A SELF-GUIDE TO O-MINIMALITY
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1. How to read these notes?

These notes were written for the tutorial in the Camerino Modnet Summer school. The main source I used was v.d Dries' book [1] and many of the proofs here are based on ideas from that book. I recommend also Macpherson's survey article [2]. For material on definable groups, see Otero's survey [3].

Naturally, the notes cover much more than could be discussed in five lectures, so I chose to leave the proofs of many small results and observations as problems to the reader. Thus, these notes can serve as a source for independent study of the basics of o-minimality. While I expect that most of the problems are "doable" without much background it is quite possible that some turn out to be more difficult than I had expected.

Because of lack of time I totally avoided bibliography, but the above texts contain a very good bibliography list for almost all results mentioned here (except the very last section).

Please feel free to contact me (see e-mail at the end) for any question on these notes. They were written quite hastily so contain, I am certain, many errors.

2. The basics of o-minimality

Convention In these notes, the notion of a definable set always allows parameters in the definition. We sometimes refer to these parameters by writing *definable over* A.

2.1. **Linearly ordered structures.** *Assume here that* $\mathcal{M} = \langle M, <, \cdots \rangle$ *is a linearly ordered structure, with a dense ordering.*

Topology We take the order topology on M and the product topology on M^n.

Problems 2.1. *(1) Show that there is a definable family of subsets of* M^n *which forms a basis for the topology on* M^n*. Is every open subset of* M^n *definable?*
(2) For $X \subseteq M^n$ *a definable set, show that the topological interior and closure of* X *are definable. Show that that the sets of isolated points of* X *and of cluster points of* X *are definable.*
(3) Assume that $f : M \to M$ *is definable.*
(a) Define what $\lim_{t\to a} f(t)$ *means (without using the notion of a converging sequence). Show that the function* $x \mapsto \lim_{t\to x} f(t)$ *is definable at every point* x *at which this limit exists.*
(b) Show that the set

$$\{x \in M : f \text{ is continuous at } x\}$$

is definable.
(5) Show that every linearly ordered (dense) structure $\mathcal{M}$ has an elementarily equivalent structure $\mathcal{N}$ such that NO sequence in N is convergent, NO definable subset of N of more than one point is connected, and NO definable infinite subset of N is compact.

Definably connected sets

Definition 2.2. A subset of $X \subseteq M^n$ is called *definably connected* if there are no definable open sets $U_1, U_2 \subseteq M^n$ such that $X \cap U_1 \cap U_2 = \emptyset$, and $X \cap U_1$, $X \cap U_2$ are nonempty.

Problems 2.3. *(6) Show: If $\mathcal{M}$ expands the ordered real numbers then a definable $X \subseteq \mathbb{R}$ is connected iff it is definably connected. What about subsets of $\mathbb{R}^n$?*
(7) Show that the image of a definably connected set under a definable continuous function is definably connected.
(8) Show that if $X_1, X_2 \subseteq M^n$ are each definably connected and $X_1 \cap X_2 \neq \emptyset$ then their union is also definably connected.
(9) Show that if $X_1, X_2 \subseteq M^n$ are definable and definably connected and $Cl(X_1) \cap X_2 \neq \emptyset$ then $X_1 \cup X_2$ is definably connected.
(10) Assume that a definably connected set $X \subseteq M^n$ can be written as an arbitrary union of definable, pairwise disjoint open sets. Then X must be equal to one of these sets.
(11) (Uniform definability) Let $\{X_a : a \in M^k\}$ be a uniformly definable family of subsets of M^n (by that we mean that there exist a definable set $X \subseteq M^{k+n}$ such that for every $a \in M^n$, we have $X_a = \{b \in M^n : (a, b) \in X\}$).

Show that the sets of all $a \in M^k$ such that X_a is a discrete set, closed set, open set, nowhere dense, bounded, are definable.

What about the set of a's such that X_a is definably connected?

2.2. O-minimal structures.

Definition 2.4. We assume that $\mathcal{M} = \langle M, <, \cdots \rangle$ is a linearly ordered structure (by $<$). $\mathcal{M}$ is called *o-minimal* if every definable subset of M is a finite union of points, and open intervals whose endpoints lie in $M \cup \{\pm\infty\}$.

Although the theory of o-minimal structures does not require any other assumptions on the linearly ordered set $\langle M, < \rangle$, it is convenient to focus on the case where $\langle M, < \rangle$ is densely ordered (it turns out that this is basically the only case where the structure of definable sets is of real interest).

We assume from now on that $\mathcal{M}$ is a densely ordered o-minimal structure.

Problems 2.5. *(1) Give several examples of a linearly ordered sets $\langle N, < \rangle$ which are not o-minimal (in the pure $<$-language).*
(2) Try to formulate a necessary and sufficient condition on a linearly ordered set $\langle N, < \rangle$ to be o-minimal.
(3) Show that the following are equivalent:
(a) $\mathcal{M}$ is o-minimal.
(b) Every definable discrete subset of M is finite and every definable subset of M which is bounded from above has a least upper bound.
(c) For every definable $X \subseteq M$, if $\emptyset \neq X \neq M$ then its boundary $(Cl(X) \setminus Int(X))$

is finite and nonempty.

(4) Let $\mathcal{V} = \langle V, <, +, \lambda_a \rangle_{a \in F}$ be an ordered vector space over an ordered field F. Show how quantifier elimination for $\mathcal{V}$ implies that $\mathcal{V}$ is o-minimal.
(5) Let $\mathcal{R} = \langle \mathbb{R}, <, +, \cdot \rangle$ be the ordered field of real numbers. Show how quantifier elimination for $\mathcal{R}$ implies that $\mathcal{R}$ is o-minimal.
(6) (o-minimality and QE) Find an expansion of $\langle \mathbb{R}, < \rangle$ which has quantifier elimination but is not o-minimal.

Find an expansion of $\langle \mathbb{R}, < \rangle$ which is o-minimal but does not have QE (this is probably more difficult).

We now assume that $\mathcal{M}$ is o-minimal:
(7) Show that every interval in M and more generally, every definable rectangular box in M^n is definably connected.
(8) Assume that X is a definably connected subset of M^n and $f : X \to M$ is definable and continuous. Show that graph(f) is definably connected.
(9) Let $\{X_a : a \in M^n\}$ be a uniformly definable family of subsets of M^k. Show that the set $\{a \in M^n : X_a$ is finite $\}$ is definable (this should be done just based on the definition of o-minimality).
(10) **The Intermediate Value Theorem** *Assume that $f, g : I \to M$ are two definable continuous functions on an open interval, such that for every $x \in I$ we have $f(x) \neq g(x)$. Show that either $f < g$ on I, or $g < f$ on I.*

3. O-minimality and elementary equivalence

A Fundamental Question *Is o-minimality preserved under elementary equivalence of structures?*

Here are several related observations which are not difficult to verify. We denote by $\pi_1 : M^{n+1} \to M^n$ the projection on the first n-coordinates. For $X \subseteq M^{n+1}$, and $a \in M^n$, we let $X_a = \{b \in M : (a, b) \in X\}$.

We still assume that $\mathcal{M}$ is o-minimal.

Problems 3.1. *(1) Assume that for every $\mathcal{M}$-definable family $\{X_a : a \in M^k\}$ there exists a number K such that every X_a is the union of at most K-many intervals and points. Then every structure which is elementary equivalent to $\mathcal{M}$ is also o-minimal.*
(2) Assume that for every definable family $\{X_a : a \in M^k\}$ of finite subsets of M there exists a number K such that for every $a \in M^k$, $|X_a| \leq K$. Then every structure elementarily equivalent to $\mathcal{M}$ is o-minimal.
(3) If $\mathcal{M}$ is ω -saturated then every elementarily equivalent structure is o-minimal.
(4) Show that "minimality" (in contrast to o-minimality) is NOT preserved under elementary equivalence. Find infinite structures $\mathcal{N}_1 \equiv \mathcal{N}_2$ such that every definable subset of N_1 is either finite or co-finite, but this fails for $\mathcal{N}_2$.

The goal of the next few sections is to establish the assumptions in Problems (1) and (2) above, and to conclude that o-minimality is indeed preserved under elementary equivalence.

3.1. **The monotonicity-continuity theorem.**

We use here $]a, b[$ to denote open intervals and (a, b) to denote ordered pairs.

Our goal is to prove the monotonicity theorem, denoted by MCT. The main ingredient is the following.

Lemma 3.2. *(**Main Lemma**) Let $f : I \to M$ be a definable function on some open interval I and assume that for every $x \in I$ we have $f(x) > x$. Then there exists $d \in I$ such that the set $\{x < d : f(x) > d\}$ is infinite.*

Proof. Let $I =]a, b[$ (where a and b could be $-\infty, +\infty$, respectively), and define

$$B = \{x \in I : \forall t \in]a, x[\ f(t) < f(x)\}.$$

Case 1 B is infinite.

In which case, by o-minimality, B contains an open interval J. Take $c \in J$ and let $d = f(c) > c$. Now, for every x in the infinite interval $J \cap]c, d[$, we have $f(x) > f(c) = d$, as required.

Case 2 B is finite.

Let $b_0 = minB$. By the definition of B, for every $t \in]a, b_0[$ there exists $t' < t$ such that $f(t') > f(t)$. If we take $c \in]a, b_0[$ then there exists an infinite sequence $\cdots < t_2 < t_1 < c$ such that $f(c) > f(t_1) > f(t_2) > \cdots$. In particular, if we let $d = f(c)$ then there exist infinitely many $x < d$ such that $f(x) > d$. □

Corollary 3.3. *(**A definable "Ramsey" theorem**) Let $I \subseteq M$ be an open interval and assume that $I^2 = X_1 \cup \cdots \cup X_r$, for definable X_i's. Then there exists an open (nonempty) subinterval $J \subseteq I$ and an $i = 1, \ldots, r$ such that for every $a < b$ in J, $(a, b) \in X_i$.*

Proof. By o-minimality, for every $a \in I$ there exists $a' > a$ and an $i = i(a) \in \{1, \ldots, r\}$ such that

$$\forall t \in]a, a'[\text{ we have } (a, t) \in X_i \tag{3.1}$$

Again because of o-minimality, there exists an open interval $J' \subseteq I$ and a fixed i_0 such that for all $a \in J'$, we have $i(a) = i_0$. Given $a \in J'$, let $s(a)$ equal the supremum of all $a' > a$ satisfying (3.1) (this supremum may equal $+\infty$). We have $s(x) > x$ for all $x \in J'$, hence we may apply the main lemma and get an open interval $J \subseteq J'$ and $d > J$ such that for all $x \in J$, $s(x) > d$. It is now easy to see that for all $a < b$ in J, we have $(a, b) \in X_{i_0}$. □

Theorem 3.4. MCT *Let $f : I \to M$ be a definable function on some open interval $I =]a, b[$ (where the endpoints could be $\pm\infty$). Then there are $a_0 = a < a_1 < \cdots < a_n = b$ such that on each $]a_i, a_{i+1}[$ the function f is either constant or strictly monotone and continuous.*

Proof. **Monotonicity**

Consider the following three definable subsets of M, for $\Box \in \{=, <, >\}$:

$$A_\Box = \left\{ x \in I : \begin{array}{ll} f \text{ is locally constant in a neighbrohood of x} & (\text{if } \Box \text{ is } =) \\ f \text{ is strictly increasing in a neighborhood of x} & (\text{if } \Box \text{ is } <) \\ f \text{ is strictly decreasing in a enighborhood of x} & (\text{if } \Box \text{ is } >) \end{array} \right\}.$$

We claim that $A_= \cup A_< \cup A_>$ covers all but finitely many points in M: Indeed, assume towards contradiction that the complement B of this union is infinite, so contains an open interval, call it B again. Consider the sets:

$$X_\Box = \{(x, y) \in B^2 : f(x) \Box f(y)\},$$

with $\Box \in \{=, <, >\}$. These sets clearly form a partition of B^2. By applying Corollary 3.3, we may find an interval $J \subseteq B$, and $\Box \in \{=, <, >\}$, such that for every $x < y$ in J, we have $(x, y) \in X_\Box$. If $\Box$ is $=$ then f is constant on J, hence $J \subseteq A_=$. If $\Box$ is $<$ or $>$ then $J \subseteq A_<$ or $J \subseteq A_>$, contradicting our choice of B. Hence, B is indeed finite, and therefore there are $a_0 = a < a_1 < \cdots, a_n = b$, such that each interval $]a_i, a_{i+1}[$ is contained in one of the $A_\Box$'s.

It is left to see that if f is locally constant or locally strictly monotone near every x in $]a_i, a_{i+1}[$ then it is globally so, on the whole interval. We leave it as an exercise. □

Continuity

We may assume that $f : I \to M$ is strictly increasing. As above, it is sufficient to prove that I contains an interval on which f is continuous. Since f is not constant, its image contains an open interval $J =]c, d[$. But now, for every $c < c_1 < d_1 < d$, the pre-image of $]c_1, d_1[$, must be the interval $]f^{-1}(c_1), f^{-1}(d_1)[$ (notice that f is one-to-one on I). In particular, f is continuous on $f^{-1}(J)$. □

Problems 3.5. *(1) Assume that for some open interval J, a definable function f is locally constant (strictly monotone) near every $x \in J$. Show that f is constant (strictly monotone) on the whole of J.*

(2) Let f be a definable continuous function from an interval $]a, b[$ into M^k. Use MCT to show that if $f(t)$ is bounded then it has limits in M^k as t tends to a and to b.

3.2. **Sparse subsets of M^2.** By problem 3.1 (2), in order to prove that o-minimality is preserved under elementary equivalence it is sufficient to prove that any definable family $\{X_a : x \in M^k\}$ of finite subsets of M has a uniform bound on their size. Our goal in the next section is to prove that for the case $k = 1$, namely for 1-parameter definable families of subsets of M.

Definition 3.6. A definable set $X \subseteq M^2$ is called *sparse* if for all but finitely many $a \in M$, the fiber X_a is finite.

And by induction, a definable $X \subseteq M^{n+1}$ is *sparse* if for all $a \in M^n$ outside a sparse subset of M^n the fiber X_a is finite.

Notice that the union of finitely many sparse sets is again a sparse set. Our goal is to prove:

Theorem 3.7. *Given $X \subseteq M^2$ sparse, there is $K \in \mathbb{N}$ such that for every $a \in M$, if X_a is finite then $|X_a| \leq K$.*

We prove the theorem through a sequence of lemmas.

Lemma 3.8. *For $X \subseteq M^2$ a definable set, the following are equivalent:*

1. *X is sparse.*
2. *The interior of X is empty.*
3. *X is nowhere dense in M^2.*

Proof. We first show that (2) implies (1): If X is not sparse then, by o-minimality, there is an interval $I \subseteq M$ such that for every $a \in I$, the fiber X_a contains a whole open interval. Given $a \in I$, we let $s_1(a) \in M \cup \{-\infty\}$ be the infimum of the first open interval in X_a and let $s_2(X) \in M \cup \{+\infty\}$ be the supremum of the first open interval in X_a. The functions are definable (why?) and we have $s_1(a) < s_2(a)$. By MCT, we may assume that they are both continuous on a subinterval $J \subseteq I$.

But then the set $\{(x,y) : x \in J \& s_1(x) < y < s_2(x)\}$ is an open subset of X, so $Int(X) \neq \emptyset$.

(1) $\Rightarrow$ (3): Let $U \subseteq M^2$ be a definable open set. Because X is sparse the set $X^c \cap U$ is not sparse. But then, by what we just proved, $Int(X^c \cap U) \neq \emptyset$. It follows that X is not dense in U, so X is nowhere dense. The fact that (3) implies (2) is immediate. □

One corollary of the above is that the definition of a sparse set in M^2 does not depend on the ordering of the coordinates, hence if X is sparse then after permutating of the first and second coordinate we still have a sparse set.

Corollary 3.9. *If $X \subseteq M^2$ is sparse then $Cl(X)$ is sparse.*

Proof. X sparse $\Rightarrow$ X is nowhere dense $\Rightarrow$ $Cl(X)$ is nowhere dense $\Rightarrow$ $Cl(X)$ is sparse. □

Lemma 3.10. *Let $X \subseteq M^2$ be definable and sparse.*
(i) If $\pi_1(X)$ is infinite then X contains the graph $\{(x, f(x)) : x \in I\}$, of a continuous definable function f on an open interval I.
(ii) If X contains the graph of a definable continuous function $f : I \to M$ as above then there exist $x_0 \in I$ and an open rectangular box $U \ni (x_0, f(x_0))$ such that $X \cap U = Graph(f) \cap U$.

Proof. (i) We may assume that $\pi_1(X)$ is an open interval $I_0 \subseteq M$. For every $x \in I_0$ let $f(x)$ be the minimal y such that $(x,y) \in X$. By MCT, there exists an interval $I \subseteq I_0$ on which f is continuous.
(ii) For every $x \in I$, let

$f_1(x) = max\{y < f(x) : (x,y) \in X\}$ and if there is no such y, take $f_1(x) = -\infty$.

$f_2(x) = min\{y > f(x) : (x,y) \in X\}$ and if there is no such y, take $f_1(x) = +\infty$.

Again, by MCT, there exists an open interval $J \subseteq I$ on which all three functions, $f_1 < f < f_2$ are continuous, and moreover, for every $a \in J$, we have $X_a \cap]f_1(a), f_2(a)[= \{f(a)\}$. The result now easily follows from the continuity of the three functions. □

Lemma 3.11. *Let $X \subseteq M^2$ be a definable sparse set, and $X_1 \subseteq X$ definable such that $\pi_1(X)$ is infinite. Then there exists an open definable $U \subseteq M^2$ such that $(U \cap X) \subseteq X_1$ and furthermore this intersection is the graph of a continuous function on some open (nonempty) interval.*

Proof. We first apply Lemma 3.10(i) to X_1 and obtain a continuous function f whose graph is contained in X_1. We then apply (ii) to X and to f. □

An immediate corollary is:

Corollary 3.12. *Let $X \subseteq M^2$ be a definable sparse set. Then $Cl(X) \setminus X$ is finite.*

Proof. By Lemma 3.9, $Cl(X)$ is also sparse. We now consider $X_1 = Cl(X) \setminus X$. If X_1 was infinite then either $\pi_1(X_1)$ or $\pi_2(X_1)$ were infinite. Because of the symmetry we discussed above we may assume that $\pi_1(X_1)$ is infinite. If we now apply Lemma 3.11 to $X_1 \subseteq Cl(X)$, we obtain an open U such that $(U \cap Cl(X)) \cap X = \emptyset$ (while $U \cap Cl(X) \neq \emptyset$), contradiction. □

Given $X \subseteq M^2$, let $G(X)$ be the set of all $(x,y) \in X$ for which there exists an open rectangular box $I \times J \ni (x,y)$ such that $X \cap (I \times J)$ is the graph of a continuous function $f : I \to M$. Notice that if X is definable then so is $G(X)$.

Lemma 3.13. *If $X \subseteq M^2$ is definable and sparse then the projection of $X \setminus G(X)$ onto the first coordinate is finite.*

Proof. Let $X_1 = X \setminus G(X)$. By Lemma 3.11, we must have $\pi_1(X_1)$ is finite. □

Given $X \subseteq M^2$ and $a \in \pi_1(X)$, we say that X *is bounded near a* if there exists an open interval $I \ni a$ such that $X \cap (I \times M)$ is bounded in M^2.

Problems 3.14. *Assume that $X \subseteq M^2$ is a sparse set. Show that for all but finitely many $a \in \pi_1(X)$, the set X is bounded near a (hint: use the functions $s_M(x) = sup(A_x)$ and $s_m(x) = inf(A_x)$).*

We now complete the proof of Theorem 3.7 as follows:

Given a sparse $X \subseteq M^2$, we remove all points $a \in \pi_1(X)$ such that (1) $a \in \pi_1(Cl(X) \setminus X)$ and (2) X is not bounded near a. By Lemma 3.11 and Problem 3.14 there are only finitely many such points. By partitioning $\pi_1(X)$ into finitely many open intervals, we may assume:
(i) $\pi_1(X)$ is an open interval I.
(ii) X is relatively closed in $I \times M$.
(iii) X is bounded near every $a \in \pi_1(X)$.

The proof of the theorem will be finished once we prove:

Lemma 3.15. *If $X \subseteq M^2$ is a sparse set satisfying (i)-(iii) above and if $X = G(X)$ then for all $a_1, a_2 \in I$, we have $|X_{a_1}| = |X_{a_2}|$.*

Proof. For $n \in \mathbb{N}$, let $X(n) = \{a \in I : |X_a| = n\}$. We claim that $X(n)$ is open:

Indeed, let $a \in X(n)$ and let $X_a = \{b_1, \ldots, b_n\}$. Because each (a, b_i) is in $G(X)$, there are pairwise-disjoint open intervals $J_1, \ldots, J_n$, $b_i \in J_i$, and an open interval $I' \ni a$ such that for each $i = 1, \ldots, n$, $X \cap (I' \times J_i)$ is the graph of a continuous function f_i whose domain is I'. Moreover, we may assume that X is bounded in $I' \times M$ (by (iii)).

Because $X \cap (I' \times M)$ is bounded and relatively closed, by shrinking I' even further we may assume that

(3.2) $$X \cap (I' \times M) \text{ is contained in } \cup_{i=1}^{n} (I' \times J_i).$$

But now, for every $x \in I'$, $X_x = \{f_1(x), \ldots, f_n(x)\}$, thus showing that $I' \subseteq X(n)$ and therefore $X(n)$ is open.

We showed that I is a union of pairwise disjoint definable open sets $\{X(n) : n \in \mathbb{N}\}$. Because I is definably connected, it follows (see Problem 2.3(10)) that $I = X(n)$ for one of these n's, thus ending the proof of Lemma 3.15, and of Theorem 3.7. □

Problems 3.16. *Prove that we may indeed assume (3.2) above.*

The proof of Theorem 3.7 gives more than just a uniform bound on the size of the fibers in sparse sets. It shows the following:

Theorem 3.17. *For every definable sparse set $X \subseteq M^2$, there are $a_1 < \cdots < a_n$ such that for each i, the set $X \cap]a_i, a_{i+1}[$ is a union of k_i-many pair-wise disjoint graphs of continuous definable functions from $]a_i, a_{i+1}[$ into M (where k_i could be equal to zero).*

Notice that by the o-minimal version of the Intermediate Value Theorem (see Problem 2.5 (10)), there is an ordering on the continuous functions on each of the intervals.

Consider now an arbitrary definable set X in M^2, and let $X^\star$ be the set of all (a,b) such that b is on the boundary of X_a. Because of o-minimality, $X^\star$ is a sparse set. Because of Theorem 3.17, we may assume, after partitioning $\pi_1(X^\star)$ into finitely many open intervals, that $\pi_1(X^\star)$ is an open interval I, and that $X^\star$ is the union of k-many graphs of continuous functions $f_i : I \to M$, $i = 1, \ldots, k$.

Problems 3.18. *Assuming that $X^\star$ satisfies the above, prove that $X \cap (I \times M)$ is a finite union of sets of the form:*
(i) Graph of f_i for some $i \in \{1, \ldots, k\}$, or
(ii) $\{(x,y) : x \in I \& f_i < y < f_{i+1}\}$, for some $i \in \{1, \ldots, k\}$.

This is the first step towards proving the fundamental Cell Decomposition Theorem.

3.3. **Cell Decomposition.**

Definition 3.19. For every $n \in \mathbb{N}$, we define *k-cells* in M^n as follows:

In M: Every point is a 0-cell. Every open interval is a 1-cell.

And by induction: A k-cell in M^{n+1} is a set $X \subseteq M^{n+1}$ whose projection on M^n is a cell C and such that:

(i) Either C is a k-cell and X is the graph of a definable continuous function $f : C \to M$;

(ii) Or C is a $k-1$ cell and there are definable continuous functions $f, g : C \to M$ with $f(x) < g(x)$ for all $x \in C$ (and f, g possibly taking the value $-\infty, +\infty$, respectively, on the whole of C) such that X is the region "trapped" between the graphs of f and g. Namely:

$$X = \{(x,y) : x \in C \& f(x) < y < g(x)\}.$$

A cell decomposition of a definable set $X \subseteq M^n$ is a partition of X into finitely many pairwise disjoint cells such that for any two cells C_1, C_2, either $\pi(C_1) = \pi(C_2)$ or $\pi(C_1) \cap \pi(C_2) = \emptyset$ (where π is the projection on the first $n-1$ coordinates). A cell decomposition of X is said to be *compatible with* $Y \subseteq X$ if every cell is either contained in Y or disjoint from it.

Remark The notion of a cell is defined with respect to a particular ordering of the coordinate axes. It is not invariant under a permutation of the coordinates.

Problems 3.20. *(1) Show that every definable cell is definably connected (use induction on the construction of the cell).*

We say that two cells C_1, C_2 are adjacent if either $C_1 \cap Cl(C_2) \neq \emptyset$ or $C_2 \cap Cl(C_1) \neq \emptyset$.
(2) Let X be a finite union of the cells $C_1, \ldots, C_k$. Show that X is definably connected if and only there is an ordering of the cells such that any two cells in this ordering are adjacent.
(3) Show: If $C \subseteq M^n$ is a k-cell then it is definably homeomorphic to an open subset of M^k. Moreover, this homeomorphism is given by a projection of C onto k of the coordinates (hint: Use induction on n).
(4) If $X \subseteq M^n$ is a union of finitely many $n-1$ cells then $M^n \setminus X$ is dense in M^n and has a nonempty interior.
(5) Give an example of a cell $C \subseteq \mathbb{R}^2$ (in the language of real closed fields) which is not a cell after a permutation of the coordinates. Can every cell in $\mathbb{R}^2$ be decomposed

into finitely many sets which are cells with respect to both orderings of the coordinate axes?

The following theorem can probably be called "The Fundamental Theorem of O-minimality".

Theorem 3.21. *If $X \subseteq M^n$ is definable then there is a cell decomposition of M^n into cells which is compatible with X.*

Proof. Note that Theorem 3.17 and Problem 3.18 establish the theorem for the case $n = 2$. The proof of the general result is too long for these notes. However, the ambitious reader can try to push forward the arguments in Section 3.2 to sparse subsets of M^3 (and then similarly by induction). The missing ingredient which needs to be established is: Every definable function $f : M^2 \to M$ is continuous outside a sparse subset of M^2. □

Given a definable set X, *a definably connected component of* X is a maximal definably connected subset of X (A-priori, such components might not be definable).

Corollary 3.22. *Every definable $X \subseteq M^n$ has finitely many definably connected components, each definable itself.*

Proof. We write $X = \bigcup_i C_i$ using the cell decomposition. Since C_i is definably connected, each definably connected component of X either contains the cell or disjoint from it. Hence, each component is a union of some of the cells in the decomposition. □

Corollary 3.23. *Let $\{X_a : a \in M^k\}$ be a uniformly definable family of subsets of M^n. Then the set $\{a \in M^k : X_a$ is definably connected $\}$ is definable.*

Moreover, there is a bound on the number of definably connected components of each X_a, as a varies in M^k.

Proof. Consider the set $X \subseteq M^{k+n}$ which defines the family, and take a partition of X into cells $C_1, \ldots, C_r$. Notice that for every $a \in M^k$, the fiber $(C_i)_a$ is a cell in M^n. Now use Problem 3.20(2) above. □

Corollary 3.24. *Let $f : U \to M$ be a definable function on a definable open set $U \subseteq M^n$. Then there are finitely many k-cells $C_1, \ldots, C_r \subseteq U$, with $k \leq n-1$, such that f is continuous on $U \setminus \bigcup_{i=1}^r C_i$.*

Proof. We apply Cell Decomposition to the graph of f. □

Problems 3.25. *(1) Given a definable $X \subseteq M^{n+1}$ there exists $K \in \mathbb{N}$ such that for every $a \in M^n$, if X_a is finite then $|X_a| \leq k$. Moreover, the number K is not larger than d where d is the number of cells in a cell decomposition of X.*
(2) Prove the "moreover" clause in Corollary 3.23.
(3) Prove that the notion of "definable connectedness" does not depend on how rich the langauge is: Assume that $\mathcal{N}$ is an o-minimal expansion of $\mathcal{M}$; show that a $\mathcal{M}$-definable subset of M^n is definably connected in the sense of $\mathcal{M}$ iff it is definably connected in the sense of $\mathcal{N}$ (hint: prove it first for cells).
(4) If $\mathcal{M}$ expands the ordered real numbers then a definable $X \subseteq \mathbb{R}^k$ is connected iff it is definably connected (prove it first for cells).

Finally, using our earlier observations we can conclude:

Theorem 3.26. *If $\mathcal{M}$ is o-minimal and $\mathcal{N} \equiv \mathcal{M}$ then $\mathcal{N}$ is also o-minimal.*

4. Additional features of o-minimality

4.1. **Dimension.** Recall that for $A \subseteq M$ and $b \in M$, we say that b is in the *algebraic closure of* A ($b \in acl(A)$) if b belongs to an A-definable finite set. We say that b is in the *definable closure of* A if the singleton $\{b\}$ is A-definable.

Notice that in any linearly ordered structure, $b \in acl(A)$ iff $b \in dcl(A)$ (why?).

In order to develop dimension theory using $acl()$ one needs to prove the Exchange Principle:

Theorem 4.1. *If $\mathcal{M}$ is o-minimal, $A \subseteq M$, $b, c \in M$ and $c \in acl(Ab) \setminus acl(A)$ then $b \in acl(Ac)$.*

Proof. Since $c \in dcl(Ab)$ there exists a formula $\phi(x, y)$, with implicit parameters from A, such that c is the unique solution of $\phi(b, y)$. The point b must be an interior point of the set $\{x \in M : \exists! y \phi(x, y)\}$, for otherwise b is in $acl(A)$, in which case $c \in acl(A)$, contradiction. It follows that there exists an A-definable open interval I containing b and an A-definable function $f : I \to M$ such that $f(b) = c$.

We claim that f must be strictly monotone locally near b. Indeed, if not, then either b belongs to an A-definable finite set (the endpoints of the intervals given by MCT), or f is locally constant near b. In in the first case, we have $b \in acl(A)$ which we already saw to be impossible. In the second case, there is an A-definable interval containing b on which f takes the value c. In particular, $c \in dclA()$, contradiction.

Therefore, again by MCT, there exists an A-definable interval $I \ni b$ on which f is strictly monotone. The function $g = (f|I)^{-1}$ is then A-definable and sends c to b, thus $c \in dcl(Ab)$. □

The Exchange Principle allows us to define a model theoretic notion of dimension for every set $B \subseteq M$:

Given $A, B \subseteq M$ (not assumed to be definable), the dimension of B over A, $\dim(A/B)$ is the cardinality of a maximal acl-independent over A subset of B. The Exchange principle guarantees, just like for a basis for vector spaces, and transcendence basis for fields, that this cardinality is independent of the maximal set we choose.

4.2. **Geometric structures.** A κ-saturated structure $\mathcal{N}$ is called *a geometric structure* if the algebraic closure satisfies the Exchange Principle and for every uniformly definable family $\{X_a : a \in N^k\}$ of subsets of N^n there is a number $K \in \mathbb{N}$ such that every finite X_a in the family has at most K elements.

Examples for such structures are strongly minimal, o-minimal structures and the p-adic numbers.

We assume now that $\mathcal{N}$ is a κ-saturated geometric structure and define the notion of $\dim(B/A)$ for any small (i.e. of cardinality less than κ) sets $A, B \subseteq N$, as above. The following properties can be established (all subsets $A, B, C \subseteq N$ are assumed to be small):

(1) Assume that $B \subseteq acl(A, C)$. Then $\dim(B/A) \leq \dim(C/A)$. In particular, if B and C are inter-algebraic over A then $\dim(B/A) = \dim(C/A)$.

(2) Let $a_1, \ldots, a_n \in N$, $A \subseteq N$. Then: $\dim(a_1, \ldots, a_n/A) \leq k$ if and only if there exists an A-definable set $X \subseteq N^n$ containing $\bar{a} = (a_1, \ldots, a_n)$ and a projection π on some k of the coordinates such $\pi|X$ is finite-to-one.

Given $X \subseteq M^n$ A-definable, we define

$$\dim(X) = \max\{\dim(\bar{a}/A) : \bar{a} \in X\}.$$

(3) The definition of $\dim(X)$ does not depend on A. Namely, if X is definable over A and over B then

$$\max\{\dim(\bar{a}/B) : \bar{a} \in X\} = \max\{\dim(\bar{a}/A) : \bar{a} \in X\}.$$

(hint: One first needs to establish that for every $A \subseteq N$, $\max\{\dim(\bar{a}/A) : \bar{a} \in N^k\} = k$.)
(4) If $f : X \to Y$ is an definable surjection then $\dim(Y) \leq \dim(X)$. In particular, the notion of dimension is preserved under definable bijections. (Here we use (1) above).
(5) If $\{X_a : a \in N^k\}$ is a definable family of subsets of N^n then for every d, the set $\{a \in N^k : \dim(X_a) = d\}$ is definable. More precisely, a definable set $X \subseteq N^n$ has dimension $\geq d$ if and only if for some projection π of N^n onto d of the coordinates $\{i_1, \ldots, i_d\}$, we have

$$\exists^\infty x_{i_1} \exists^\infty x_{i_2} \cdots \exists^\infty x_{i_d} (x_{i_1}, \ldots, x_{i_d}) \in \pi(X).$$

(The fact that this condition is definable follows the second property of geometric structures).
Remark Even though the definition of dimension requires a saturated structure, using Property (5), one can define dimension of sets in any elementarily equivalent structure.

(6) Assume that $X \subseteq N^n$ is definable. Then $\dim(X) \geq k$ if and only if there exists a projection π on k of the coordinates such that $\dim(pi(X)) = k$.
(7) The dimension formula: For every $\bar{a}, \bar{b}$ tuples of elements from M and $A \subseteq M$, we have:

$$\dim(\bar{a}\bar{b}/A) = \dim(\bar{a}/A\bar{b}) + \dim(\bar{b}/A).$$

The following notion is essential to the study of geometric structures, whether stable or not:

Generic points We still assume that N is an ω-saturated geometric structure.

Definition 4.2. Assume that $X \subseteq N^k$ is A-definable, for a finite A. A point $a \in X$ is called *generic in X over A* if $\dim(a/A) = \dim(X)$.

Notice that by definition of a geometric structure, every definable set contains a generic point.
(8) If X is A-definable, and a is generic in X over A then for every A-definable Y containing a we have $\dim(X \cap Y) = \dim X$.

Problems 4.3. *Prove Properties (1)-(9) of geometric structures (if one prefers, this can be first proved for o-minimal structures using what has been established thus far).*

4.3. **O-minimality and dimension.** In the o-minimal context, the notion of dimension defined above has a natural topological characterization.

Here are some basic observations, mostly based on the definition of dimension and on the its characterization given in Section 4.2 (5).
(1) Every k-cell in M^n has dimension k.

(2) If $X \subseteq M^n$ is a definable set and $X = \bigcup_i C_i$ is a cell-decomposition of X then $\dim(X) = \max\{k : C_i \text{is a } k\text{-cell}\}$.

(3) If $X \subseteq M^n$ is a definable set then $\dim(X) \geq k$ if and only if there is some projection π of X onto k of the coordinates such that $\pi(X)$ contains an open subset of M^k.

The following result follows from the cell decomposition theorem and is fundamental to the development of dimension theory in the o-minimal context.

Theorem 4.4. *For every definable $X \subseteq M^n$, we have $\dim(Cl(X) \setminus X) < \dim X$.*

The following corollary of the above replaces the uniqueness of generic types in ω-stable theories (for sets of Morley degree 1).

(4) Let $X \subseteq M^n$ be A-definable and a generic in X over A. If $a \in Y \subseteq M^n$ another A-definable set then there exists an open definable $U \subseteq M^n$ such that $U \cap X = U \cap Y$. In particular, if a is generic in M^n over A then it belongs to the interior of every A-definable set containing a.

(5) If a is generic over A in an A-definable set X then for every open set U containing a, we have $\dim(U \cap X) = \dim(X)$ (this can be first proved for A-definable U, and then one can show that the assumption that U is A-definable can be omitted).

Problems 4.5. *Prove properties (1)-(5).*

4.4. **Definable choice, EI and Curve Selection.** It is common in the study of o-minimal structures to add this (seemingly artificial) assumption:

Assume that $\langle M, <, +, \ldots \rangle$ is an o-minimal expansion of an ordered divisible abelian group (actually, it is sufficient to assume the existence of group structure on M; the rest follows). The first reason for this addition is the following elementary observation:

If $\mathcal{M}$ expands an ordered group as above then every $a \in M^n$ has a 1-parameter definable family of neighborhoods $\{N_t : t > 0\}$, where

$$N_t = \{(x_1, \ldots, x_n) : \forall 1 \leq i \leq n \ |x_i - a_i| < t\}.$$

(where $|x - y|$ is defined in an obvious way).

The second observation is that every definable family of nonempty sets has a choice function which is definable as well:

Theorem 4.6. Definable Choice *Let $\{X_a : a \in M^k\}$ be a definable family of nonempty subsets of M^n. Then there is a definable function $f : M^k \to M^n$ such that $f(a) \in X_a$ for every $a \in M^k$. Furthermore, if $X_a = X_b$ then $f(a) = f(b)$.*

Proof. Fix an element $1 > 0$ in M and use induction on n:

$n = 1$: If $X_a \subseteq M$ then let I be the leftmost interval of X_a. If $|I| = 1$, we let $f(a)$ be this point. If $I = M$ let $f(a) = 0$; if $Int(I) =]c, \infty[$ let $f(a) = c + 1$. If $Int(I) =]-\infty, d[$ let $f(a) = d - 1$; if $Int(I) =]c, d[$ let $f(a) = \frac{c+d}{2}$. It is easy to see that this definition is uniform.

For $n + 1$, we first obtain, by induction, a definable $g : M^k \to M^n$ such for every $a \in M^k$, $g(a)$ is in the projection of X_a on M^n. We now consider the family of subsets of M: $\{(X_a)_{g(a)} : a \in M^k\}$. Using the case $n = 1$, we have a definable function $f : M^k \to M$ such that for every $a \in M^k$, the element $(g(a), f(a)) \in X_a$, as required. □

An immediate corollary of the above is Elimination of imaginaries. I.e, in o-minimal expansions of groups, there is no need to develop a separate theory for T^{eq} since every definable equivalence relation has a definable set of representatives for the quotient structure.

The following corollary (which can also be proved without definable choice) adds another feature to o-minimality which it shares with strongly minimal structures but not with other geometric structures such as the p-adic numbers: It is sometimes called the *E-property.*

Corollary 4.7. *Let E be a definable equivalence relation on a definable set X. Then there are at most finitely many E-classes whose dimension equals to* $\dim(X)$.

Proof. We may assume that $\mathcal{M}$ is $|T|^+$-saturated (WHY?). Let $n = \dim(X)$. Since the dimension of a definable set is a definable property, the set $Y = \{x \in X : \dim(x/E) = n\}$ is definable.

By Definable Choice, there exists a definable set of representatives $Y_0 \subseteq Y$ for E-classes in Y. Assume that X, Y and Y_0 are all A-definable. Let y be a generic element in Y_0 over A and let x be a generic element over Ay in the E-class of y. By the Dimension Formula,

$$\dim(x, y/A) = \dim(x/Ay) + \dim(y/A) = \dim(y/Ax) + \dim(x/A).$$

By our assumptions on the E-classes in Y, we have $n = \dim(x/Ay) \le \dim(x/A) \le \dim(X) = n$, hence also $\dim(x/A) = n$. Because Y_0 contains a single element of every E-class, $\dim(y/Ax) = 0$. It follows from the formula that $\dim(Y_0) = 0$, hence there are finitely many equivalence classes. □

Finally, the last important corollary of Definable Choice is:

Theorem 4.8. Curve Selection *Given a definable $X \subseteq M^n$ and given $a \in Cl(X)$, there exists a definable function $\sigma :]0, \epsilon[\to X$ such that $\lim_{t\to 0} \sigma(t) = a$.*

Proof. Consider a 1-parameter family of neighborhoods $\{N_t : t > 0\}$ of a. By definable choice, there exists a definable function $\sigma : M^{>0} \to X$ such that for every $t > 0$, we have $\sigma(t) \in X \cap N_t$. The restriction of σ to any subinterval $]0, \epsilon[$ will give the desired function. □

Problems 4.9. *What are other examples of geometric structures which have definable choice?*

What can be said about general (saturated) linearly ordered structure with definable choice and Curve Selection?

4.5. **The Non Independence Property.** One of the most important dividing line between theories, introduced by Shelah, is the Independence Property versus the Non Independence Property (known as NIP).

Definition 4.10. We fix a sufficiently saturated model of a theory T. A formula $\phi(\bar{x}, \bar{y})$ *has the Independence property* if for every $n \in \mathbb{N}$ there exist tuples $\bar{a}_1, \ldots, \bar{a}_n$, $|\bar{a}_i| = |\bar{y}|$, such that every boolean combination of $\{\phi(\bar{x}, \bar{a}_i) : i = 1, \ldots, n\}$ is consistent (where we don't allow a formula $\phi(\bar{x}, \bar{a}_i)$ to appear twice in this combination).

T is said to have the Independence Property if some formula has it. T is said to have NIP if no formula has the independence property.

A fundamental theorem of Shelah says that every unstable theory has either the strict order property or the Independence Property (or both). Another theorem of his states: *If T has the independence property then some formula $\phi(x, \bar{y})$ with $|x| = 1$ has the independence property.*

Theorem 4.11. *If T is o-minimal then no formula has the independence property. Namely, T has NIP.*

Proof. By Shelah's theorem it is sufficient to prove that for every formula $\phi(x, \bar{y})$ there exists an n such that for every $\bar{a}_1, \ldots, \bar{a}_n$ some boolean combination of the $\phi(x, \bar{a}_i)$'s (with non-repeating occurrences) is inconsistent. It is clearly sufficient to consider conjunctions of $\phi(x, \bar{a}_i)$ and their negations.

The proof uses induction on the maximal number of definably connected components in $\phi(M, \bar{a})$ (which exists by o-minimality), as $\bar{a}$ varies:

Basic case: For every $\bar{a} \in M^k$, $\phi(M, \bar{a} = \{b \in M : \phi(, \bar{a})\}$ has at most one interval (of any kind)

We take $n = 3$ and note that given any 3 intervals, it must be the case that the intersection of some pair of the intervals is either contained in the third interval or disjoint from it. In both cases, we obtain an inconsistent boolean combination ($(I_1 \cap I_2) \setminus I_3$ or $I_1 \cap I_2 \cap I_3$).

Problems 4.12. The induction step: *Assume that for every $\phi(x, \bar{y})$ with at most k-intervals in every $\phi(M, \bar{a})$ the number $n = n(k)$ witnesses that $\phi(x, \bar{y})$ is not independent. Show that $n(k + 1) = 3n(k)$ works for any $\phi(x, \bar{y})$ with at most $k + 1$-intervals in every $\phi(M, \bar{a})$.*

□

5. Definable groups

We still assume that $\mathcal{M}$ is an o-minimal, densely ordered structure.

Definition 5.1. A definable group is a definable subset G of M^n together with a definable binary function from $G \times G$ into G which makes G into a group.

Here are some examples of groups that are definable in o-minimal structures:

- *One dimensional groups*:
  Divisible, torsion-free abelian groups. (These can be equipped with an ordering $<$ such that $\langle G, <, + \rangle$ is o-minimal);
  The circle group (definable in the real field);
  The group $\langle [0, a), +(mod\ a) \rangle$ (definable in any ordered divisible abelian group, for $a > 0$).
- $Gl(n, \mathbb{R})$ and any semi-algebraic subgroup of $Gl(n, \mathbb{R})$.
- $Gl(n, \mathbb{C})$ and any complex algebraic subgroup of it (these can be viewed as semialgrbaic subgroups of $Gl(2n, \mathbb{R})$).
- The R-points of an R-algebraic group, for R an arbitrary real closed field (these are obviously definable in the field $\langle R, <, +, \cdot \rangle$).
- Certain solvable linear groups in $Gl(n, \mathbb{R})$ which are definable only in $\langle \mathbb{R}, <, +, \cdot, e^x \rangle$ (and not group-isomorphic to any semialgebraic group).

As we will below, the group $\langle \mathbb{Z}, + \rangle$ cannot be definable in any o-minimal structure.

The fundamental theorem on definable groups in o-minimal structures says that every such groups admits a topology t whose basis is definable, such that G is a topological group with respect to this topology:

Theorem 5.2. *Let $\langle G, \cdot \rangle$ be a definable group in an o-minimal structure, $G \subseteq M^k$ and* $\dim(G) = n$. *Then:*

(1) *Given any generic g in G and a definable basis $\{U_s : s \in S\}$ for the neighborhoods of g in M^k, the family $\{hU_s : h \in G\}$ forms a basis for a topology t on G such that $\langle G, \cdot, t \rangle$ is a topological group (i.e. the group operation and the inverse function are continuous).*

(2) *Moreover, there exist a finite cover of definable and large* t-open sets $G = \bigcup_{i=1}^{r} U_i$, and for each $i = 1, \ldots, r$ a definable homeomorphism ϕ_i from U_i (with its t-topology) to an open $V_i \subseteq M^n$. The triple $\{\langle U_i, V_i, \phi_i \rangle : i = 1, \ldots, r\}$ is called an atlas for G.*

$\star$ By *a large subset of G* we mean a definable $X \subseteq G$ such that $\dim(G \setminus X) < \dim(G)$.

Remarks (1) Notice that the t-topology defined on G may not agree everywhere with the topology induced on G by M^k. For example, in the group $\langle [0,1), (+mod1) \rangle \subseteq \mathbb{R}$, a neighborhood of 0 in the t-topology will be of the form $[0, \epsilon) \cup (1 - \delta, 1)$.

(2) The atlas on G can be considered as a manifold structure on G with respect to the "Euclidean" topology on M^n. Indeed, the above theorem is usually formulated without Clause (1), and with an additional requirement, that the transition maps $\phi_i \phi_j^{-1}$ are continuous in the M^n-topology. One can then read the t-topology from the manifold structure: $X \subseteq G$ is t-open if and only if $\phi_i(X)$ is open in M^n for every i.

Problems 5.3. *1. Why is $\langle [0,1), +(mod1) \rangle$ not a topological group with respect to the $\mathbb{R}$-topology? Find an atlas on this group as in (2) above.*

2. Find a definable basis for a topology on $\langle \mathbb{R}, + \rangle$, such that $+$ is continuous but the group-inverse is not continuous.

Theorem 5.2 allows us to translate the following properties, via the atlas, from the M^n-topology to the t-topology.

Corollary 5.4. *Let G be a definable group, and let t be the topology above. If $X \subseteq G$ is a definable set then:*

(i) X has finite number of definably t-connected components.

(ii) $\dim(Cl_t(X) \setminus X) < \dim(X)$ *(where $Cl_t(X)$ is the closure of X in the t-topology).*

Problems 5.5. *Prove the last corollary.*

Combining the topological facts above with the group structure we obtain the following group theoretic properties:

Theorem 5.6. *Let G be a definable, n-dimensional group. Then:*

(1) *Every definable subgroup $H < G$ is t-closed.*

(2) *The definably t-connected component of G which contains the identity is a normal subgroup of G of finite index; we denote it by G^0. The cosets of G^0 are the definably t-connected components of G. In particular, all t-components of G have the same dimension as G.*

(3) *The following are equivalent for any definable subgroup* $H \subseteq G$*: (i)* H *has finite index in* G*. (ii)* $G^0 \subseteq H$*. (iii)* $\dim H = \dim G$*.*

(4) *(***DCC***- The descending Chain Condition for definable subgroups) There is no infinite descending chain of definable subgroups of* G*.*

Proof. (1) If H is not t-closed then the t-frontier of H, $Fr(H) = Cl(H) \setminus H$, is invariant under multiplication by $h \in H$ (because such a multiplication is a t-homeomorphism of G that fixes H set-wise). It follows that given $g \in Fr(H)$, we have $Hg \subseteq Fr(H)$. But then $\dim(Fr(H)) \geq \dim(Hg) = \dim(H)$ (because the dimension is preserved under definable bijections), contradicting the fact that $\dim Fr(H) < \dim H$.

(2) By continuity, the map $(g,h) \mapsto g \cdot h^{-1}$ sends $G^0 \times G^0$ to a definably t-connected set which contains G^0, hence equals to G^0. For the same reason, each definably t-connected component of G is a coset of G^0. Why is G^0 normal in G?

(3) Assume that a definable $H \subseteq G$ has finite index in G. By (1), the group H (and therefore each coset of H) is t-closed. But then the complement of H is a finite union of closed sets hence H is also open in G. Since $e \in H$ and G^0 is definably connected, we must have $G^0 \subseteq H$.

If $G^0 \subseteq H$ then clearly, by (2), $\dim H = \dim G$. Assume that $H \subseteq G$ is a definable subgroup and $\dim(H) = \dim(G)$. Then every coset of H has the same dimension. By the so-called E-property, proved above, there can be only a finitely many such cosets therefore H has finite index.

(4) Assume that $G \supset H_1 \supset H_2 \supset \cdots H_n \supset \cdots$ is a descending chain of definable subgroups. Because $\dim H_i$ is finite, the dimension of the groups must eventually stabilize at some H_k. By (3), for every $m \geq k$ the group H_m has finite index in H_k hence, again by (3), H_m is a union of some of the cosets of H_k^0. If $[H_k : H_k^0] = d$ then there can be at most d-many subgroups of H_k of the same dimension so the chain of groups must eventually stabilize. □

As always in the model theory of groups, the DCC property is very powerful. It implies in particular that the centralizer of ANY subset of G is definable (see problem below).

Problems 5.7. *(1) Show that the group* $\langle \mathbb{Z}, + \rangle$ *is not definably isomorphic to any group definable in an o-minimal structure.*

(2) Show that the group

$$\left\{ \begin{pmatrix} z & 0 \\ 0 & e^z \end{pmatrix} : z \in \mathbb{C} \right\}.$$

(identified with a subset of $Gl(4, \mathbb{R})$ *in the obvious way) is not definable in any o-minimal expansion of the real field, but is group-isomorphic to a definable group in an o-minimal structure.*

(3) Let $f : G \to H$ *be a definable group homomorphism. Show that* f *is necessarily continuous with respect to the* t*-topology on* G *and* H*.*

(4) Let G *be a definable group in an o-minimal structure,* A *an arbitrary subset of* G*. Show that* $C_G(A) = \{g \in G : \forall h \in A gh = hg\}$ *is definable.*

(5) Show that if G *contains an infinite abelian subgroup* A *(not necessarily definable) then it contains a definable infinite subgroup. Conclude that if* G *contains an element of infinite order then* G *contains a definable infinite abelian subgroup.*

(6) **One-dimensional groups***: Let* G *be a 1-dimensional definably connected definable group in an o-minimal structure. Show that* G *is abelian and has no*

infinite definable subgroups. (Hint: note first that G has no infinite definable subgroup).

5.1. **Torsion elements in definable groups.** Using the notion of an Euler characteristic of a definable set (upon which we did not touch here), one can prove a variety of results on torsion points in definable groups. Here are two of the most important ones.

(1) If G is infinite then there exists an element $g \in G$ of infinite order.
(2) If G is abelian and $E(G) = 0$ then for every $p \geq 0$ there exists an element in G of order p.

5.2. **Definable compactness and groups.** As was pointed out in the earlier problems, the topology of an o-minimal is in general not locally connected or locally compact. As we saw, by replacing "connected" with "definably connected" and by restricting ourselves to definable sets, the topology becomes well-behaved (or "tame") from this point of view.

It turns out that by replacing limits of converging sequence with limits of definable curves one obtain a definable replacement to compactness as well.

Definition 5.8. A definable group G (or a subset X of M^k) is called *definably compact* if for every definable map continuous function $f(x)$ from an interval $]a, b[\subseteq M$ into G (or into X), the t-limits (or the M^k-limits) of $f(x)$, as x tends to a and to b, exist in G (or in X).

The following is easy to verify using definable choice, but it turns out to be true even without assuming it (see also Problem 3.5(2)):

Lemma 5.9. *A definable $X \subseteq M^k$ is definably compact iff it is closed and bounded.*

In particular, if $\mathcal{M}$ expands the ordered real numbers then the notions of compactness and definable compactness coincide for definable sets.

Problems 5.10. *(1) Show that the following definable variant of compactness fails in a sufficiently saturated o-minimal structure: Every uniformly definable open cover of $[0, 1]$ has a finite subcover.*

(2) Let $f : X \to Y$ be a definable continuous surjection. Show that if X is definably compact then so is Y (assume that $\mathcal{M}$ has definable choice).

Just like in the theory of Lie groups, the dichotomy between definably compact groups and those which are not definably compact is crucial. There are still very interesting open problems on both sides of this dividing line, but most of the difficulties seem to involve definably compact groups.

Here is one reason why this is so.

Theorem 5.11. *If G is a definable group that is not definably compact then it has a definable 1-dimensional, torsion-free group.*

Proof. **The idea:** For simplification we assume definable choice. Since G is not definably compact, there is a definable continuous $\gamma : (a, b) \to G$ with no limit at b. To improve intuition we will use 0∞ instead of b.

The idea is to define H as the set of all limit points of $\gamma(t)\gamma(t)^{-1}$, as t tends to ∞. More precisely,

$$H = \{g \in G : \forall \text{ open } V \ni g \, \forall d > a \, \exists t_1, t_2 > d \; (\gamma(t_1)\gamma(t_2)^{-1} \in V)\}.$$

Why is H a subgroup of G?

By definable choice, h is in H iff there exists a definable function $\sigma_h : (a, \infty) \to (a, \infty)$ tending to ∞ such that $\gamma(x)\gamma(\sigma_h(x))^{-1}$ tends to h as x tends to infinity.

Now, given $g, h \in H$, the function $\sigma_h \circ \sigma_g$ witnesses the fact that gh is in H, and the compositional inverse of σ_g witnesses the fact that $g^{-1} \in G$.

The fact that $\dim(H) \leq 1$ is obtained by viewing H as a subset of the "frontier" of the two-dimensional set $\{\gamma(x)\gamma(y)^{-1} : x, y > a\}$ (one can make this precise by identifying ∞ with 0).

To see that H is infinite, one shows that it intersects the boundary of every sufficiently small ball around $e \in G$ (or a rectangular box): Given such a ball B, notice that for every $x > a$, $\gamma(x)\gamma(x)^{-1} = e$ while for there exists $y > x$ such that $\gamma(x)\gamma(y)^{-1}$ is outside the ball (here we use the fact that γ has no limit in G). It follows that for some $y(x) > x$, the group element $\gamma(x)\gamma(y(x))^{-1}$ is on the boundary of B. Taking the limit of these, as x tends to infinity we obtain an element of H on the boundary of B.

Eventually, one can show that H is torsion-free. □

5.3. **Rings and fields in o-minimal structures.**

Theorem 5.12. *If K is a definable field then there is a definable real closed field R (of o-minimal dimension 1) such that K is definably isomorphic to R or to the algebraic closure of R (and then* $\dim(K) = 2$*).*

Proof. It follows from the fundamental theorem on groups that there is a definable topology making all field operations continuous.

First note that $\langle K, + \rangle$ is not definably compact: Indeed, Take $\gamma(x)$ a curve tending to 0. Then $1/\gamma(x)$ cannot have a limit in K as $\gamma(x)$ tends to 0 (WHY?).

By Theorem 5.11, K has a definable, 1-dimensional, definably connected, torsion-free subgroup H. Multiplying H by $1/h$, for some $h \in H$, we may assume that $1 \in H$. Given $a \in H$, the set of all b such that $ba \in H$ is a subgroup of H containing all integers hence it is infinite, and therefore equals to H (H is 1-dimensional and definably connected hence has no definable subgroups). Similarly, the set aH must equal H for every $0 \neq a \in H$, so all nonzero elements are invertible in H. Hence, H is a 1-dimensional subfield of K, call it R.

R is a real closed field

We will use: R is a real closed field iff it is orderable, every positive element has a square root and every polynomial of odd degree has a root.

Let X and Y be the components of $R \setminus \{0\}$, and assume that $1 \in X$. We show that X is closed under addition, multiplication and the field-inverse, and moreover $-X = Y$.

The map $x \mapsto 1/x$ sends X into X (the image is a definably connected set containing 1) but because it is an involution its image is the whole of X. Because $X \cdot X$ is definably connected and contains 1 it must be contained in X. Similarly, $X + X$ contains elements near 1 (because $1 + \alpha$ is close to 1 for $\alpha \in X$ near 0) hence $X + X \subseteq X$. Finally, because X is closed under $+$, we must have $-X = Y$.

It follows that X is a positive cone in R. I.e. the set of positive elements in an ordered field (where $x < y$ iff $y - x \in X$).

Because $\langle X, \cdot \rangle$ is a 1-dimensional definably connected group, it is divisible so every positive element of R has a square root. The fact that every polynomial of odd degree has a root follows from the intermediate value theorem which holds in R because of the definable connectedness. We thus showed that R is real closed.

K is an algebraic extension of R

Indeed, if $\alpha \in K$ is transcendental over R then for every

n, we have $R + \alpha R + \cdots + \alpha^n R \neq K$. However, the dimension of the set on the left is n so the dimension of K will not be finite, which is impossible.

Because real closed fields have only one algebraic extension, K must equal $R(\sqrt{-1})$ and its dimension is 2. □

Problems 5.13. *Show that the last theorem still holds if we only assume that K is an integral domain.*

6. On R-differentiability

In most interesting cases one assumes that our o-minimal $\mathcal{M}$ is already an expansion of a real closed field whose underling ordering is that of M. This will be our assumption from now on.

We now can define a notion of derivative with respect to to R:

Definition 6.1. A definable function $f :]a,b[\to R$ is R-differentiable at $x_0 \in]a,b[$, with derivative $d \in R$, if

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = d$$

(where the limit is taken in R).

f is said to be a C^n-function on $]a,b[$ if its n-th derivative exists and continuous there.

Notice that if f is R-differentiable on $]a,b[$ then its derivative is definable as well. (But it is not true in general that the primitive of a definable function is also definable!)

The analogous result to MCT is:

Theorem 6.2. *Let $f :]a,b[\to R$ be a definable function. Then for every $n \in \mathbb{N}$ there exist $a = a_0 < a_1 < \cdots < a_r = b$ such that f is C^n on each $]a_i, a_{i+1}[$.*

Proof. For $x \in]a,b[$ we let

$$f'(x^+) = \lim_{h \downarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \; ; \; f'(x^-) = \lim_{h \uparrow 0} \frac{f(x_0 + h) - f(x_0)}{h} = d.$$

As we already saw in Problem 3.5 (2), these limits must exist in $R \cup \{\pm\infty\}$. Moreover, $f'(x)$ exist if and only if $f'(x^+) = f'(x^-) \in R$.

Using MCT for these two definable functions we may assume that both are continuous (possibly taking values in $\pm\infty$).

Claim (1) If $f'(x^+) > c \in R$ for all x in some subinterval I then there exists a sub-interval J such that for every $y > x$ in J, $f(y) - f(x) > c(y - x)$.

(2) If $f'(x^-) < c \in R$ for all x in some subinterval I then there exists a sub-interval J such that for every $y < x$ in J, $f(y) - f(x) > c(y - x)$.

Indeed, the assumption in (1) implies that for every x there exists $x' > x$ such that for every $y \in]x, x'[$ we have $f(y) - f(x) > c(y - x)$. Let $\sigma(x)$ be the supremum of all these x'. By Lemma 3.2 there exists $d \in I$ and an interval $J < d$ such that for all $x \in J$ we have $\sigma(x) > d$. It implies that for every $x < y$ in J we have $f(y) - f(x) > c(y - x)$.

The proof of (2) is the same (noting that switch of the sign as we multiply by $y - x$ for $y < x$). End of Claim.

Clearly, (1) and (2) cannot take place on the same interval therefore, $f'(x^+)$ and $f'(x^-)$ can differ only on finitely many points in I.

Claim $f'(x^+)$ (and hence also $f'(x^-)$) can take the value $\pm\infty$ at most finitely often.

Indeed, assume toward contradiction that $f'(x^+) = +\infty$ on an open interval $J \subseteq I$. Given $x_1 < x_2 \in J$, let $\ell(x)$ be the affine function connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Because $f'(x_1^+) = +\infty$, we have $f(x) > \ell(x)$ for $x > x_1$ sufficiently close to x_1. Because $f'(x_2^-) = +\infty$, we have $\ell(x) < f(x)$ for $x < x_2$ and sufficiently close to it. By o-minimality, there exists $x_3 \in]x_1, x_2[$ such that $\ell(x_3) = f(x_3)$ and $f(x) > \ell(x)$ for all $x < x_3$ and sufficiently close to it, contradicting the fact that $f'(x_3^-) = -\infty$.

By removing finitely many points we have $f'(x^+) = f'(x^-) \in R$ hence f is R-differentiable. We now repeat this process for the definable function $f'(x)$ and so on. □

Problems 6.3. *(1) (Rolle's theorem) Assume that a definable $f : [a, b] \to R$ is continuous on $[a, b]$, R-differentiable on $]a, b[$ and $f(a) = f(b)$. Show that there exists $c \in]a, b[$ such that $f'(c) = 0$.*
(2) Prove the Mean Value Theorem for definable R-differentiable functions.

Definition 6.4. A definable map $f : U \subseteq R^n \to R^m$ is called a C^1-function if all the partial derivatives $\partial f_i / \partial x_j$, of the coordinate functions exist and continuous.

For such an f, we denote by $d_x f \in M_{m \times n}$ the matrix of partial derivatives.

As for continuity, one can prove that definable functions are "generically" C^n.:

Theorem 6.5. *For every definable $f : U \to R^m$ on an open set $U \subseteq R^n$, there exist finitely many definable open sets $U_1, \ldots, U_k \subseteq U$ such that f is a C^1-map on each U_i and $\dim(U \setminus \cup_{i=1}^r U_i) < n$.*

All the basic results on real differential maps, such as the implicit and inverse function theorems, can be proved in this context as well.

7. The analogue of complex differentiability

We still assume that $\mathcal{M}$ is an o-minimal expansion of a real closed field. In this case, the algebraic closure of R, call it K, is an algebraic extension of degree 2. By fixing $i = \sqrt{-1}$ we can identify K with R^2 via $a + bi \sim (a, b)$, just like $\mathbb{C}$ is identified with $\mathbb{R}^2$. The topology of R^2 makes K into a topological field.

Every definable function $f : K \to K$ is given, coordinate-wise, by definable functions from R^2 into R.

Fact 7.1. *The field operations on K are definable in $\langle R, +, \cdot \rangle$, when we identify the underlying K with R^2 as above. Moreover, they are C^∞-maps in the sense of R.*

Proof. Addition is just point-wise addition and multiplication is given by $(a, b) \cdot (c, d) = (ac - bd, bc + ad)$ (just check the real and imaginary parts of complex multiplication). Since the coordinate functions are R-polynomial maps they are C^∞. □

We can now imitate the usual definition of complex differentiability:

Definition 7.2. Let $U \subseteq K$ be an open set. A function $f : U \to K$ is K-holomorphic at $z_0 \in K$, with derivative d, if

$$\lim_{h \to 0} \frac{f(z - 0 + h) - f(z_0)}{h} = d,$$

where the field operations and limits are now taken in K.

Problems 7.3. *(i) Let* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a* 2×2 *matrix over* R. *Show that the map* $T : K \to K$ *induced by* A *is* K*-linear if and only if* $a = d$ *and* $b = -c$.

(i) Show that every polynomial over K *is* K*-holomorphic on the whole of* K.

(ii) Show that if f *is* K*-holomorphic on* $U \subseteq K$ *then its real and imaginary parts, call them* $u(x, y), v(x, y)$, *are* C^1*-maps from* R^2 *into* R *which satisfy the Cauchy-Riemann equation:*

$$\partial u/\partial x = \partial v/\partial y \ ; \ \partial u/\partial y = -\partial v/\partial x.$$

Hint: As in the complex case, this is done by choosing h, *in the definition of derivative, to be first real and then imaginary.*

The converse of the above is also true: If $f : U \to K$ is a definable C^1-map whose partial derivatives at every point satisfy the Cauchy-Riemann equation then f is K-holomorphic on U.

The above basic properties could be expected to hold in o-minimal structures, since their classical proofs do not use in any deep way the underlying properties of $\mathbb{R}$ and $\mathbb{C}$.

The rest of classical development of the theory of complex functions uses either convergent power series or integration (or both) none of which is available in the o-minimal setting. However, it is turns out that one can by-pass these tools and still develop the basic theory in this context:

Theorem 7.4. *Let* $f : U \to K$ *be a definable* K*-holomorphic function. Then:*

(1) $f'(z)$ *is* K*-holomorphic as well.*
(2) $f(z)$ *satisfies the maximum principle.*
(3) *If* f *is bounded then it is a constant function.*

The proofs of these results goes beyond the scope of this tutorial.

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