G-linear sets and torsion points in definably compact groups

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Abstract

Let G be a definably compact group in an o-minimal expansion of a real closed field. We prove that if $\dim(G \setminus X) < \dim G$ for some definable $X \subseteq G$ then X contains a torsion point of G. Along the way we develop a general theory for the so-called *G-linear sets*, and investigate definable sets which contain abstract subgroups of G.

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1 Introduction

We prove:

Theorem 1.1. *Let G be a definably compact group in an o-minimal expansion of a real closed field. If $X \subseteq G$ is a definable large set (i.e. $\dim(G \setminus X) < \dim G$) then X contains a torsion point of G.*

This is a weak approximation to the conjecture that every generic set in a definably compact group contains a torsion point, a conjecture which itself follows from the "compact domination conjecture" from [7].

The proof starts with the abelian case, where we first develop, in o-minimal expansions of an ordered group, a notion of a coset of a definable

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local subgroup (a G-linear set). We then use tools from algebraic topology, in expansions of real closed fields, to conclude the theorem.

We assume some familiarity with results on groups definable in o-minimal structures (see [8]).

While this paper was under review, the "compact domination conjecture" has been proved for abelian groups in [6], making use of some results of this paper.

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2 G-linear sets

In this section $\mathcal{M}$ will denote an o-minimal expansion of an ordered group. By definable we mean definable in $\mathcal{M}$. G will denote a definable abelian group and 0_G its identity element. All topological concepts are with respect to the group topology such G is equipped with. (We believe that most, if not all, of the work below can be developed in an arbitrary definable group, but for simplicity we limit ourselves to the commutative setting.)

Definition 2.1. 1) *Given two sets $X, Y \subseteq G$ and $a \in G$, we say that X* and Y have the same germ at a, *in notation $X =_a Y$, if there exists an open neighbourhood U of a such that $X \cap U = Y \cap U$. We say that* the germ of X at a is contained in the germ of Y at a, *in notation $X \subseteq_a Y$, if there exists an open $U \ni a$ such that $X \cap U \subseteq Y \cap U$.*

2) *Given $X \subseteq G$ and $g, h \in G$, we say that* the germ of X at g is G-equivalent to the germ of X at h *if $X - g =_{0_G} X - h$. (Note that if X is a definable set then we obtain in this way a definable equivalence relation on G.)*

Definition 2.2. *Let X be a definable subset of G.*

1) *X is called* G-linear *if for every $g, h \in X$ we have $X - g =_{0_G} X - h$. Given $g \in X$, we say that X* is locally G-linear at g *if there exists an open $U \ni g$ such that for every $h \in U \cap X$ we have $X - g =_{0_G} X - h$.*

Two G-linear sets $X, Y \subseteq G$ are called G-equivalent *if for every $g \in X$, $h \in Y$, we have $X - g =_{0_G} Y - h$.*

2) *Let X be a G-linear set. A definable $Y \subseteq G$ is called* a G-subset of X *if for every $h \in Y$ and $g \in X$, we have $Y - h \subseteq_{0_G} X - g$.*

Notice that a translate of a G-linear set (resp. a G-subset of X) is also a G-linear set (resp. a G-subset of X). Also, that any subset of a G-linear set is a G-subset of it, but a G-subset of X may not be a subset of X.

Example 2.3. *Let G be a definable abelian group and H a definable subgroup. If X is a definable relatively open subset of a coset of H then X is G-linear. If Y is any definable subset of a coset of H then Y is a G-linear subset of X.*

We observe without a proof (we will not be using it): If $\mathcal{M}$ expands a real closed field then Y is a G-subset of the G-linear set X if and only if at every smooth point $y \in Y$, the tangent space of Y at y is a subset of the tangent space of X at any of its points.

Lemma 2.4. *If X is definably connected and locally G-linear at every $g \in X$ then X is G-linear.*

Proof. Because X is locally G-linear at every point, for every $g \in X$, the set $\{h \in X : X - g =_{0_G} X - h\}$ is relatively open in X. But then it is also relatively closed (since its complement is a union of open sets), so by definable connectedness must be the whole of X, so X is G-linear. □

Notation. Given $X \subseteq G$, we denote by X_{lin} the set

$$X_{lin} = \{g \in X : \ X \text{ is locally } G\text{-linear at } g\},$$

and X_{lin}^{max} denotes the union of all components of X_{lin} of maximal dimension.

Notice that X_{lin} is a relatively open (possibly empty) subset of X. By the last lemma, every definably connected component of X_{lin} is a G-linear set, but X_{lin} itself might not be G-linear (*E.g.*, if $X = H_1 \cup H_2$ is the union of two definably connected subgroups of G, none containing the other, then $X_{lin} = X \setminus (H_1 \cap H_2)$ is not a G-linear set). It is easy to see that X_{lin} is itself G-linear if and only if all its definably connected components are G-equivalent.

Infinitesimals

It might be easier here to use the language of infinitesimals: We move to an $|M|^+$-saturated elementary extension $\mathcal{N}$ of $\mathcal{M}$. For $g \in G(\mathcal{M})$, we denote by ν_g the intersection of all $\mathcal{M}$-definable open neighbourhoods of g (with respect to the group topology) in the structure $\mathcal{N}$. For $X \subseteq G$, we write $\nu_g(X) = \nu_g \cap X$. Notice that for $X, Y \subseteq G$ definable and $g, h \in G$, $\nu_g(X) = \nu_g(Y)$ iff $X =_g Y$, and $\nu_g(X) - g = \nu_h(X) - h$ iff $X - g =_{0_G} X - h$.

We will be using the following simple observation: If X, Y are A-definable sets, and x is generic in X and in Y over A then $\nu_x(X) \subseteq Y$.

Lemma 2.5. *X is locally G-linear at $g \in X$ if and only if $\nu_g(X) - g$ is a subgroup of G. In particular, under these conditions, X is a submanifold of G near g and hence locally closed near g.*

Proof. We may assume $g = 0_G$. If $\nu_{0_G}(X)$ is a subgroup then for every $h \in \nu_{0_G}(X)$, we have $h + \nu_{0_G}(X) = \nu_{0_G}(X)$. It easily follows that X is locally G-linear at 0_G.

Conversely, assume that X is locally G-linear at 0_G. We pick $h \in X$ close to 0_G and generic in X over 0_G such that $X =_{0_G} X - h$. We take open $U \ni 0_G$, definable over parameters independent of $0_G, h$, such that $X \cap U = (X - h) \cap U$. Because of the genericity of h, there exists an open neighbourhood $V \ni h$, which we may assume is contained in $h + U$, such that for all $h' \in X \cap V$,

$$(X - h) \cap U = X \cap U = (X - h') \cap U.$$

By adding h to both sides of the last equation, we obtain: For all $h' \in X \cap V$, $X \cap (h + U) = (X - h' + h) \cap (h + U)$, hence (since $V \subseteq (h + U)$) also for all $h' \in X \cap V$,

$$X \cap V = (X - h' + h) \cap V.$$

In particular, if $h'', h' \in X$ are sufficiently close to h (so that $h'' - h' + h \in V$) then we have $(h'' - h') + h$ in X. It follows, that $\nu_h(X) - h$ is a subgroup of G. Because $\nu_{0_G}(X) = \nu_h(X) - h$, it also follows that $\nu_{0_G}(X)$ is a subgroup of G.

Finally, suppose X is locally G-linear at g. Then the germ of X at g is G-equivalent to the germ of X at some generic point near g. By homogenity of the subgroup $\nu_g(X) - g$ it follows that X is a submanifold near g. □

For a G-linear X, we denote by $\nu(X)$ the group $\nu_x(X) - x$, for some (all) $x \in X$. Notice that Y is a G-subset of X iff $\nu_h(Y) - h$ is a subset of the group $\nu(X)$, for every $h \in Y$. Note also that two G-linear sets X_1, X_2 are G-equivalent if and only if $\nu(X_1) = \nu(X_2)$.

Lemma 2.6. *Let $X, Y \subseteq G$ be definable sets of dimension d. If $\dim(X + Y) = d$ then X_{lin} is large in X, Y_{lin} is large in Y and X_{lin}^{max}, Y_{lin}^{max} are G-linear sets that are G-equivalent.*

Proof. We assume that X and Y are $\emptyset$-definable and write $Z = X + Y$. Take g generic in X over $\emptyset$ and h generic in Y over g. Then $g + h$ is generic in Z over each g and h and hence for every two $x, y \in \{g, h, g + h\}$ we have $\dim(x/y) = d$.

For every $\emptyset$-definable neighbourhood U of g, we have $(U \cap X) + h \subseteq Z$. Moreover $g + h$ is generic in $(U \cap X) + h$ over h and therefore $(U \cap X) + h$ contains $\nu_{g+h}(Z)$. It follows that $\nu_g(X) + h \supseteq \nu_{g+h}(Z)$. By symmetry we may conclude that $\nu_g(X) + h = \nu_{g+h}(Z)$, or equivalently, $\nu_g(X) - g = \nu_{g+h}(Z) - (g + h)$.

In the very same way we see that $\nu_h(Y) - h = \nu_{g+h}(Z) - (g + h)$. Hence, we showed that for *every* generic $g \in X$, $h \in Y$ which are independent from each other we have $\nu_g(X) - g = \nu_h(Y) - h$.

Fixing a generic $h \in Y$, this implies that $\nu_{g'}(X) - g'$ is constant as g' varies over all elements g' which are generic in X over h. In particular, X_{lin} contains all those elements g' and moreover all the components of X_{lin} of dimension d are G-equivalent. The same argument shows that all components of Y_{lin} of dimension d are G-equivalent to each other and that X_{lin}^{max} is G-equivalent to Y_{lin}^{max}. □

Corollary 2.7. *Let $X, Y, Z \subseteq G$ be definable sets of dimension d. Assume that for every g in a large set $X_0 \subseteq X$, we have* $\dim((g + Y) \cap Z) = d$. *Then X_{lin} is large in X.*

Proof. Without loss of generality, X, Y, Z, X_0 are $\emptyset$-definable, and hence every generic element $g \in X$ belongs to X_0. For every such $g \in X_0$ there exists $c \in (g + Y) \cap Z$ which is generic in Z over g. It follows that $h = c - g$ is in Y and, as before, the dimension of any two of $\{g, h, c\}$ is $2d$. We can now find an open neighbourhood V of h such that $g + (V \cap Y) \subseteq Z$. Because of the genericity of g there is a neighbourhood U of g such that $(U \cap X) + (V \cap Y) \subseteq Z$. In particular, $\dim((U \cap X) + (V \cap Y)) = d$. We can therefore apply Lemma 2.6 to $U \cap X$ and $V \cap Y$ and conclude that $(X \cap U)_{lin}$ is large in $X \cap U$. Because this is true for every generic g, the set X_{lin} is large in X. □

Note that we cannot conclude, under the assumption of the last corollary, that all components of X_{lin}^{max} are G-equivalent.

Definition 2.8. *If U is an open symmetric ($U = -U$) neighbourhood of* 0_G *and $Y \subseteq G$ a definable set, we say that $h_1, h_2 \in Y$ are* U-connected in Y *if $h_2 \in h_1 + U$ and there exists a definable path in Y connecting h_1 and h_2, which is contained in $h_1 + U$.*

Lemma 2.9. *Let X be G-linear and Y a G-subset of X. Take $g \in X$, and assume that U is an open symmetric neighbourhood of* 0 *such that* $(g+U) \cap X$ *is relatively closed in $g + U$. Then, for any $h_1, h_2 \in Y$ that are U-connected in Y, we have $g + (h_1 - h_2) \in X$.*

Proof. Because h_1, h_2 are U-connected in Y, there exists a definable path $\gamma : [a, b] \to Y$ such that $\gamma(a) = h_1, \gamma(b) = h_2$ and for every $t \in [a, b]$, $\gamma(t) - h_1 \in U$. Consider the set T of all $t \in [a, b]$ such that $g + (h_1 - \gamma(t')) \in X$, for all $t' \leq t$.

We claim that T is both open and closed (with respect to the order topology in M) in $[a, b]$. Indeed, because $(g + U) \cap X$ is relatively closed in $g + U$, the set T is closed in $[a, b]$.

To see that it is open, assume that $t_0 \in T$. In particular, $k = g + (h_1 - \gamma(t_0))$ is in X. For $t' \in [a, b]$ close to t_0 we have $(\gamma(t_0) - \gamma(t')) \in X - k$ (because $\nu_{\gamma(t_0)}(Y) - \gamma(t_0) \subseteq \nu_k(X) - k = \nu(X)$ is a subgroup of G), therefore $t' \in T$. It follows that T is open and closed in $[a, b]$, hence $T = [a, b]$ and therefore $g + (h_1 - h_2) \in X$. □

The following technical lemma ensures that we can extend every G-linear set and every G-subset of a G-linear set beyond its frontier (where the frontier of X is $Fr(X) = Cl(X) \setminus X$).

Lemma 2.10. *Assume that X is G-linear and Y is a G-subset of X. Then $Cl(Y)$ is also a G-subset of X.*

Proof. Fix h in $Cl(Y) \setminus Y$. We first prove that $\nu_h(Y) - h \subseteq \nu(X)$. Namely, we show: For all $h' \in Y$ sufficiently close to h and every $g \in X$ we have $g + h' - h \in X$.

If the above fails then we have a curve $\gamma : [a, b) \to Y$, with $\lim_{t \to b} \gamma(t) = h$ such that for all $t \in [a, b)$, $g + \gamma(t) - h \notin X$. Fix U an open symmetric neighbourhood of 0 such that $X \cap (g + U)$ is relatively closed in $g + U$ (by Lemma 2.5, X is locally closed). By choosing $\gamma(a)$ sufficiently close to h, we may assume that for every $t \in [a, b]$, we have $\gamma(t) \in \gamma(a) + U$ (so in particular, $h \in \gamma(a) + U$). It follows that for every $t \in (a, b)$, we have $\gamma(a)$ and $\gamma(t)$ are U-connected (as witnessed by γ) and therefore, by Lemma 2.9, $g + \gamma(a) - \gamma(t) \in (g + U) \cap X$. Because $X \cap (g + U)$ is closed in $g + U$, we may take t to be b and conclude that $g + \gamma(a) - \gamma(b) = g + \gamma(a) - h \in X$, contradicting our assumption.

We therefore showed that for all $h \in Cl(Y) \setminus Y$, $\nu_h(Y) - h \subseteq \nu(X)$. Because Y is a G-subset of X this is true for every $h \in Cl(Y)$. Assume now, without loss of generality, that $0_G \in X$. Given $h \in Cl(Y)$, we can find an open $U \ni 0_G$ such that (1) $X \cap U$ is relatively closed in U, and (2) $(Y - h) \cap U \subset X \cap U$. Taking closure of both sides in (2), we see that $Cl(Y) - h \subseteq X \cap U$. This implies that $Cl(Y)$ is a G-subset of X. □

Lemma 2.11. *Let $X \subseteq G$ be a definable set. Assume that*
(i) X_{lin} is large in X (i.e. $\dim(X \setminus X_{lin}) < \dim(X)$) and
(ii) for every $h \in X$ and $g \in X_{lin}$, $X - g \subseteq_{0_G} X - h$.
Then X is G-linear.

Proof. Assume that $\dim(X) = d$. First notice that by (ii), all the components of X_{lin} are G-equivalent to each other and therefore X_{lin} is itself a G-linear set of dimension d.

Because of (ii) the local dimension of X at every point is d and therefore, by (i), X_{lin} is dense in X. By Lemma 2.10, X is a G-subset of X_{lin}.

Given any $h \in X$, we have

$$\nu_h(X) - h \subseteq \nu(X_{lin}) \subseteq \nu_h(X) - h.$$

Indeed, the left inclusion follows from the fact that X is a G-subset of X_{lin} while the right one is by assumption (ii).

Therefore, for every $g, h \in X$, $\nu_g(X) - g = \nu(X_{lin}) = \nu_h(X) - h$. That is, X is G-linear. □

Before the next lemma we make a small observation.

Remark 2.12. *Let $Z \subseteq G$ and $W \subseteq G \times G$ be definable sets and let $f : W \to Z$ be a definable continuous map (all are 0-definable). Then for every $z \in f(W)$ that is locally generic in Z and $w \in f^{-1}(z)$, we have $\nu_z(Z) = f(\nu_w(W))$. (By "locally generic" we mean that for some open $V \ni z$, we have $dim(z/\emptyset) = \dim(V \cap Z)$).*

Proof. Fix V as above and let $U \subseteq G \times G$ be a definable open neighbourhood of w such that $f(U \cap W) \subseteq V \cap Z$ and z is still generic in $V \cap Z$ over the parameters (say A) defining U. Then, because $z \in f(U \cap W)$, we have

$$\dim(V \cap Z) = \dim(z/A) \leq \dim f(U \cap W) \leq \dim(V \cap Z).$$

Therefore, z is generic in both $V \cap Z$ and $f(U \cap W)$, so $\nu_z(Z) \subseteq f(U \cap W)$. Because this is true for every neighbourhood U of w, it follows that $\nu_z(Z) \subseteq f(\nu_w(W))$. The converse follows by continuity. □

Lemma 2.13. *Let X and Y be two G-linear sets. Then:*
(i) $X + Y$ is G-linear and we have

$$\nu(X + Y) = \nu(X) + \nu(Y).$$

In particular, if X and Y are G-equivalent G-linear sets then $X + Y$ is G-equivalent to them as well.
(ii) If X and Y are G-equivalent then $X \cup Y$ is a G-linear set and G-equivalent to X and Y.

Proof. (i) First, observe that, by continuity of $+$, for any $x \in X$, $y \in Y$, and $z = x + y$ we have $\nu_x(X) + \nu_y(Y) \subseteq \nu_z(X + Y)$. It follows that for all $z \in X + Y$, we have $\nu(X) + \nu(Y) \subseteq \nu_z(X + Y) - z$.

By Remark 2.12, if $z = x + y$ is a locally generic element of $X + Y$ then $\nu_z(X + Y) - z = \nu_x(X) + \nu_y(Y) - z = \nu(X) + \nu(Y)$. Therefore, $X + Y$ is locally G-linear at every locally generic $z \in X + Y$ and we have $\nu_z(X + Y) - z = \nu(X) + \nu(Y)$.

It follows that $(X + Y)_{lin}$ is large (actually dense) in $X + Y$ and the germs of $X+Y$ at all points in $(X+Y)_{lin}$ are G-equivalent. Taken together with our first observation, we see that $X + Y$ satisfies the assumptions of Lemma 2.11, hence it is G-linear and we have $\nu(X + Y) = \nu(X) + \nu(Y)$.

If X and Y are G-equivalent then $\nu(X) = \nu(Y)$, hence $\nu(X) + \nu(Y) = \nu(X)$, so $X + Y$ is G-equivalent to both X and Y.

(ii) It is easy to see that for all $z \in X \cup Y$, if $z \notin Fr(X) \cup Fr(Y)$, then we have $(X \cup Y) - z =_{0_G} X - z$ (if $z \in X$) or $(X \cup Y) - z =_{0_G} Y - z$ (if $z \in Y$). Because X and Y are locally G-linear at every point, it follows that $(X \cup Y)_{lin}$ contains $(X \cup Y) \setminus (Fr(X) \cup Fr(Y))$ and hence large in $X \cup Y$. Moreover, because X and Y are G-equivalent, for every $g \in (X \cup Y)_{lin}$, $x \in X$ and $y \in Y$, we have

$$(X \cup Y) - g =_{0_G} X - x =_{0_G} Y - y.$$

In particular, for every $g \in (X \cup Y)_{lin}$ and $h \in X \cup Y$, we have $(X \cup Y) - g \subseteq_{0_G} (X \cup Y) - h$. We therefore can apply Lemma 2.11 and conclude that $X \cup Y$ is G-linear, and moreover G-equivalent to X and Y. $\square$

Lemma 2.14. *Let X be a G-linear set and Y_1, Y_2 two G-subsets of X. Then*
(i) $Y_1 + Y_2$ is a G-subset of X.
(ii) $Y_1 \cup Y_2$ is a G-subset of X.

Proof. (i) Because Y_1 and Y_2 are G-subsets of X, for all $y_1 \in Y_1$, $y_2 \in Y_2$, we have

$$(\nu_{y_1}(Y_1)+\nu_{y_2}(Y_2))-(y_1+y_2) = (\nu_{y_1}(Y_1)-y_1)+(\nu_{y_2}(Y_2)-y_2) \subseteq \nu(X)+\nu(X) = \nu(X).$$

Again, by Remark 2.12, if $z = y_1 + y_2$ is a locally generic element of $Y_1 + Y_2$ then

$$\nu_z(Y_1 + Y_2) - z = \nu_{y_1}(Y_1) + \nu_{y_2}(Y_2) - z \subseteq \nu(X)$$

(the last inclusion is a result of Y_1, Y_2 being G-subsets of X). Hence, the set

$$Y_0 = \{z \in Y_1 + Y_2 : \nu_z(Y_1 + Y_2) - z \subseteq \nu(X)\}$$

is dense in $Y_1 + Y_2$. Because $Y_0 \subseteq Y_1 + Y_2$ it follows from its definition that Y_0 is itself a G-subset of X. Hence, by Lemma 2.10 $Y_1 + Y_2$ is a G-subset of X.

(ii) By 2.10, we may assume that Y_1, Y_2 are closed. It follows that for every $y \in Y_1 \cup Y_2$, the set $\nu_y(Y_1 \cup Y_2)$ is either equal to $\nu(Y_1)$, or to $\nu_y(Y_2)$ (if y belongs to only one of the Y_i's) or, if $y \in Y_1 \cap Y_2$ then $\nu_y(Y_1 \cup Y_2) = \nu_y(Y_1) \cup \nu_y(Y_2)$. Because each Y_i is a G-subset of X it follows that $\nu_y(Y_1 \cup Y_2) \subseteq \nu(X)$. □

We recall the following definition from [10]:

Definition 2.15. *Given a definable group G in an κ-saturated $\mathcal{M}$, a subgroup $\mathcal{H}$ of G is called* $\bigvee$-definable, *if it can be written as the directed union of A-definable sets with $|A| < \kappa$, $\mathcal{H} = \bigcup\{X_i : i \in I\}$.*

We say that $\mathcal{H}$ is connected *if the X_i's can all be chosen to be definably connected.*

The dimension of a $\bigvee$-definable group is taken to be $max\{\dim X_i : i \in I\}$. Notice that if $\mathcal{H}$ is connected then it is actually definably path connected in the sense that any two points can be connected by a definable path contained in $\mathcal{H}$. Notice that the notion of connectness for such groups is quite subtle, see [1]. The following claim is easy to verify:

Claim 2.16. *If $\mathcal{H} = \bigcup\{X_i : i \in I\}$ is a $\bigvee$-definable group and $g \in \mathcal{H}$ then there exists $i \in I$ such that $g \in X_i$ and*

$$\nu_g(\mathcal{H}) = g + \nu_{0_G}(\mathcal{H}) = \nu_g(X_i).$$

Lemma 2.17. *Every $\bigvee$-definable subgroup $\mathcal{H}$ of G can be written as the directed union of G-linear sets.*

Proof. Without loss of generality all X_i's in the definition of $\mathcal{H}$ have maximal dimension d. By Claim 2.16, for every $g \in \mathcal{H}$ there exists $i \in I$ such that $\nu_g(\mathcal{H}) = \nu_g(X_i)$. For such an i we have: $g \in (X_i)_{lin}$, the local dimension of X_i at g equals d (hence, $g \in (X_i)^{max}_{lin}$) and all components of $(X_i)^{max}_{lin}$ are G-linear and G-equivalent. If we let $X'_i = (X_i)^{max}_{lin}$ then we have $\mathcal{H} = \bigcup_{i\in I} X'_i$.

To see that $\{X'_i : i \in I\}$ is a directed system of sets: Given $i, j \in I$, we take $k \in I$ such that $X_i \cup X_j \subseteq X_k$ and claim that $X'_i \cup X'_j \subseteq X'_k$. Indeed, if $g \in X'_i$ then $\nu_g(X'_i) = \nu_g(\mathcal{H})$ and hence $\nu_g(X_i) = \nu_g(X_k)$. In particular, $g \in (X_k)^{max}_{lin} = X'_k$. □

Lemma 2.18. *We assume $\mathcal{M}$ is an ω-saturated structure. Let X be a definable G-linear set, $0_G \in X$. Then the following hold.*

(1) The group $\langle X \rangle$ generated by X is $\bigvee$-definable and $\nu_{0_G}(\langle X \rangle)$ equals to $\nu(X)$. In particular, $\dim(\langle X \rangle) = \dim X$.

(2) If Y is a G-subset of X containing 0_G, then the group $\langle Y \rangle$ generated by Y is a $\bigvee$-definable group of dimension $\leq \dim X$ and $\nu_{0_G}(\langle Y \rangle)$ is contained in $\nu(X)$.

Proof. (1) The group $\langle X \rangle$ generated by X is a countable increasing union of the sets $X_0 = X$, $X_1 = X - X$, $X_2 = (X - X) + (X - X), \ldots$, so $\bigvee$-definable. By Lemma 2.13, each X_n is G-linear and G-equivalent to X (so in particular, has the same dimension as X). Given $g \in \langle X \rangle$, there exists $n \geq 0$ such that $g \in X_n$. Because of the G-linearity, for every $k \geq n$, we have $X_n =_g X_k$. Because of saturation, there exists a neighbourhood U of g such that $U \cap \langle X \rangle = U \cap X_n$. It follows that the germ of $\langle X \rangle$ at this point equals to that of X_n and in particular, $\nu_{0_G}(\langle X \rangle) = \nu(X)$.

(2) The group $\langle Y \rangle$ generated by Y is a countable increasing union of the sets $Y_0 = Y$, $Y_1 = Y - Y$, $Y_2 = (Y - Y) + (Y - Y), \ldots$. Because $-Y$ is also a G-subset of X, we can apply Lemma 2.14 and conclude that each Y_n is a G-subset of X whose germ at 0_G is contained in $\nu(X)$. It follows that the dimension of $\langle Y \rangle$ is at most $\dim X$ and that the germ of $\langle Y \rangle$ at 0_G is contained in that of $\langle X \rangle$. ☐

We end this section with a small observation on $\bigvee$-definable subgroups:

Lemma 2.19. *Let $\mathcal{H}_1, \mathcal{H}_2$ be $\bigvee$-definable subgroups of G. Then $\mathcal{H}_1 + \mathcal{H}_2$ is a $\bigvee$-definable group whose germ at 0 equals to the sum of the germs of $\mathcal{H}_1$ and $\mathcal{H}_2$ at 0.*

Proof. Let

$$\mathcal{H}_1 = \bigcup_{i \in I} X_i \ ; \ \mathcal{H}_2 = \bigcup_{j \in J} Y_j.$$

By Lemma 2.17, we may assume that the X_i's Y_j's are all G-linear. By Lemma 2.13(i), the sets $X_i + Y_j$ are all G-linear and $\nu(X_i + Y_j) = \nu(X_i) + \nu(Y_j)$. By Lemma 2.18(1),

$$\nu_{0_G}(\mathcal{H}_1 + \mathcal{H}_2) = \nu(X_i + Y_j) = \nu(X_i) + \nu(Y_j) = \nu_{0_G}(\mathcal{H}_1) + \nu_{0_G}(\mathcal{H}_2).$$

☐

3 Definable sets containing abstract subgroups

In this section $\mathcal{M}$ will denote an ω-saturated o-minimal expansion of an ordered group.

Theorem 3.1. *Let G be a definable abelian group, $\Gamma \subseteq G$ an abstract subgroup (i.e., Γ is not necessarily definable). Let d be the minimal dimension of definable subsets of G containing Γ. Then there exist a definable set X of dimension d, a connected $\bigvee$-definable subgroup $\mathcal{H}$ of dimension d, and $g_1, \ldots, g_k \in \Gamma$ such that $\Gamma \subseteq X \subseteq \bigcup_{i=1}^{k} \mathcal{H} + g_i$.*

Proof. Assume the nontrivial case $d > 0$. Let $X \subseteq G$ be a definable set containing Γ of dimension d. For every $g \in \Gamma$, the set $X \cap (g+X)$ contains Γ and, hence by minimality, has dimension d. It follows that the set $Y = \{g \in X : \dim(X \cap (X+g)) = d\}$ contains Γ and therefore, again by minimality, has dimension d.

We now apply Corollary 2.7 to Y, Y, X and X (for X_0, X, Y and Z, respectively), and conclude that Y_{lin} is large in Y.

Since Y contains Γ we may replace X with Y and assume from now on that X_{lin} is large in X (however, it need not be the case that X or even X_{lin} is G-linear). Moreover, we pick $X \supseteq \Gamma$ such that X_{lin} has the minimal number of definably connected components of dimension d. Note that each component of dimension d has infinitely many elements of Γ (otherwise, we can replace the component by finitely many points).

Claim 1 All the components of X_{lin} of dimension d are G-equivalent to each other.

Proof. Let X_1, X_2 be two such components. For $g \in X_1$, let $X_2(g) = \{h \in X_2 : g+h \in X\}$. If $g \in X \cap \Gamma$ then $X_2 \cap \Gamma \subseteq X_2(g)$ and hence $\dim X_2(g) = d$. Thus, the set $X_1' = \{g \in X_1 : \dim X_2(g) = d\}$ contains $X_1 \cap \Gamma$ and hence has dimension d as well.

We now pick $g \in X_1'$ generic and $h \in X_2(g)$ generic over g (note that g and h are generic in X_1, X_2, respectively). We can now find open neighbourhoods $U \ni g$ and $V \ni h$ such that $(X_1 \cap U) + (X_2 \cap V) \subseteq X$. By Lemma 2.6 (and the fact that each X_i is G-linear), the sets $U \cap X_1$ and $V \cap X_2$ are G-equivalent. Because X_1 and X_2 are G-linear it follows that they are G-equivalent as well. **End of Claim 1**.

We therefore showed that X_{lin}^{max} is G-linear. We shall construct $\mathcal{H}$ as the sum of two $\bigvee$-definable subgroups, one obtained via de components of X_{lin}^{max} and the other one via the rest of the components of X. Let $X^* = X \setminus X_{lin}^{max}$. Fix one of the components X_0 of X_{lin}^{max}.

As before, for every $g \in \Gamma \cap X^*$, the set $\{h \in X_0 : g + h \in X\}$ contains $X_0 \cap \Gamma$ and hence has dimension d. Therefore, after possibly replacing X^* by a smaller set we may assume that for all $g \in X^*$ the set $(g + X_0) \cap X$ has dimension d.

Claim 2 X is a G-subset of X_{lin}^{max}.

Proof. By abuse of notation we let $\nu(X)$ be the infinitesimal subgroup associated to the G-linear set X_{lin}^{max}. By Lemma 2.10, it is enough to see that a dense subset of X is G-subset of X_{lin}^{max}. Namely, we will show that for every locally generic $g \in X$, we have $\nu_g(X) - g \subseteq \nu(X)$. It is clearly sufficient to consider $g \in X^*$.

We fix an open set $U \subseteq G$ and $g \in X^*$ which is generic in $U \cap X$. By our assumption on X^*, there exists h generic in X_0 over g such that $g + h$ is (generic) in X. As in the proof of Lemma 2.6, it follows that $\nu_g(X) + h \subseteq \nu_{g+h}(X)$, hence

$$\nu_g(X) - g \subseteq \nu_{g+h}(X) - (g + h) = \nu(X).$$

(the right-most equality follows from the fact that $g+h$ is generic in X and hence belongs to X_{lin}^{max}). **End of Claim 2.**

Let $X_{lin}^{max} = X_1 \cup \dots \cup X_r$ be the union of those components of X_{lin} of dimension d. By Lemma 2.4 each X_i is G-linear. For each X_i, pick $h_i \in \Gamma \cap X_i$, and consider the set $X'_i = X_i - h_i$. By Claim 1 and Lemma 2.13, the union $X' = \bigcup_{i=1}^r X'_i$ is G-linear (and $\nu(X') = \nu(X_i)$ for every i). It also contains 0_G and it is definably connected. Let $\mathcal{H}'$ be the subgroup of G generated by X'. By Lemma 2.18(i), $\dim \mathcal{H}' = \dim X' = d$. We thus have

$$X_{lin}^{max} \subseteq \bigcup_{i=1}^r \mathcal{H}' + h_i.$$

Let $X^1, \dots, X^t$ be the definably connected components of X^*. For every such X^j, we may assume that $X^j \cap \Gamma \neq \emptyset$ (for otherwise we may omit this component), take $h^j \in \Gamma \cap X^j$, and let $X'' = \bigcup_{j=1}^t X^j - h^j$. By Claim 2, each X^j (and therefore also $X^j - h^j$) is a G-subset of X_{lin}^{max}. By Lemma 2.14 (ii), the set X'' is also a G-subset of X_{lin}^{max}. It is also definably connected and contains 0_G.

Let $\mathcal{H}''$ be the subgroup of G generated by X''. By Lemma 2.18(ii), the germ of $\mathcal{H}''$ at 0_G is contained in that of X' and therefore of $\mathcal{H}'$. We now let $\mathcal{H} = \mathcal{H}' + \mathcal{H}''$. By Lemma 2.19, we have $\nu_{0_G}(\mathcal{H}) = \nu_{0_G}(\mathcal{H}' + \mathcal{H}'') = \nu_{0_G}(\mathcal{H}')$, so in particular the dimension of $\mathcal{H}$ is d.

Putting the above facts together we obtain:

$$\Gamma \subseteq X \subseteq \bigcup_{i=1}^{k} \mathcal{H} + g_i,$$

for some $g_1, \dots, g_k \in \Gamma$.

With that we end the proof of the theorem. $\square$

Corollary 3.2. *Let G be a definably connected abelian group. Let Γ be a divisible subgroup of the subgroup of torsion points $Tor(G)$. Let $X \subset G$ be a definable set containing Γ. Then, there is a connected $\bigvee$-definable subgroup $\mathcal{H}$ of G with $dim\, \mathcal{H} \leq dim\, X$ such that $\Gamma \subset \mathcal{H}$.*

Proof. By Theorem 3.1, taking X of minimal dimension, there is a definably connected $\bigvee$-definable group $\mathcal{H}$ subgroup of G, with $dim\, \mathcal{H} = dim\, X$, and $g_1, \dots, g_k \in \Gamma$ such that $\Gamma \subseteq \bigcup_{i=1}^{k} \mathcal{H} + g_i$. Then $\Gamma \subset \mathcal{H}$. Indeed, let $g \in \Gamma$ and let $m = lcm(ord(g_1), \dots, ord(g_k))$, since Γ is divisible there is $h \in \Gamma$ such that $g = mh$. For such h there is an i such that $h = h' + g_i$, for some $h' \in \mathcal{H}$. Then $g = mh = mh' \in \mathcal{H}$. $\square$

4 The main result: commutative case

In this section $\mathcal{M}$ will be an ω-saturated o-minimal expansion of a real closed field and G will denote a definably compact definably connected abelian group of dimension n. Such G is divisible and hence the subgroup $Tor(G)$ and any p-Sylow of G are also divisible (the p-Sylow of G is $G_p = \bigcup_{n>0} G[p^n]$, where $G[m] = \{g \in G \colon mg = 0_G\}$).

By Theorem 1.1 in [5] we have $G[m] \cong (\mathbb{Z}/m\mathbb{Z})^n$ for any $m > 0$, and $\pi_1(G) \cong \mathbb{Z}^n$, where $\pi_1(G)$ is the o-minimal fundamental group of G.

Let p be a prime number and let $x_1, \dots, x_n \in G[p]$. We say that $x_1, \dots, x_n$ are n *independent p-torsion points* if they are $\mathbb{F}_p$-independent under the isomorphism $G[p] \cong {\mathbb{F}_p}^n$.

Theorem 4.1. *Let X be a definable subset of G. If X contains a p-Sylow of G, then $dim\, X = n$.*

Lemma 4.2. *Let p be a prime number and let $x_1, \dots, x_n \in G$ be n independent p-torsion points. For each $i = 1, \dots, n$ let τ_i be a path in G from 0_G to x_i, and let $p\,\tau_i$ denote the loop at 0_G, $t \mapsto p\,\tau_i(t)$. Then, $[p\,\tau_1], \dots, [p\,\tau_n] \in \pi_1(G)$ are $\mathbb{Z}$-linearly independent.*

Proof. Let $\varphi\colon \pi_1(G) \to G[p]$ be the homomorphism $[\gamma] \mapsto \tilde{\gamma}(1)$, where $\tilde{\gamma}$ is the unique path in G starting at 0_G and such that $p\tilde{\gamma} = \gamma$ (for the existence of such $\tilde{\gamma}$, see the proofs of propositions 2.10 and 2.11 in [5]). Suppose $[p\tau_1], \dots, [p\tau_n]$ are $\mathbb{Z}$-linearly dependent and let $m_1[p\tau_1] + \cdots + m_n[p\tau_n] = [k_{0_G}]$ with $(m_1, \dots, m_n) = 1$, where k_{0_G} is the constant loop at 0_G. Applying φ to this equality we get $m_1x_1 + \cdots + m_nx_n = 0_G$ and hence $p|(m_1, \dots, m_n)$, a contradiction. $\square$

Lemma 4.3. *Let $[\gamma_1], \dots, [\gamma_n] \in \pi_1(G)$ be $\mathbb{Z}$-linearly independent. Then,*

$$\left\{\sum_{i=1}^{n} \gamma_i(t_i)\colon t_i \in [0,1), 1 \leq i \leq n\right\} = G.$$

Proof. Consider the definably compact definably connected n-dimensional abelian group $\mathbb{T} = [0,1)^n$ and the definable map

$$\begin{array}{cccc} f\colon & \mathbb{T} & \longrightarrow & G \\ & (t_1, \dots, t_n) & \mapsto & \sum_{i=1}^{n} \gamma_i(t_i). \end{array}$$

It suffices to prove that f is onto. Since f is continuous with respect to the manifold topology of $\mathbb{T}$, we can see f as a definable continuous map between definable manifolds.

Claim It suffices to prove that f induces an isomorphism $f_*\colon H_1(\mathbb{T};\mathbb{Q}) \to H_1(G;\mathbb{Q})$.

Proof. The manifolds $\mathbb{T}$ and G –being definably compact groups– are orientable manifolds (see [3]). Fix an orientation in each one of $\mathbb{T}$ and G. By Proposition 4.1 in [5] applied to $N = \mathbb{T}$ (respectively, $N = G$) we obtain a generator $\zeta_{\mathbb{T}}$ of the top o-minimal homology group $H_n(\mathbb{T}) \cong \mathbb{Z}$ (resp. ζ_G generator of $H_n(G) \cong \mathbb{Z}$). The generator $\zeta_{\mathbb{T}}$ (resp. ζ_G) is called the fundamental class of $\mathbb{T}$ (resp. of G) for the given orientation. Therefore, f is a definable continuous map between oriented definably compact definable manifolds of dimension n, so we can define (as in section 4 of [5]) the degree of f by the following equation: $f_*(\zeta_{\mathbb{T}}) = deg(f)\zeta_G$, where $f_* : H_n(\mathbb{T}) \to H_n(G)$ is the map induced by f in o-minimal homology.

Now suppose f is not onto and let $q \in G \setminus Imf$. Then, by definition of *degree of f over $\{q\}$* (see section 4 of [5]) $deg_{\{q\}}f = 0$. On the other hand, since G is definably connected $degf = deg_{\{q\}}f = 0$. Therefore, it suffices to prove that $degf \neq 0$.

Next, we argue as in the proof of the Structure Theorem (section 5 of [5]). We first consider the o-minimal cohomology classes $\omega_{\mathbb{T}} \in H^n(\mathbb{T}, \mathbb{Q})$ and $\omega_G \in H^n(G, \mathbb{Q})$ corresponding to $\zeta_{\mathbb{T}}$ and ζ_G, respectively, i.e., $\omega_{\mathbb{T}}(\zeta_{\mathbb{T}}) = 1$

and $\omega_G(\zeta_G) = 1$. Then, we have that $f^*(\omega_G) = deg(f)\omega_{\mathbb{T}}$, where f^* is the map induced by f in o-minimal cohomology. Since ω_G (respectively $\omega_{\mathbb{T}}$) generates the 1-dimensional $\mathbb{Q}$-vector space $H^n(G, \mathbb{Q})$ (resp. $H^n(\mathbb{T}, \mathbb{Q})$), to prove that $deg f \neq 0$ suffices to show that $f^*\colon H^n(G; \mathbb{Q}) \to H^n(\mathbb{T}; \mathbb{Q})$ is a $\mathbb{Q}$-vector space isomorphism.

Now, by Theorem 3.4 in [5], $f^*\colon H^*(G; \mathbb{Q}) \to H^*(\mathbb{T}; \mathbb{Q})$ is a $\mathbb{Q}$-algebra morphism. (Note that by abuse of notation we are denoting with the same f^* both the map between the $\mathbb{Q}$-algebras and its restriction to the $\mathbb{Q}$-vector space $H^m(G; \mathbb{Q})$, for each $m = 1, \dots, n$.)

On the other hand, since both $\mathbb{T}$ and G are abelian groups, the cohomology algebras $H^*(G; \mathbb{Q})$ and $H^*(\mathbb{T}; \mathbb{Q})$ are generated by elements of degree one, i.e., elements in $H^1(G; \mathbb{Q})$ and $H^1(\mathbb{T}; \mathbb{Q})$, respectively. Let $\omega_G = x_1 \cdot \dots \cdot x_n$, so that $\{x_1, \dots, x_n\}$ is a basis of the $\mathbb{Q}$-vector space $H^1(G; \mathbb{Q})$. Hence $f^*(\omega_G) = f^*(x_1) \cdot \dots \cdot f^*(x_n)$. Finally, if f induces an isomorphism $f_*\colon H_1(\mathbb{T}; \mathbb{Q}) \to H_1(G; \mathbb{Q})$ of $\mathbb{Q}$-vector spaces, then $f^*\colon H^1(G; \mathbb{Q}) \to H^1(\mathbb{T}; \mathbb{Q})$ is also a $\mathbb{Q}$-vector space isomorphism by duality, and so the set $\{f^*(x_1), \dots, f^*(x_n)\}$ is a basis of the $\mathbb{Q}$-vector space $H^1(\mathbb{T}; \mathbb{Q})$ and hence their product $f^*(x_1) \cdot \dots \cdot f^*(x_n)$ generates $H^n(\mathbb{T}, G)$, so it must be a nonzero multiple of $\omega_{\mathbb{T}}$. Therefore, $deg f \neq 0$ as required.**End of Claim**.

Now, let $\delta_i\colon [0, 1) \to \mathbb{T}\colon t \mapsto \delta_i(t) = (0, \dots, \overset{i}{1}, \dots, 0)$. The map f induces a map $f_*\colon \pi_1(\mathbb{T}) \to \pi_1(G)\colon [\delta_i] \mapsto f_*([\delta_i]) = [\gamma_i]$, for each $i = 1, \dots, n$, which is one to one and has finite cokernel ($= \pi_1(G)/Im(f_*)$) because we have n $\mathbb{Z}$-linear independent $[\gamma_i]$'s. Identifying the π_1's with the H_1's via the Hurewicz isomorphism, we have the following exact sequence

$$0 \to H_1(\mathbb{T}) \xrightarrow{f_*} H_1(G) \to H_1(G)/Im(f_*) \to 0.$$

Tensoring with $\mathbb{Q}$ we obtain that $f_*\colon H_1(\mathbb{T}; \mathbb{Q}) \to H_1(G; \mathbb{Q})$ is an isomorphism. □

Corollary 4.4. *Let $\mathcal{H}$ be a definably connected $\bigvee$-definable subgroup of G. Suppose $\mathcal{H}$ contains $G[p]$, for some prime p. Then $\mathcal{H} = G$.*

Proof. Let $x_1, \dots, x_n$ be n independent p-torsion points and for each $i = 1, \dots, n$ let τ_i be a path in $\mathcal{H}$ from 0_G to x_i. Let γ_i denote the loop at 0_G defined by $t \mapsto p\tau_i(t)$. Since $\mathcal{H}$ is a group $\gamma_i([0, 1]) \subset \mathcal{H}$. By Lemma 4.2 $[\gamma_1], \dots, [\gamma_n] \in \pi_1(G)$ are $\mathbb{Z}$-linearly independent.

By Lemma 4.3, $S = \{\sum_{i=1}^n \gamma_i(t_i)\colon t_i \in [0, 1), 1 \le i \le n\} = G$. Again since $\mathcal{H}$ is a group, $S \subset \mathcal{H}$. □

Proof of Theorem 4.1. Let $X \subseteq G$ be a definable set containing a p-Sylow G_p of G. Since G_p is divisible, by Corollary 3.2, there exists a definably connected $\bigvee$-definable group $\mathcal{H}$ of dimension $\leq dim\, X$ containing G_p. Since G_p contains $G[p]$, we can apply Corollary 4.4 to get $\mathcal{H} = G$ and hence $dim\, X = dim\, G$ $\square$

Corollary 4.5. *Let X be a definable subset of G. If X is large in G then for any prime p and any l there is an $m > l$ and a $g \in X$ of order p^m.*

Proof. Note that this corollary is equivalent to Theorem 4.1. Indeed, for the nontrivial case suppose there is a prime p and an l such that $X \cap G_p \subset G[p^l]$. So the set $X' = X \setminus G[p^l]$ is still large in G and $X' \cap G_p = \emptyset$. But then $G \setminus X'$ contains G_p and $\dim(G \setminus X') < \dim G$, a contradiction with the theorem. $\square$

5 The main result: general case

We work in an ω-saturated structure $\mathcal{M}$ which is an o-minimal expansion of a real closed field $\mathcal{M}$. We will use multiplicative notation for groups, and denote by p_k the map $x \mapsto x^k$. A definably connected group which is either abelian or definably compact is divisible (see [2] or [4]), hence the map p_k is onto. Recall that a *generic subset* Y of a definable abelian group G is a definable subset of G such that finitely many translates of Y cover G.

Lemma 5.1. *Let T be a definably connected abelian group. Suppose that for every generic $Y \subseteq T$, $Y \cap Tor(T) \neq \emptyset$. Then, for each generic $Y \subseteq T$ there is a $k \in \mathbb{N}$ such that $p_k(Y) = T$.*

Proof. First note that for each $Y \subseteq T$ generic, there are $g_1, \ldots, g_l \in Tor(T)$ such that $T = \bigcup_{i=1}^{n} g_i Y$. Indeed, if not, by compactness, there is $h \notin Tor(T)Y$, *i.e.*, $hY^{-1} \cap Tor(T) = \emptyset$, but hY^{-1} is also generic, a contradiction. Now taking $k = lcm(ord(g_1), \ldots, ord(g_l))$ we have $p_k(Y) = p_k(T) = T$ as required. $\square$

We recall that a maximal definably connected abelian subgroup T of a definably connected definably compact group G is called a *maximal torus of* G and that G is the union of the conjugates of T, *i.e.*, $G = \bigcup_{g \in G} T^g$ (see Theorem 6.12 in [2] or [4]).

Lemma 5.2. *Let G be a definably compact definably connected group. Let T be a maximal torus of G and $X \subseteq G$ definable. Assume the following:*

(i) For every $g \in G$ and for every generic $Y \subseteq T^g$, $Y \cap Tor(T^g) \neq \emptyset$, and

(ii) For every $g \in G$, $X \cap T^g$ is generic in T^g.

Then there is a $k \in \mathbb{N}$ such that $p_k(X) = G$.

Proof. Given $g \in G$, by (ii), we can apply Lemma 5.1, for $Y = X \cap T^g$, to get a $k(g) \in \mathbb{N}$ such that the map $y \mapsto y^{k(g)}$ sends $X \cap T^g$ onto T^g (note that by divisibility of T^g any multiple of $k(g)$ has the same property). Then, by compactness, there is $k \in \mathbb{N}$ such that for every $g \in G$, the map $y \mapsto y^k$ sends $X \cap T^g$ onto T^g. To finish the proof suffices to make use of the equality $G = \bigcup_{g \in G} T^g$. □

For the next lemma we will use the notions of *compact domination* and *very good reduction* introduced in [7]. Recall that a definably simple group has very good reduction and that a definably compact group which has very good reduction has also compact domination (see Theorem 10.7 in [7]).

Lemma 5.3. *Let G be a definably compact definably connected group. Let $X \subseteq G$ be a definable set such that $Tor(G) \subseteq X$. Then the following holds.*

(i) If G has very good reduction then there is $k \in \mathbb{N}$ such that $p_k(X) = G$.

(ii) If N is a finite (central) normal subgroup of G and G/N has very good reduction then there is $k \in \mathbb{N}$ such that $p_k(X) = G$.

Proof. The result can probably be read off standard Lie theory. However, since we could not find a reference we give a complete proof.

(i) Without lost of generality we may assume that $G = G(M)$ is defined over the reals. By our assumptions on G, $G(\mathbb{R})$ is a compact connected Lie group. Let T_0 be a standard maximal torus of $G(\mathbb{R})$, hence T_0 is also a Lie group and hence definable over the reals. Let $T = T_0(\mathcal{M})$. So T is still a maximal torus of G and it has very good reduction and hence it is compactly dominated. The same happens for every conjugate T^g of T, for $g \in G$. Since all T^g's are also abelian, we can apply Proposition 10.6 in [7], to get that for every $g \in G$, every generic subset of T^g has a torsion point, so condition (i) of Lemma 5.2 is satisfied. On the other hand $Tor(G) \subseteq X$, so the set $T^g \setminus X$ has no torsion and hence, by Lemma 5.2 (i), it is non-generic in T^g. It follows that $T^g \cap X$ is generic in T^g and therefore Lemma 5.2 (ii) holds. Now we can apply Lemma 5.2 to get the required result.

(ii) Let $\pi : G \to G/N$ be the projection map. Because N is finite, all torsion elements of G/N are in $\pi(X)$, hence by (i), there is k such that $p_k(\pi(X)) = G/N$. If we take $k' = k \cdot |N|$ then $p_{k'}(X) = G$ (because N is central and G divisible). □

Notice, in the setting of the above lemma, that if $p_k(X) = G$ then in particular, $\dim X = \dim G$ (since the image of X under a definable map cannot increase in dimension).

We can now prove the main result:

Theorem 5.4. *Let G be a definably compact group and assume that $X \subseteq G$ is a definable set containing all torsion points of G. Then $\dim X = \dim G$.*

Proof. By 4.1 in [9] and 5.4 in [12], after modding out the semisimple group $G_1 = G/Z(G)$ by its finite centre, the group we obtain is a direct product of definably simple groups. In particular, it has very good reduction (see the proof of 5.1 in [11]) and hence (ii) of Lemma 5.3 holds for $G/Z(G)$. We therefore have:

(a) For every definable $X_1 \subseteq Z(G)$, if $Tor(Z(G)) \subseteq X_1$ then $\dim X_1 = \dim Z(G)$ (by Theorem 4.1 this is true for subsets of $Z(G)^0$, but then it clearly follows for subsets of $Z(G)$ as well).

(b) For every definable $X_1 \subseteq G$, if $Tor(G/Z(G)) \subseteq X_1/Z(G)$ then $\dim(X_1/Z(G)) = \dim(G/Z(G))$. (by Lemma 5.3 and our previous observation).

We now proceed as follows. Let $X \subseteq G$ be a definable set containing $Tor(G)$. Given $g \in Tor(G)$ and $h \in Tor(Z(G))$, we have $gh \in Tor(G) \subseteq X$. Hence, for every such g, we have

$$X_g = \{h \in Z(G) : gh \in X\} \supseteq Tor(Z(G)),$$

so, by (a), $\dim X_g = \dim Z(G)$. Since $gX_g = (gZ(G)) \cap X$, we have, for every $g \in Tor(G)$,

$$\dim((gZ(G)) \cap X) = \dim Z(G).$$

Let

$$X_1 = \{g \in G : \dim((gZ(G)) \cap X) = \dim Z(G)\}.$$

It follows from the above that $Tor(G) \subseteq X_1$, and therefore $Tor(G/Z(G)) \subseteq X_1/Z(G)$. (Indeed, if $hZ(G) \in Tor(G/Z(G))$ then for some $n \in \mathbb{N}$, $h^n \in Z(G)$, and therefore for some $k \geq n$, $h^k \in Z(G)^0$. Because $Z(G)^0$ is divisible there exists $h_1 \in Z(G)$, with $h_1^k = h^k$. Because h_1 is central, $(hh_1^{-1})^k = e$, hence $hh_1^{-1} \in X_1$ and therefore $hZ(G) \in X_1/Z(G)$). By (b), $\dim(X_1/Z(G)) = \dim(G/Z(G))$.

To finish the proof, consider the set $Y = (X_1 Z(G)) \cap X$. The cosets of $Z(G)$ partition Y into equivalence classes, each of dimension $\dim Z(G)$ (by definition of X_1). Since every $Z(G)$-coset of an element in X_1 intersects X nontrivially, we have

$$\dim(Y/Z(G)) = \dim(X_1/Z(G)) = \dim(G/Z(G)).$$

Summarizing, we have

$$\dim Y \geq \dim Z(G) + \dim(G/Z(G)) = \dim G,$$

and hence (since $Y \subseteq X$) $\dim X = \dim G$. □

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