.*

Proof. Let $N \ge 10$ be a large integer. Consider the execution of BF on the following input. The input consists of $N > d$ items, where the items of indices which are of the form $\frac{k(k+1)}{2}$ (for some integer k) have a size of γ (where $\gamma < \frac{1}{2N}$), and all other items are of size 2γ. For a sequence of length N, an optimal solution can use two bins in total, each containing the items of one of the sizes. We show by induction that after the arrival of the items of indices $j(j-1)/2+1, \ldots, j(j+1)/2$, for some $j \ge 0$, the algorithm has j bins, each bin has some number of items of size 2γ (at most $j-1$ such items) and one item of size γ, thus the total size of items packed into each bin is at most $(2j-1)\gamma$. Assume that this is true for $j = i$.

Next, the items of indices $i(i+1)/2+1, \ldots, (i+1)(i+2)/2-1$ arrive, all of which have size 2γ. By the hypothesis, all bins contain an item of size γ, so a new bin is opened for the first such item, and the other items join it. The item of size γ arrives when the bin contains items of a total size of $2i \cdot \gamma$. Since each one of the previous bins contain items of a total size of at most $(2i-1) \cdot \gamma$, the item of index $(i+1)(i+2)/2$ is packed into the last bin which contains the largest total size of items, as claimed. As a result, each bin now contains a total size of items of at most $(2i+1)\gamma$, and each bin contains at item of size γ.

The number of bins is therefore $\Theta(\sqrt{N})$. In order to let the optimal cost tend to infinity (to comply with the definition of competitive ratio), the construction is repeated with increasing values of γ, that is, it is created for $\gamma = \frac{1}{N4^{N+1}}, \frac{1}{N4^N}, \ldots, \frac{1}{N \cdot 4^2}$. The instances for the different values of γ are independent, as no algorithm can combine items of more than one instance in a bin. Thus we get a cost of $\Theta(N^{\frac{3}{2}})$, and an optimal cost of $\Theta(N)$. ■

Theorem 6 WF *and* AWF *have unbounded competitive ratios for bin packing with* LIB *constraints, for any value of d.*

Proof. Consider WF and an input of length N which alternates between items of the sizes 2γ and γ, starting with the first type. After each assignment of an item of size 2γ, the next item of size γ can be assigned to any previous bin. However, the last bin has a total size of 2γ whereas all other bins are occupied by 3γ, therefore it is packed in the last bin. The items of size 2γ can never join a previously created bin. Thus the cost of the algorithm is $\Theta(N)$, while the optimal cost is 2. The same method as in the previous example is applied to achieve the situation where WF has a cost of $\Theta(N^2)$, and the optimal cost is $\Theta(N)$.

Next, consider AWF, and an input of length N which is similar to the previous one, but it starts from an item of size γ. At each time, after a larger item is assigned, there is one bin containing an item of size γ, some number of bins (zero or more), each of which contains a larger item followed by a smaller item, and a bin with the item of size 2γ. The smaller item can be packed in any bin. The worst fit is the bin which has a total size of γ. There is one bin with a total of 2γ and all other bins with a total of 3γ, thus the new item joins the bin which contains of item of size 2γ, to create an additional bin with a total of 3γ, and the next item of size 2γ needs to be packed into a new bin. As in the previous case, we can use this type of construction to get a cost of $\Theta(N^2)$, and an optimal cost of $\Theta(N)$. ■

We next consider the class of algorithms HARM of Lee and Lee [8]. Each algorithm has an integer parameter $k \geq d$, and it partitions items into classes which are packed independently. Class i, for $d \leq i \leq k-1$, contains all items of size in $(\frac{1}{i+1}, \frac{1}{i}]$. Class k contains all other items, that is, items of size in $(0, \frac{1}{k}]$. In the original definition, each class is packed using NF, but using FF instead of NF does not degrade the performance. In the case where the algorithm is used for standard bin packing, all bins of each class $i < k$ contain exactly i items, except for possibly the last bin opened for this class. In the case of packing with LIB constraints, this is not necessarily the case. In particular, due to the unsatisfactory behavior of NF for packing with LIB constraints [9, 10], and similarly to [11], we consider the variant which applies FF to each class.

As opposed to standard bin packing, we can show that the performance of Harm only degrades as the value of k grows.

Theorem 7 *Harm has an unbounded competitive ratio for bin packing with* LIB *constraints for any $d \geq 1$. For a specific value of k, the competitive ratio of* Harm *is $\Theta(k-d+1)$.*

Proof. We first show that the competitive ratio of Harm for specific values of d and k is finite, namely, that it is $O(k-d+1)$. Consider first the bins which contain a total size of items of at least $\frac{1}{2}$. Clearly, the number of such bins is at most $2 \cdot \text{Opt}$. Remove these bins and their contents from the input. As a result, no items of size in $(\frac{1}{2}, 1]$ remain, so we can assume $d > 1$. We further consider the modified input, which we denote by ν, while the original input is denoted by μ.

Consider the sub-sequence of ν which is restricted to class i, and denote it by ϕ_i. The items of ϕ_i are packed independently from other items. Let $B_{x_1^i}, B_{x_2^i}, \ldots, B_{x_{k_i}^i}$ denote the bins used for the packing of ϕ_i. We can define a sub-sequence σ_i of ϕ_i, which is a sequence of representatives, similarly to the definition of σ in Theorem 2. σ_i contains one item per packed bin, and the items appear in the order of the bins. Specifically, we define the sequence $\sigma_i = \{z_{x_1^i}, z_{x_2^i}, \ldots, z_{x_{k_i}^i}\}$ as follows. The item $z_{x_{k_i}^i}$ is the last item packed into bin $B_{x_{k_i}^i}$. For $j < k_i$, the item $z_{x_j^i}$ is the last item packed into bin $B_{x_j^i}$, among the items of ϕ_i which arrived before the item $z_{x_{j+1}^i}$.

Since no bin, packed with items of ν, is ever occupied by a total size exceeding $\frac{1}{2}$, and all item sizes are in $(0, \frac{1}{2}]$, then for any $j < k_i$, the reason that item $z_{x_{j+1}}$ is not packed into bin B_{x_j} must be $z_{x_{j+1}} > z_{x_j}$, so σ_i is a a good sequence of representatives, and we have $\text{Opt} \geq |\sigma_i| = k_i$. Thus, the total number of bins is $\sum_{i=d}^{k} k_i \leq (k-d+1)\text{Opt}$. This gives a total cost of $O(k-d+1) \cdot \text{Opt}$, including all bins used for μ.

For the lower bound, consider specific values of d and k. If $k = d$, then we are done. Otherwise $k > d \geq 1$, and therefore $k \geq 2$. We consider three cases as follows. Let $N = k-d$, if $k \leq 2d$, $N = d$ if $2d < k \leq 3d$, and if $k > 3d$, we let $j \geq 1$ be an integer such that $2^j d < k-d \leq 2^{j+1}d$, and $N = 2^{j-1}d$.

Let M be a large integer. For every $d \geq 1$, we show an input for which $\text{Opt} = M$, and the cost of Harm is MN. In the first case, where $k \leq 2d$, we have $N = k-d \geq 1$. In the second case, where $2d < k \leq 3d$, we have $N < k-d \leq 2N$ and in the third case, where $k > 3d$, we have $2N < k-d \leq 4N$, so $\frac{k-d}{4} \leq N \leq k-d$ holds for all cases, and therefore $N = \Theta(k-d)$. Note that $k - N \geq d \geq 1$ in all cases.

Let $\varepsilon = \frac{1}{k(k-1)MN(N+1)}$. There are N batches of items, where each batch contains M items, and the i-th item in the j-th batch has a size of $\frac{1}{k-N+j} + i\varepsilon$.

The items of the j-th batch all have a size larger than $\frac{1}{k-N+j} \geq \frac{1}{k}$. Using $i\varepsilon \leq M\varepsilon = \frac{1}{k(k-1)N(N+1)} < \frac{1}{k(k-1)}$, and $j \leq N$, we get

$$\frac{1}{k-N+j} + i\varepsilon \quad < \quad \frac{1}{k-N+j} + \frac{1}{k(k-1)}$$

$$\leq \quad \frac{1}{k-N+j} + \frac{1}{(k-N+j-1)(k-N+j)} = \frac{1}{k-N+j-1} \leq \frac{1}{d}.$$

Thus, all items have a size in the required interval, and the items of batch j are exactly the items of class $k-N+j-1$. The items of each such class form an increasing sequence, and therefore, each item is packed by HARM into a separate bin, which results in NM bins.

An optimal solution packs the items of the form $\frac{1}{k-N+j}+i\varepsilon$, for a fixed value of i, and $1 \leq j \leq N$, into a dedicated bin. The sub-sequence σ_i, which is defined to contain all the items of the form $\frac{1}{k-N+j}+i\varepsilon$, for a fixed value of i, is a decreasing size sequence as a function of j. We show that in all cases, one bin is sufficient for the packing of σ_i. Since the sequence of items sizes in σ_i is decreasing, in order to show the last claim, we need to show that the total size of items in σ_i does not exceed 1. In the first two cases, $N \leq d$, and each item has a size of at most $\frac{1}{d}$. In the last case, the size of each item is at most $\frac{1}{k-N+1}+i\varepsilon \leq \frac{1}{k-N+1}+M\varepsilon \leq \frac{1}{k-N+1}+\frac{1}{N(N+1)} \leq \frac{1}{2^j d+d-2^{j-1}d+1}+\frac{1}{2^{j-1}d(2^{j-1}d+1)} < \frac{1}{2^{j-1}d+1}+\frac{1}{2^{j-1}d(2^{j-1}d+1)} = \frac{1}{2^{j-1}d} = \frac{1}{N}$. In this last case, the total size of N items does not exceed 1. Thus, the total number of bins used by an optimal solution is at most M.

Therefore, the competitive ratio of HARM is $\Omega(N) = \Omega(k-d)$. To show that it is unbounded, we let k tend to infinity for a constant value of d. ■

4 Lower bound

In this section we present a non-trivial lower bound on the competitive ratio of any algorithm. We prove the following theorem.

Theorem 8 *The competitive ratio of any online algorithm for bin packing with* LIB *constraints is at least 2, for any parameter d.*

Proof. Let $N \geq d+1$ be a large integer. All the items in the sequence will have a size in $[\frac{1}{N}-\frac{1}{2N^2}, \frac{1}{N}]$. Since $(N+1)(\frac{1}{N}-\frac{1}{2N^2}) > 1$, the maximum number of items that can be packed in one bin is N. In fact, if a bin contains less than N items, it can receive an additional item, provided that this item is smaller than the items already packed into that bin.

We construct the sequence in phases, so that each phase contains either N items or $N+1$ items. Moreover, the sizes of items are chosen so that the items of phase j cannot be packed into any of the bins used for items from phases $1, 2, \ldots, j-1$. If phase j contains exactly N items, then all these items are of the same size a_j, which is defined based on the previous phases. Otherwise, the first N items are of size a_j, and one additional item has a size of $b_j < a_j$. In order to define the sizes, the output of each phase is an interval $[\alpha_j, \beta_j]$ (with $\alpha_j < \beta_j$) of allowed sizes for the future phases. We let $[\alpha_0, \beta_0] = [\frac{1}{N}-\frac{1}{2N^2}, \frac{1}{N}]$, and we define the intervals so that $[\alpha_{j+1}, \beta_{j+1}] \subseteq [\alpha_j, \beta_j]$ holds for any $j \geq 0$.

We define the phases by induction as follows. For some $j \geq 1$, N items of size $a_j = \frac{1}{3}(\alpha_{j-1}+2\beta_{j-1})$ arrive. Due to the invariant that they must be packed into new bins, there are two options. The items are either packed into at least two bins, or else they are packed into a single bin.

In the first case, where at least two bins are used for the items of size a_j, all future items are larger than the items of size a_j, and therefore it would not be possible to use the bins of the current phase again. We let $\alpha_j = \frac{1}{4}(\alpha_{j-1} + 3\beta_{j-1})$, and $\beta_j = \beta_{j-1}$. Since $\alpha_{j-1} < \beta_{j-1}$, we have $a_j < \alpha_j < \beta_j$.

In the second case, where all items of size a_j are packed into one bin, one additional item of size $b_j = \frac{1}{3}(2\alpha_{j-1} + \beta_{j-1}) < a_j$ arrives. This item must be packed into a new bin, since it cannot share a bin with items of previous phases, and the only bin created in the current phase already contains N items. All future items will be larger than this item (i.e., larger than b_j), so this bin will not be used again. However, all future items would be smaller than a_j. Since a bin which contains a maximum number N of items (of size a_j) was created, this bin cannot be used again either. Thus we let $\alpha_j = \frac{1}{12}(7\alpha_{j-1}+5\beta_{j-1})$, $\beta_j = \frac{1}{12}(5\alpha_{j-1}+7\beta_{j-1})$. Since $\alpha_{j-1} < \beta_{j-1}$, we get $b_j < \alpha_j < \beta_j < a_j$.

Consider an input created using M phases, so that M is divisible by N. In each phase, the algorithm uses at least two bins, which are never used again. Thus its cost is at least $2M$. We next construct an offline solution which uses at most $M + \frac{M}{N}$ bins. This would result in a lower bound of $\frac{2}{1+\frac{1}{N}}$, which tends to 2 for large enough values of N. To construct the offline solution, we do as follows. For each phase which contains only N identical items, pack the items of this phase into one bin. Let $j_1, j_2, \ldots, j_k$ be the phases which contain $N + 1$ items. For each such phase, we pack $N - 1$ items of size a_j followed by one item of size b_j into one bin. This is possible since $b_j < a_j$ for all $j \in \{j_1, j_2, \ldots, j_k\}$. We next prove that $a_{j_1} > a_{j_2} > \cdots > a_{j_k}$, which allows us to pack the remaining k items of these sizes, N items to a bin (except for possibly the last such bin), resulting in $\lceil \frac{k}{N} \rceil \le \lceil \frac{M}{N} \rceil = \frac{M}{N}$ additional bins. To prove $a_{j_s} > a_{j_{s+1}}$ for some $s < k$, recall that since phase j_s contains $N + 1$ items, then $\beta_{j_s} < a_{j_s}$. Therefore, all further items after phase j_s are no larger than β_{j_s}, so we get $a_{j_{s+1}} \le \beta_{j_s} < a_{j_s}$. ■

5 Conclusion

We have studied the general case of online bin packing with LIB constraints, as well as the parametric case, where all items sizes are in $(0, \frac{1}{d}]$, for some integer $d \ge 2$.

A noticeable difference with standard bin packing is that the best competitive ratio is a constant which is at least 2, even for large values of the parameter d. For standard bin packing, a competitive ratio of at most 1.588 is achievable already for $d = 1$ [12], and the competitive ratio of FF is 1.7 for $d = 1$ and $\frac{d+1}{d}$ for $d > 1$ [13, 7]. It is an open problem to find whether FF achieves the best possible competitive ratio for online bin packing with LIB constraints. It seems difficult to find another algorithm of better performance (or even of equally good performance). Finding the tight competitive ratio as a function of d is left as an open problem.

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